Hamburger Beiträge
zur Angewandten Mathematik

Root Images of Median Filters

Ulrich Eckhardt

Reihe A
Preprint 165
August 2001

Hamburger Beiträge zur Angewandten Mathematik

Reihe A	Preprints
Reihe B	Berichte
Reihe C	Mathematische Modelle und Simulation
Reihe D	Elektrische Netzwerke und Bauelemente
Reihe E	Scientific Computing
Reihe F	Computational Fluid Dynamics and Data Analysis

Root Images of Median Filters

Ulrich Eckhardt
University Hamburg
Department of Mathematics
— Optimization and Approximation —
Bundesstrasse 55
D–20 146 Hamburg

Abstract

Median filters are frequently used in signal analysis because of their smoothing properties and their insensitivity with repect to outliers in the data. Since median filters are nonlinear filters, the tools of linear theory are not applicable to them. One approach to deal with nonlinear filters consists in investigating their root images (fixed elements or signals transparent to the filter). Whereas for one–dimensional median filters the set of all root signals can be completely characterized, this is not true for higher dimensional filters.

In 1989, Döhler stated a result on certain root images for two–dimensional median filters. Although Döhlers results are true for a wide class of median filters, his arguments were not correct and his assertions do not hold universally. In this paper we give a rigorous proof of Döhlers results. Moreover, his approach is generalized to the d–dimensional case.

1 Introduction

Median filters are widely used in digital signal processing (see e.g. [7, 11, 13]). They are members of a large class of nonlinear filters, the so–called stack filters. Since median filters have very attractive properties, they were investigated by numerous researchers.

There are mainly two directions of research in this field. The first one is concerned with convergence of iterated median filtering. Typically, weak convergence results (i.e. convergence for compactly supported signals) were obtained. Tyan [11] gave a characterization of signals leading to nonconvergent sequences. A number of examples for phenomena occuring with two dimensional median filters was presented by Ronse [10]. Goles and Olivos [6] provided a theory which in the case of median filters asserts that for each signal (of finite support) the iterated application of the median filter leads after a finite number of steps either to a signal which is unchanged under median filtering or else the sequence of iterates becomes 2–periodic.

The other main direction of research is devoted to those signals which remain unchanged under median filtering. Such signals are usually termed *root signals*. They provide an analogon of the passband signals of linear theory. Gallagher and Wise [4] characterized one–dimensional root signals. In higher dimensions, the situation is more complicated. Astola, Heinonen and Neuvo [1] gave a construction rule for obtaining multidimensional root signals from one–dimensional ones. Using concepts of digital geometry, Döhler [2] provided an explicit rule for obtaining so–called “smallest surviving objects” in the two–dimensional case. The aim of the present paper is to correct some errors of Döhler's paper and to generalize its results. A characterization of all digitally convex root signals (for "normal" windows) is presented.

We adopt here the terminology of Krabs' book on signal theory [8]. A *signal* is a function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$. A *filter* is a function which assigns to an input signal f an output signal g. The *median filter* is characterized by a *window* which is a finite set $U \in \mathbb{R}^d$ containing the zero element $\Theta_d \in \mathbb{R}^d$. Usually the number of points in the window is assumed to be odd. The result of median filtering the signal f is then found in the following way: First all function values $f(x+u)$ for $u \in U$ are sorted. The function value which has the central position in the sorted set provides the value $g(x)$. It can be shown that it is sufficient to consider only bivalued signals [3]. Such signals can be interpreted as characteristic functions of sets in $\mathbb{R}^d$. In this setting, the application of a median filter with window U to a set $S \subseteq \mathbb{R}^d$ results in a set $\mathrm{Med}_U S$ whose points can be found in the following way: A point $x \in \mathbb{R}^d$ belongs to $\mathrm{Med}_U S$ if and only if more than one half of the points in $x + U$ belongs to S.

Usually median filters are defined for discrete signals [8, Chapter 1]. That means that the domain of the signal function f is a "discretely generated" subset of $\mathbb{R}^d$. In most applications one is using the space $\mathbb{Z}^d$ for this purpose, which is the set of all vectors in $\mathbb{R}^d$ having integer components.

In his article, Döhler [2] states some results on root images of median filters in $\mathbb{Z}^2$. He explicitly requires

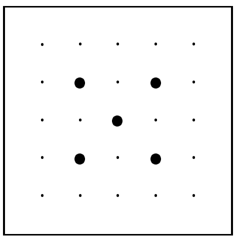
> ... our approach is restricted on a special root signal, which we call the smallest surviving object (SSO) of a median filter. This SSO is a convex digital polygon without holes.

Then he formulates as the main result of his paper

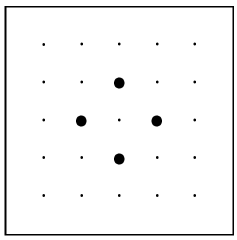
> **Proposition 3.** *The contour of the SSO results from the linkage of all possible straight line segments that connect the center of the window with some other lattice point within the window. This linkage must be done in such a manner that all straight line segments are sorted with respect to their slopes. If there are segments with the same slope, only the longest of them is to be considered.*

Looking closer at the situation, one suspects that this cannot be true in general. We start with two counterexamples.

Example 1.1 *In $\mathbb{Z}^2$ we consider the following neighborhood and a root image of the median filter corresponding to it. The latter was constructed by applying Döhler's algorithm for finding smallest surviving objects.*



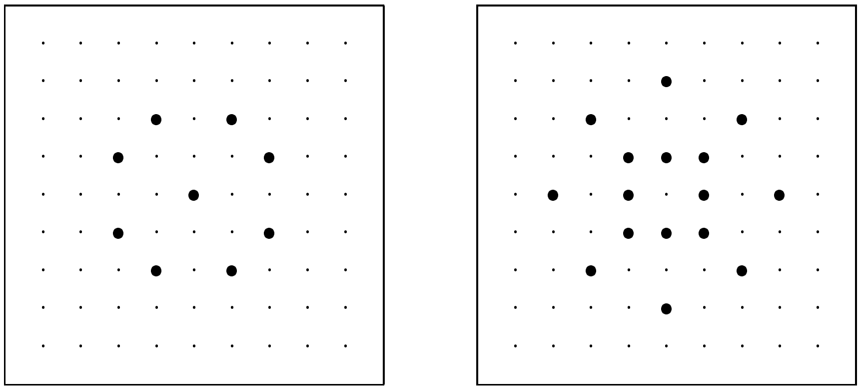
Neighborhood



Root Image

The root image is not convex. Its convex hull is not a root image with respect to the neighborhood. This is not surprising since it is not possible to join any two points in $\mathbb{Z}^2$ by a path consisting of directions from U. That means, $\mathbb{Z}^2$ is not connected in the "topology" generated by U

Example 1.2 *A more interesting example is given by the "knight's neighborhood" in $\mathbb{Z}^2$.*



Neighborhood Root Image

It is easily seen that any two points in $\mathbb{Z}^2$ can be joined by a series of knight's moves. However, the root image given in the picture is not convex and its convex hull is not a root image. The eight outer points of the root image were obtained by Döhler's construction. Neither these points alone nor their convex hull constitute a root image.

2 Convex Sets

We assume that our discrete spaces are embedded into $\mathbb{R}^d$, the vector space of d–tuples of real numbers which is equipped with the usual Euclidean geometry induced by the usual scalar product $\langle \cdot, \cdot \rangle$. We need some facts from the theory of digital convexity. For a rigorous and detailed treatment of the theory of convex sets in discrete spaces the reader is referred to Lateckis book [9].

A subset $S \subseteq \mathbb{R}^d$ is *convex* if the following implication is true: $x, y \in S$, $0 \leq \lambda \leq 1 \Longrightarrow \lambda x + (1-\lambda) y \in S$. The theory of convex sets is a very powerful tool in mathematics. For details see e.g. Valentine's book [12].

For a vector $x^* \in \mathbb{R}^d$ which is not the zero vector Θ_d and a real number α, the set

$$[x^* : \alpha] = \{x \in \mathbb{R}^d \mid \langle x^*, x \rangle = \alpha\}$$

is termed a *hyperplane* in $\mathbb{R}^d$. The set

$$[x^* : \alpha]^+ = \{x \in \mathbb{R}^d \mid \langle x^*, x \rangle \geq \alpha\}$$

is the *half–space* which is bounded by the hyperplane $[x^* : \alpha]$. Hyperplanes and half–spaces are convex sets.

For any set $S \subseteq \mathbb{R}^d$, the *convex hull* conv S is defined to be the intersection of all half–spaces containing S:

$$\text{conv } S = \bigcap_{S \subseteq [x^* : \alpha]^+} [x^* : \alpha]^+.$$

The convex hull of a set is a convex set.

A fundamental tool in convexity theory is the Separation Theorem for Convex Sets. In its simplest form it states (see [12])

Theorem 2.1 (Separation Theorem for Convex Sets) *Let $S \subseteq \mathbb{R}^d$ be a convex set and $x_0 \in \mathbb{R}^d$ not in the topological closure of S. Then there exists a vector $x^* \in \mathbb{R}^d$ and a number $\varepsilon > 0$ such that*

$$\langle x^*, x \rangle \geq \langle x^*, x_0 \rangle + \varepsilon \qquad \text{for all } x \in S.$$

It is a consequence of the Separation Theorem, that conv S is the smallest convex set containing S and that conv S is the set of all *convex combinations* from elements of S:

$$\text{conv } S = \left\{ x = \sum_{j=1}^{n} \lambda_j s_j \;\middle|\; s_j \in S,\ \sum_{j=1}^{n} \lambda_j = 1 \text{ and } \lambda_j \geq 0 \text{ for all } j = 1, 2, \cdots, n \right\}.$$

Definition 2.1 *Let $U = \{u_1, u_2, \cdots, u_n\}$ be a finite subset of $\mathbb{R}^d$ with $\Theta_d \in U$.*

The discrete space $\mathbb{R}^d_U$ *generated by U is the set of all $x \in \mathbb{R}^d$ which can be generated by finite linear combinations of elements of U with integer coefficients:*

$$\mathbb{R}^d_U = \left\{ x = \sum_{j=1}^{n} \lambda_j u_j \;\middle|\; u_j \in U \text{ and } \lambda_j \in \mathbb{Z} \text{ for all } j = 1, 2, \cdots, n \right\}.$$

U is referred to as zero neighborhood *of $\mathbb{R}^d_U$. The neighborhood of a point $x \in \mathbb{R}^d_U$ is then defined by translation $U(x) = x + U$.*

Remark In Example 1.1 is $\mathbb{R}^2_U \neq \mathbb{Z}^2$, whereas in Example 1.2 we have $\mathbb{R}^2_U = \mathbb{Z}^2$.

Definition 2.2 *A set $S \subseteq \mathbb{R}^d_U$ is termed* digitally convex *(or* convex *for short), whenever*

$$S = \mathbb{R}^d_U \cap \text{conv } S.$$

Remark The root image in Example 1.1 is convex in $\mathbb{R}^d_U$, however, it is not convex in $\mathbb{Z}^2$.

3 Median Filtering

Generally, we now assume that U is centrally symmetric. That means, $u \in U$ implies $-u \in U$. Since a centrally symmetric finite set has an odd number of elements, we can define:

Definition 3.1 *Given a set $S \subseteq \mathbb{R}^d_U$, U centrally symmetric. Then the set obtained by* median filtering *the set S is*

$$\mathrm{Med}_U S = \left\{ x \in \mathbb{R}^d_U \mid \#(S \cap U(x)) \geq \frac{\#U + 1}{2} \right\}.$$

Here $\#S$ denotes the number of elements of the (finite) set S.

Lemma 3.1 *Given a centrally symmetric zero neighborhood and a set $S \subseteq \mathbb{R}^d_U$.*

If $x_0 \notin \mathrm{conv}\ S$ (and $x_0 \in \mathbb{R}^d_U$), then $x_0 \notin \mathrm{Med}_U S$.

Proof Assume $x_0 \notin \mathrm{conv}\ S$. By the Separation Theorem 2.1 there is an $x^* \in \mathbb{R}^d$ and an $\varepsilon > 0$, such that $\langle x^*, x \rangle \geq \langle x^*, x_0 \rangle + \varepsilon$ for all $x \in \mathrm{conv}\ S$, hence also for all $x \in S$. This means, $S \subseteq [x^* : \langle x^*, x_0 \rangle + \varepsilon]^+$.

For $x_0 + x \in S$ is $\langle x^*, x_0 + x \rangle \geq \langle x^*, x_0 \rangle + \varepsilon$, consequently $\langle x^*, x \rangle \geq \varepsilon$, hence

$$\langle x^*, x_0 - x \rangle = \langle x^*, x_0 \rangle - \langle x^*, x \rangle \leq \langle x^*, x_0 \rangle - \varepsilon < \langle x^*, x_0 \rangle + \varepsilon,$$

consequently $x_0 - x \notin [x^* : \varepsilon]^+$ and $x_0 - x \notin S$.

For $x_0 + x \in S$, $x \in U \setminus \{\Theta_d\}$, is $x_0 - x \notin S$. Consequently, from both points $x_0 + x$ and $x_0 - x$ in U at least one is not in S. If the center point x_0 is also not contained in S, at least $(\#U + 1)/2$ points in U do not belong to S, hence $x_0 \notin \mathrm{Med}_U S$. □

Lemma 3.2 *Let H^+ be a half–space in $\mathbb{R}^d$. The set $S = H^+ \cap \mathbb{R}^d_U$ is termed a* digital half–space.

If S is a digital half–space, then $\mathrm{Med}_U S = S$.

Proof Since $S = H^+ \cap \mathbb{R}^d_U$, we have $\mathrm{conv}\ S \subseteq H^+$ and

$$S = \mathbb{R}^d_U \cap S \subseteq \mathbb{R}^d_U \cap \mathrm{conv}\ S \subseteq \mathbb{R}^d_U \cap H^+ = S,$$

hence S is digitally convex.

Let $x_0 \notin S$, but $x_0 \in \mathbb{R}^d_U$. Then $x_0 \notin H^+$, and by Lemma 3.1 is $x_0 \notin \mathrm{Med}_U S$.

If $x_0 \in S$ then $x_0 \in H^+$. Assume that $x_0 + u \notin S$ for some $u \in U$. Then $x_0 + u \notin H^+$. If $H^+ = [x^* : \alpha]^+$, then $\langle x^*, u \rangle < 0$. This implies $\langle x^*, x_0 - u \rangle > \alpha$, hence $x_0 - u \in H^+$ and thus $x_0 - u \in S$. By $x_0 \in S$ we finally get $x_0 \in \mathrm{Med}_U S$. □

Remark The assertion of Lemma 3.2 generalizes Döhler's "Proposition 1". We do not need here the concept of a digital line (or digital hyperplane in $\mathbb{R}^d_U$) which was used by Döhler.

In order to generalize Döhler's "Proposition 3", we need a condition on the vertices of the convex hull $S^c = \mathrm{conv}\ S \subseteq \mathbb{R}^d$.

A subset $F \subseteq S^c$ (in $\mathbb{R}^d$) is termed a *facet* of S^c, if there exists a hyperplane $H = [x^* : \alpha]$ such that $S^c \subseteq H^+ = \{x \in \mathbb{R}^d \mid \langle x, x^* \rangle \geq \alpha\}$ and $H \cap S^c = F$.

A facet F of S^c is termed a *k–facet*, whenever the dimension of F is k. A 1–facet is termed a *vertex* of S^c.

It is a well–known fact from convexity theory (see [12]) that each bounded closed convex set can be represented as the convex hull of all its vertices.

Claim All vertices of S^c are in S.

Proof Let x_0 be a vertex of S^c with defining hyperplane $H = [x^* : \alpha]$. Since $x_0 \in S^c$, there exist $s_1, s_2, \cdots, s_n \in S$ and $\lambda_1, \lambda_2, \cdots, \lambda_n$ with $\lambda_j \geq 0$ for all j and $\sum_{j=1}^n \lambda_j = 1$, such that $x_0 = \sum_{j=1}^n \lambda_j s_j$. Then

$$\langle x^*, x_0 \rangle = \sum_{j=1}^{n} \lambda_j \langle x^*, s_j \rangle \geq \sum_{j=1}^{n} \lambda_j \alpha = \alpha.$$

Since $\langle x^*, x_0 \rangle = \alpha$, equality holds, hence for all j (with $\lambda_j > 0$) is $\langle x^*, s_j \rangle = \alpha$. Since x_0 is a vertex, there is only one element in S^c hving this property, namely x_0. Thus, $x_0 = s_j$ for all j with $\lambda_j > 0$, hence $x_0 \in S$. $\square$

This assertion justifies to refer to "vertices of S" instead of "vertices of S^c".

Definition 3.2 *Let $S \subseteq \mathbb{R}^d_U$ and x_0 a vertex of S.*

x_0 fulfills the vertex condition *(with respect to U), if for each $u \in U$ the following implication holds*

$$x_0 + u \notin S \qquad \Longrightarrow \qquad x_0 - u \in S.$$

The set S fulfills the vertex condition if each vertex of S fulfills the vertex condition.

We learned from Example 1.2 that not each centrally symmetric zero neighborhood yields reasonable results. Hence, we define

Definition 3.3 *Let U be a centrally symmetric zero neighborhood.*

U is normal*, if for any pair $u_1, u_2 \in U$ with $u_1, u_2 \neq \Theta_d$ and $u_1 \neq u_2$ the following condition holds: If $x = \lambda_1 u_1 + \lambda_2 u_2$ with $0 < \lambda_1, \lambda_2 < 1$ and $x \in \mathbb{R}^d_U$, then $x \in U$.*

Obviously, the knight's neighborhood is not normal. In contrast, the neighborhood from Example 1.1 is normal (however, with respect to $\mathbb{R}^2_U$).

The following Lemma is needed later:

Lemma 3.3 *Let U be a centrally symmetric normal neighborhood and let S fulfill the vertex condition.*

Assume that V is a linear subspace of $\mathbb{R}^d$. Let $U' = U \cap V$ and $S' = S \cap (V + x_0)$ for an $x_0 \in S$. Then U' is normal and S' fulfills the vertex condition with respect to U'.

Proof

1. Clearly, $\Theta_d \in U'$. Furthermore, from $u \in U \cap V$, we get $-u \in U \cap V$. Hence, U' is a centrally symmetric neighborhood.

Given now $u_1, u_2 \in U'$ and $x = \lambda_1 u_1 + \lambda_2 u_2$, $0 < \lambda_1, \lambda_2 < 1$. Then $x \in V$. If in addition, $x \in \mathbb{R}^d_U$ $(\cap V)$, then $x \in U$ by the normality condition on U. Als a consequence, also $x \in U'$. This means, U' is normal.

2. We show that S' meets the vertex condition with respect to U'. Assume that s'_0 is a vertex of S' and $u \in U'$, such that $s'_0 + u \notin S'$. Then there is a hyperplane $H = [x^* : \alpha]$ separating conv $(s'_0, s'_0 + u)$ from S (but not containing $s'_0 + u$). (Note that the existence of H is not an immediate consequence of the Separation Theorem 2.1, an additional limit argument is needed, see e.g. [12]). This hyperplane is a boundary hyperplane for S. Specifically, $S \subseteq H^+$ and $s'_0 + u \notin H^+$. There exists a representation for s'_0 as a convex combination of vertices of S and all vertices appearing in this convex combination must be in H. Let s be such a vertex. Then $s + u \notin H^+$, hence also $s + u \notin S$. By the vertex condition for S we get $s - u \in S$, consequently $s'_0 - u$, being a convex combination of all such $s - u$, is in S. □

Before treating the general case, we investigate the two–dimensional space which is also the subject of Döhler's paper.

4 The Two–Dimensional Case

Let $U = \{u_1, u_2, \cdots, u_n\} \subseteq \mathbb{R}^2$ be a finite centrally symmetric zero neighborhood of the Euclidean plane. Let $V = \{v_1, v_2, \cdots, v_k\}$ be a minimal subset of U such that for each $u_i \in U$, $u_i \neq \Theta_2$, there exists a $v_j \in V$ such that $u_i = \lambda v_j$ with $0 < \lambda \leq 1$.

In $\mathbb{R}^2$ it is possible to assume that the directions v_j are sorted by their angles with respect to a fixed direction.

Consider a set $S \subseteq \mathbb{R}^2_U$ and assume that S has vertices s_j. In $\mathbb{R}^2$ the boundary of S^c can be oriented and we assume that the vertices and edges of S^c are numbered in such a way that s_j is joined to s_{j+1} by an edge k_j. For convenience, we assume that S is a finite subset of $\mathbb{R}^2_U$. so that we have only to deal with a finite number of vertices and edges.

Döhler formulates a condition for the vertices of a set which will be stated here in a slightly more general version:

Definition 4.1 *Let s_j be a vertex of $S^c \subseteq \mathbb{R}^d$ with ingoing edge k_{j-1} $(= s_j - s_{j-1})$ and outgoing edge k_j $(= s_{j+1} - s_j)$. s_j is said to fulfill the* Döhler condition, *if there is no $v_\nu \in V$ such that*

$$v_\nu = \lambda k_{j-1} + \mu k_j \qquad mit\ \lambda, \mu > 0$$

and if in case $k_j = \lambda \cdot v_\nu$ for any $v_\nu \in V$ and $\lambda > 0$ one has $\|k_j\| \geq \|v_\nu\|$ (analogously for k_{j-1}).

The set S fulfills the Döhler condition if each vertex of S fulfills the Döhler condition.

Lemma 4.1 *A set $S \subseteq \mathbb{R}^2_U$ fulfills the Döhler condition if and only if it fulfills the vertex condition.*

Proof

1. Assume that the vertex condition holds for vertex s_j with incident edges k_{j-1} and k_j (see Figure 1). S is contained in the half–plane which is bounded by the line containing s_j and having

direction k_j. This half–plane also contains all points $s_j - \kappa_{j-1} k_{j-1}$ with $\kappa_{j-1} \geq 0$. An analogous argument holds for the half–plane bounded by a line containing s_j and having direction k_j. This means that S is contained in the convex cone with vertex s_j which is spanned by the directions $-k_{j-1}$ and k_j. This cone is given by

$$s_j + \kappa_j k_j - \kappa_{j-1} k_{j-1}, \qquad \kappa_j \geq 0,\ \kappa_{j-1} \geq 0.$$

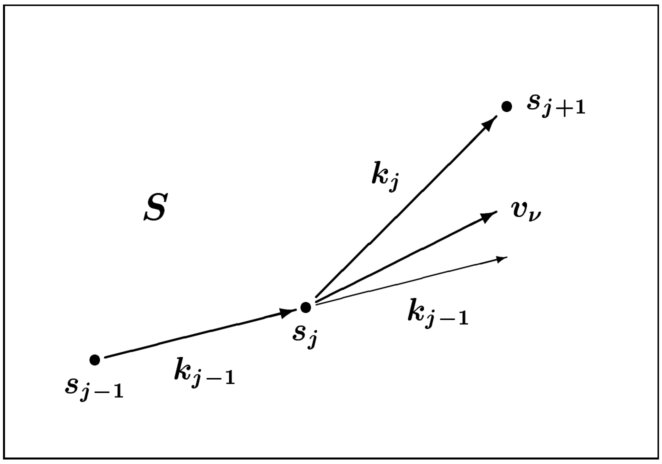


Figure 1: Döhler condition and vertex condition in the two–dimensional case.

Now assume that the Döhler condition is not fulfilled.

First, let $v_\nu = \lambda k_{j-1} + \mu k_j$ for a $v_\nu \in V$ with $\lambda, \mu > 0$. Then $s_j + v_\nu = s_j + \lambda k_{j-1} + \mu k_j$ is not in S. An analogous argument holds for $s_j - v_\nu$, thus the vertex condition is not fulfilled.

If $k_j = \lambda v_\nu$ for $v_\nu \in V$ and $\lambda > 0$, but $\lambda > 1$, then $s_j + v_\nu \notin S$ and $s_j - v_\nu \notin S$, hence also in this case the vertex condition is not fulfilled.

2. Assume the Döhler condition holds for vertex s_j, but $s_j \pm v_\nu \notin S$ for any $v_\nu \in V$. Then the line containing s_j with direction v_ν separates s_j and S. Both directions k_j and $-k_{j-1}$ (spanned from s_j) are on one side with respect to this line (see Figure 1). v_ν is not in the cone with vertex Θ_d generated by k_j and $-k_{j-1}$. Hence, either $v_\nu = \lambda k_{j-1} + \mu k_j$ with $\lambda, \mu \geq 0$ or else there is such a representation for $-v_\nu$. This contadicts the Döhler condition (The cases $k_j = \lambda v_\nu$ and $k_{j-1} = \lambda v_\nu$, $\lambda > 0$ are trivial). $\square$

The following simple Lemma will be of use:

Lemma 4.2 *Let $U \subseteq \mathbb{R}^1$ be a centrally symmetric zero neighborhood. Assume that $S \subseteq \mathbb{R}^1$ is digitally convex and fulfills the vertex condition.*

Then $\mathrm{Med}_U S = S$, *i.e. S is a root image of* Med_U.

Proof $S^c = \mathrm{conv}\ S$ is an interval with end points s_1 and s_2 in $\mathbb{R}^1_U$. Assume that $s_1 < s_2$ and $\mathrm{conv}\ U = [-h, h]$. Then the vertex condition means $s_1 + h \in \mathrm{conv}\ S$ and $s_2 - h \in \mathrm{conv}\ S$.

In $[s_1, s_1 + h]$ are $\frac{\#U+1}{2}$ points from $s_1 + U$ (vertex condition). This means $s_1 \in \mathrm{Med}_U S$.

Assume $x \in [s_1, s_1+h] \cap \mathbb{R}^1_U$. Then $[s_1, s_1+h] \subseteq x+U$, hence at least $\frac{\#U+1}{2}$ points from $x+U$ are in S, consequently $x \in \mathrm{Med}_U S$. An analogous argument holds for $x \in [s_2-h, s_2] \cap \mathbb{R}^1_U$.

Assume finally that $x \in (s_1+h, s_2-h) \cap \mathbb{R}^1_U$. Then $x+U \subseteq [s_1, s_2]$, hence $x \in \mathrm{Med}_U S$. □

Now we are ready to formulate an analogon of Döhler's "Proposition 3":

Lemma 4.3 *Let $U \subseteq \mathbb{R}^2$ be a centrally symmetric normal zero neighborhood. Let furthermore $S \subseteq \mathbb{R}^2_U$ be a bounded digitally convex set fulfilling the vertex condition.*

Then $\mathrm{Med}_U S = S$.

Proof We choose $x_0 \in S$ and show $x_0 \in \mathrm{Med}_U(S)$.

1. If x_0 is a vertex, then the assertion of the Lemma is true by the vertex condition.

2. Assume that x_0 is not a vertex but is on an edge of S^c. Let the two vertices adjacent to the edge be s_1 and s_2. Let $H = [x^* : \alpha]$ be the hyperplane (line) in $\mathbb{R}^2$ which contains the edge. Then $S \subseteq H^+$. If $x_0 + u \in H^+ \setminus H$ for a $u \in U$, then $x_0 - u \notin H^+$ and $s_i - u \notin H^+$, hence $s_i - u \notin S$ for $i = 1, 2$. By the vertex condition is $s_i + u \in S$ for $i = 1, 2$, and consequently $x_0 + u \in S$.

If $x_0 + u \notin H^+$ for a $u \in U$ then $s_i + u \notin H^+$, hence $s_i + u \notin S$, consequently $s_i - u \in S$ for $i = 1, 2$, and thus $x_0 - u \in S$.

We state as a result that from all $u \in U$ with $x_0 + u \in H^+ \setminus H$ exactly one half is contained in S.

There remains to investigate those $u \in U$ for which $x_0 + u \in H$. Recall that H is a line in $\mathbb{R}^2$. The set $U_H = U \cap \{x \mid \langle x^*, x\rangle = 0\}$ is a centrally symmetric zero neighborhood in $\mathbb{R}^1$, hence we can apply Lemma 4.2. From this Lemma we get that the vertex $[s_1, s_2]$ is a root image of the one–dimensional median filter Med_{U_H}, and this implies that more than one half of all the points in the neighborhood $x_0 + U_H$ are in S.

3. Assume finally that $x_0 \in S$ is an interior point of S^c and $x_0 + u \notin S$ for any $u \in U$. We are going to show that $x_0 - u \in S$.

There exists a boundary hyperplane (= boundary line) H of S, which separates $x_0 + u$ from S (see Figure 2). This boundary hyperplane contains at least one vertex s_1 of S, and there exists a $u_H \in U$, $u_H \neq \Theta_2$, such that $s_1 + u_H \in H$. By the vertex condition we may assume that $s_1 + u_H \in S$ (otherwise choose $-u_H$). There exists an integer i such that the intersection point $x = H \cap \mathrm{conv}\,(x_0, x_0 + u_H)$ is between $x_1 := s_1 + iu_H$ and $x_2 := s_1 + (i+1)u_H$. Note that $s_1 + iu_H \in S$. If $s_1 + (i+1)u_H \notin S$, then between x and $s_1 + (i+1)u_H \notin S$ there exists another point $s_2 \in S$, namely the second vertex of S in H. We then put $x_2 = s_2$ and $u_H = s_2 - x_1$.

a. If $x = x_1$ or $x = x_2$, then $x \in \mathbb{R}^2_U$, hence $x \in S$. In this case is $x_0 - x_1 =: u_1 \in U$.

b. By the vertex condition, $s_1 - u$ and $s_2 - u$ are in S, hence $x_1 - u \in S$. On the other hand, by construction is $x_1 + u_H \in S$. Therefore, $x_0 - x_1 = \lambda_1 u_H - \lambda_2 u$. Since U is normal, $x_0 - x_1 = u_1$ is in U.

In either case we have $x_0 = x_1 + u_1$ with $u_1 \in U$. For both end vertices s_i of $S^c \cap H$ is $s_i - u \in S$ and $s_i + u_1 \in S$. The cone with vertex s_2, which is generated by S, contains both directions $-u$ and u_1. By the Döhler condition at most one of these two directions can be an

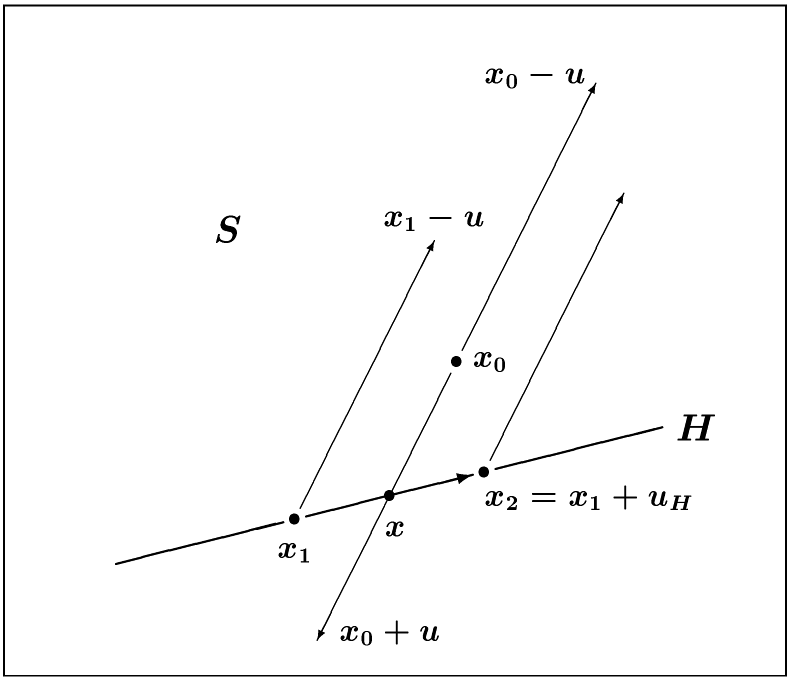


Figure 2: The two–dimensional case, x_0 interior point of S^c.

outgoing edge in s_2. From the vertex following s_2, the other direction points into the interior of S^c. This means, that also $s_2 - u + u_1$ is contained in S. An analogous argument holds for s_1.

We conclude that also $x_1 - u + u_1 = x_0 - u$ is in S. □

5 The General Case

Now it is not difficult to generalize Lemma 4.3 to $\mathbb{R}^d$.

Theorem 5.1 *Let the centrally symmetric zero neighborhood U be normal. Assume further that $S \subseteq \mathbb{R}^d_U$ is a bounded digitally convex set fulfilling the vertex condition.*

Then $\mathrm{Med}_U S = S$.

Proof We choose $x_0 \in S$ and show that $x_0 \in \mathrm{Med}_U(S)$.

1. If x_0 is a vertex then the assertion of the Theorem is true by the vertex condition.

2. Let now x_0 be a boundary point of S^c, but not a vertex. Let H be a boundary hyperplane containing x_0 and assume $S \subseteq H^+$. If $x_0 + u \notin H^+$ for any $u \in U$, then $x_0 - u \in H^+$. As in the two dimensional case, we can represent x_0 as a convex combination of vertices in H, and by the vertex condition it is seen that $x_0 - u \in S$.

There remains the case $x_0 + u \in H$. Again, Lemma 3.3 is used. Let U_H be the set of all directions in U which are parallel to H. Then by the Lemma, U_H is a normal neighborhood. $S \cap H$ is a set which fulfills the vertex condition with respect to U_H.

Now we use an inductive argument. Assume that the assertion of the Lemma is true for dimensions $< d$. Then it is true for $S \cap H$, hence $\mathrm{Med}_{U_H}(S \cap H) = S \cap H$, consequently at least $(\#U_H + 1)/2$ points in $x_0 + U_H$ are in $S \cap H$. This implies that $x_0 + U$ contains at least $(\#U + 1)/2$ points from S, hence $x_0 \in \mathrm{Med}_U S$.

3. The proof is not yet complete, since we here also have to investigate the case $x_0 \in \mathrm{int\ conv}\ S$. Assume again $x_0 + u \notin S$ for a $u \in U$. Let V be a two dimensional linear manifold which contains u. By Lemma 3.3 is $U \cap V$ normal and $S \cap (x_0 + V)$ fulfills the vertex condition. Hence we have here again the same conditions as in Figure 2, and therefore we conclude in the same way as in the two dimensional case that $x_0 - u \in S$. $\square$

6 Conclusions

Theorem 5.1 gives a sufficient condition for a digitally convex set in d–dimensional space to be a root image of a median filter (with normal neighborhood). Whenever a set is digitally convex (and the neighborhood is normal) then the vertex condition is also a necessary condition for S being a root image. Hence, Theorem 5.1 provides a complete characterization of the bounded convex root images of a median filter. It is not hard to extend the Theorem to general convex root images.

It is easily seen that there are nonconvex root images for median filters. For example, any two convex root images which are sufficiently far apart give an example for a nonconvex root image. More serious examples are provided by the construction of Astola, Heinonen and Neuvo [1]. It is an open question whether these nonconvex root images can be characterized in a similar way.

Döhler's construction process for generating minimal surviving objects can easily be modified to yield nonconvex root images

Example 6.1 *It is not true that for a normal neighborhood all root images are convex. In his paper, Döhler used a neighborhood for illustrating his construction process. By an obvious modification of this construction we can easily find a nonconvex root image for Döhler's neighborhood (see Figure 3).*

References

[1] J. Astola, P. Heinonen and Y. Neuvo, "On root structures of median and median–type filters," IEEE Trans. Acoust. Speech and Signal Process., Vol. ASSP–355, pp. 199–201, 1987.

[2] H.–U. Döhler, "Generation of root signals of two dimensional median filters," Signal Processing, Vol. 18, pp. 269–276, 1989.

[3] J. P. Fitch, E. J. Coyle and N. C. Gallagher, Jr., "Median filtering by threshold decomposition," IEEE Trans. Acoust. Speech and Signal Process., Vol. ASSP–32, 1183–1188, 1984.

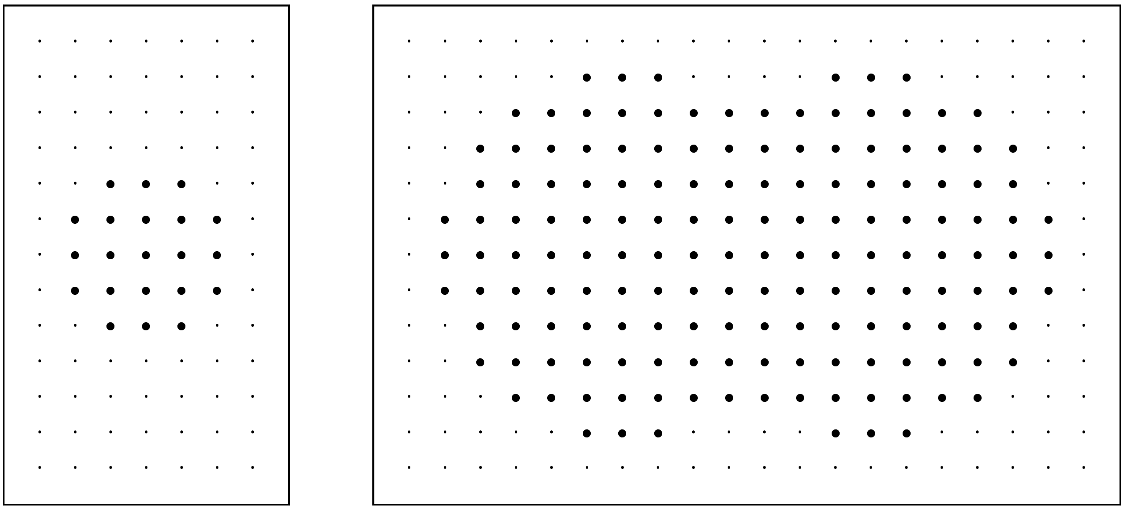


Figure 3: Nonconvex root image.

[4] N. C. Gallagher, Jr. and G. L. Wise, “A theoretical analysis of the properties of median filters,” IEEE Trans. Acoust. Speech and Signal Process., Vol. ASSP–29, 1136–1141, 1981.

[5] Z.–J. Gan and M. Mao, “Two convergence theorems on deterministic properties of median filters,” IEEE Trans. Signal Process., Vol. 39, 1689–1691, 1991.

[6] E. Goles and J. Olivos, “Comportement periodique des fonctions à seuil binaires et applications,” Discr. Appl. Math., Vol. 3, 93–105, 1981.

[7] V. Kim and L. Yaroslavskii, “Rank algorithms for picture processing,” Computer Vision, Graphics, and Image Processing, Vol. 35, 234–258, 1986.

[8] W. Krabs, *Mathematical Foundations of Signal Theory. (Sigma Series in Applied Mathematics, Volume 6)*, Heldermann: Berlin, 1995.

[9] L. J. Latecki, *Discrete Representation of Spatial Objects in Computer Vision*, *(Computational Imaging and Vision)*, Kluwer: Dordrecht / Boston / London, 1998.

[10] C. Ronse, “Lattice–theoretical fixpont theorems in morphological image filtering,” J. Math. Imaging Vis., Vol. 4, 19–41, 1994.

[11] S. G. Tyan, “Median filtering: Deterministic properties,” in T. S. Huang, ed., *Two–Dimensional Digital Signal Processing II. Transforms and Median Filters. (Topics in Applied Physics, Vol. 43)*, Springer, Berlin, pp. 197–217,

[12] F. A. Valentine: *Convex Sets*, McGraw–Hill: New York, 1964.

[13] B. Wilburn, “Theory of ranked–order filterd with applications to feature extraction and interpretive transforms,” in P. W. Hawkes, ed., *Advances in Imaging and Electron Physics, Volume 112*, pages 233–332, Academic Press: San Diego, 2000, pp. 233–332,