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Digital Lines and Digital Convexity

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Abstract. Euclidean geometry on a computer is concerned with the translation of geometric concepts into a discrete world in order to cope with the requirements of representation of abstract geometry on a computer. The basic constructs of digital geometry are digital lines, digital line segments and digitally convex sets. The aim of this paper is to review some approaches for such digital objects. It is shown that digital objects share much of the properties of their continuous counterparts. Finally, it is demonstrated by means of a theorem due to Tietze (1929) that there are fundamental differences between continuous and discrete concepts.

Keywords: digital geometry, digital lines, digital convexity, Tietzes's theorem

1 Introduction

The concept of convexity plays an important role in mathematics and also in applications. Specifically in visual perception convex sets are of importance since almost all visible objects are either convex or else composed of a finite number of convex sets (the "convex ring", see e.g. [19, 27]). Therefore a considerable part of books on digital geometry is devoted to convexity (see e.g. [2, Chapitre 5] or [31, Chapter 4.3]). In shape recognition two dimensional sets can be decomposed into perceptually meaningful parts by considering convex and concave parts of their boundaries [16]. There exist numerous generalizations of convexity [18, Chapter 6.2]. Ronse gave a bibliography on convexity which covers the years 1961 to 1988 [24]. A very detailed history of the topic is given by Hübler [10]. For a quite recent account of the subject including also detailed historical informations the reader is referred to Klette's survey article [12].

Convexity is closely related to the geometry of lines. Convex sets are defined by means of line segments and on the other hand line segments are convex hulls of two-point sets. In an abstract setting a hyperplane is a nonempty convex set whose complement is also nonempty and convex [29, Part II].

The aim of this article is to give a review on digital lines and digital convexity.

Most of the proofs are omitted in this paper. Only in cases where the proof is not easily available or if it involves some special constructions, a sketch of it is indicated.

2 The Digital Space

Given a linear space X over the real numbers $\mathbb{R}$, for example $\mathbb{R}^d$, the ordinary d-dimensional real vector space. In X we consider a certain subset X_Δ, the 'digital space'. In the standard example $X = \mathbb{R}^d$, we can take $X_\Delta = \mathbb{Z}^d$ which is the set of all vectors whose components have integer values. Although it is also possible to treat a good deal of the theory in irregular digital sets, we concentrate here exclusively on the case $(\mathbb{R}^d, \mathbb{Z}^d)$.

The topic of *digital geometry* is to translate continuous concepts, i.e. concepts in X into the digital world in X_Δ. There are basically two possibilities. The *pragmatic* way is to define a *discretization mapping* $X \mapsto X_\Delta$. A digital set $S_\Delta \subseteq X_\Delta$ is said to have a certain property if there is a continuous set $S \subseteq X$ having this property such that S_Δ is the image of S under the discretization mapping (for a related approach see [13]). Of course, one has to take care that such a definition is well-defined. One disadvantage of this approach is that it is not canonic, it depends on the discretization mapping used.

The other possible approach is the *axiomatic* way. Here, suitable characteristic properties are translated into the digital setting. The main advantage of this approach is that it is mathematically more attractive than the pragmatic way and that it allows to derive properties of the digital objects in a rigorous abstract way. In some fortunate cases both approaches lead to the same concepts.

3 Digital Lines

For sake of simplicity we now concentrate on planar sets, i. e. we consider $(\mathbb{R}^2, \mathbb{Z}^2)$. In digital (plane) geometry two metrics are popular. Let $x = (\xi_1, \xi_2)$ and $y = (\eta_1, \eta_2)$ two vectors in $\mathbb{R}^2$. Then the 4-*metric* is defined as

$$d_4(x, y) = |\xi_1 - \eta_1| + |\xi_2 - \eta_2|$$

and the 8-metric is

$$d_8(x, y) = \max\left(|\xi_1 - \eta_1|, |\xi_1 - \eta_1|\right).$$

These metrics are named according to the number of $\mathbb{Z}^2$-neighbors of distance 1 of a point in $\mathbb{Z}^2$.

A digital set $S_\Delta \subseteq \mathbb{Z}^2$ is termed σ-connected, $\sigma \in \{4, 8\}$, if there exist points $x_1, x_2, \cdots, x_n$ such that $x_i \in S_\Delta$ for all i and $d_\sigma(x_i, x_{i+1}) = 1$ for $i = 1, 2, \cdots, n-1$.

This concept of connectedness induces a 'topology' (more precisely: a graph structure, see e.g. [18] or [31] for details) on $\mathbb{Z}^2$. We define a *digital σ-curve* to be a σ-connected digital set S_Δ with the property that each point $x \in S_\Delta$ has exactly two σ-neighbors in S_Δ with the possible exception of at most two points, the so-called *end points* of the curve, having exactly one neighbor in S_Δ. A finite curve without end points is a *closed curve*.

For any set $S \subseteq \mathbb{R}^2$ and a point $x \in \mathbb{R}^2$ we define for $\sigma \in \{4,8\}$

$$d_\sigma(x,S) = \inf_{y \in S} d_\sigma(x,y).$$

Then two discretization mappings can be defined

$$\Delta_\sigma(S) = \left\{ x \in \mathbb{Z}^2 \mid d_\sigma(x,S) \leq \frac{1}{2} \right\}, \qquad \sigma \in \{4,8\}.$$

Δ_4 is sometimes referred to as *grid-intersection-discretization.*

Lemma 1. *If $S \subseteq \mathbb{R}^2$ is connnected then $\Delta_\sigma(S)$ is σ^*-connected, $\sigma \in \{4,8\}$,*

$$\sigma^* = \begin{cases} 4 & \textit{if} \;\; \sigma = 8, \\ 8 & \textit{if} \;\; \sigma = 4. \end{cases}$$

By means of the discretization mappings Δ_σ we could define a digital line as the image of a ontinuous line under one of these mappings. However, the image of a line under Δ_σ is not necessarily a digital σ-curve. If one wants to get digital lines which are also digital curves, the discretization mapping must be modified such that each point in $\mathbb{R}^2$ is mapped to exactly one point in $\mathbb{Z}^2$. If we define the *influence region* of a point $x = (i,j) \in \mathbb{Z}^2$ by

$$\pi_\sigma(x) = \left\{ y \in \mathbb{R}^2 \;\middle|\; d_\sigma(x,y) \leq \frac{1}{2} \right\},$$

then the union of all $\pi_8(x)$, $x \in \mathbb{Z}^2$ covers the plane $\mathbb{R}^2$ whereas for the union of all $\pi_4(x)$ this is not true. Even more important, the influence regions have boundary points in common. This accounts for the fact that there exist points in $\mathbb{R}^2$ which are mapped on more than one point in $\mathbb{Z}^2$ by means of the discretization mappings Δ_σ. In the next section we see how this latter situation can be remedied in the case $\sigma = 4$.

4 The Chord Property

The first systematic approach to define digital lines by means of a suitable discretization mapping dates back to 1974 when Rosenfeld published his paper on digital straight line segments [26]. For the proofs of the theorems of this section the reader is referred to Rosenfeld's original paper or the paper of Ronse [25].

For x and y in $\mathbb{R}^2$ the (continuous) line segment joining x and y is

$$[x,y] = \left\{ z \in \mathbb{R}^2 \mid z = \lambda x + (1-\lambda)y, \quad 0 \leq \lambda \leq 1 \right\}.$$

A digital set $S_\Delta \subseteq \mathbb{Z}^2$ is said to possess the *chord property* whenever for any two points x, y in S_Δ and for any $u \in [x,y]$ there exists a $z \in S_\Delta$ such that $d_8(z,u) \leq 1$. S_Δ has the *strict chord property* if the strict inequality sign holds.

Lemma 2. *A digital set S_Δ which has the strict chord property is 8-connected.*

In accordance with the pragmatic approach to digital geometry we define: a *digital (σ-) line segment* is a digital set which is a Δ_σ discretization of a line segment in $\mathbb{R}^2$.

Theorem 1. *Each 4-line segment has the chord property.*

Remark 1. It is generally not true that a 4-line segment has the strict chord property.

We define the *modified influence region* (see figure 1) of a point $x = (i, j) \in \mathbb{Z}^2$ by

$$\begin{aligned}\pi'_4(x) = &\left\{(\xi,\eta) \;\Big|\; |i-\xi| + |\eta - j| < \frac{1}{2}\right\} \cup \\ &\cup \left\{(\xi,\eta) \;\Big|\; \xi - i + \eta - j = \frac{1}{2} \quad \text{and} \quad i \le \xi \le i + \frac{1}{2}\right\} \cup \\ &\cup \left\{(\xi,\eta) \;\Big|\; \xi - i - \eta + j = \frac{1}{2} \quad \text{and} \quad i < \xi \le i + \frac{1}{2}\right\}.\end{aligned}$$

Note that each point in $\mathbb{R}^2$ belongs to at most one set $\pi'_4(x)$, $x \in \mathbb{Z}$, in contrast to the situation with influence regions $\pi_\sigma(x)$.

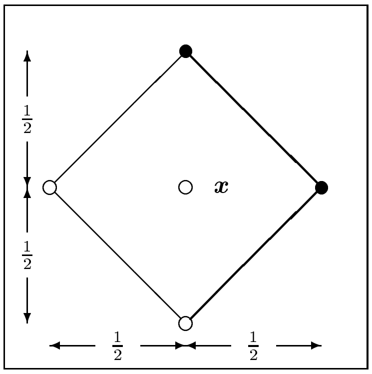


Fig. 1. The modified influence region $\pi'_4(x)$. Parts of the boundary indicated by thick lines belong to the influence region as well as the vertices marked •

The *modified grid-intersection-discretization* then is given by the mapping

$$\Delta'_4(S) = \{x \in \mathbb{Z}^2 \mid \pi'_4(x) \cap S \neq \emptyset\}.$$

Using this concept, we can formulate a stronger variant of theorem 1.

Theorem 2. *Any digital line segment generated by the modified 4-discretization mapping Δ'_4 is an 8-curve having the strict chord property.*

Remark 2. The assertion of theorem 2 is very appealing since it connects a certain discretization method with a definition for digital lines which always leads to lines having the strict chord property and being 8-curves. The price we have to pay for this greater symmetry is lack of invariance. The mapping Δ_4' is not invariant with respect to 90^o rotations of the plane.

In this form digital lines were introduced by Rosenfeld [26]. Usually, if we refer in this paper to digital lines, then a Δ_4'-line is meant.

Corollary 1. *Given a digital set S_Δ such that for any two points x and y in S_Δ there is a digital Δ_4- (Δ_4'-) line segment containing x and y which is contained in S_Δ.*

Then S_Δ has the (strict) chord property.

There is a converse of theorem 2:

Theorem 3. *Given a digital set $S_\Delta \subseteq \mathbb{Z}^2$ having the strict chord property. Let x and y be points in S_Δ.*

Then there exists a line segment $\gamma \in \mathbb{R}^2$ such that $\Delta_4'(\gamma) \subseteq S_\Delta$ and $x, y \in \Delta_4'(\gamma)$.

The proof of this theorem needs Helly's theorem [29, Part VI] from convexity theory [25, 26].

Corollary 2. *A digital 8-curve S_Δ having the strict chord property is a digital line segment (in the Δ_4' sense).*

The theory sketched here has some advantages. Specifically, it translates the common concept of a line or a line segment, respectively, in a quite straightforward way to the digital case via a quite natural discretization mapping. All constructs used here can be generalized to higher dimensions [22, 23]. However, it is not easily possible to verify the chord property for a given digital set. There exist different approaches for finding a characterization of digital lines and line segments which makes exclusively use of $\mathbb{Z}^d$-concepts. Moreover, these approaches lead to algorithms with linear time-complexity. The two main approaches in this direction are the syntactic characterization of Freeman [7], Rosenfeld [26] and Hübler, Klette and Voss [11] and the arithmetic characterization of Debled-Rennesson and Reveillès [4]. There slso exists a very interesting number-theoretic approach by Voss [30] and Bruckstein [1]. This latter approach, however, will not be treated here.

5 Syntactic Characterization of Digital Line Segments

The *chain code* for coding digital 8-curves was proposed by Freeman [6]. The 8-neighbors of a point $x = (i, j) \in \mathbb{Z}^2$ are numbered according to the following scheme.

	$i-1$	i	$i+1$
$j+1$	$N_3(x)$	$N_2(x)$	$N_1(x)$
j	$N_4(x)$	x	$N_0(x)$
$j-1$	$N_5(x)$	$N_6(x)$	$N_7(x)$

A digital 8-curve can be described by means of a simple compact linearly ordered data stucture containing the coordinates of one of its end points (or of a fixed point on it if a closed curve is treated) and a sequence of code numbers in $\{0,1,2,3,4,5,6,7\}$ indicating for each point on the curve which of its neighbors will be the next point on the curve.

The following theorem was proved by Rosenfeld [26].

Theorem 4. *Given a digital* 8*-curve consisting of more than three points which has the strict chord property. Let* $a_1, a_2, \cdots, a_n$ *be the set of chain code numbers of the curve,* $0 \leq a_i \leq 7$*. Then*

1. *There are at most two different code numbers among the* a_i*,*
2. *If there are two different code numbers among the* a_i*, say* α *and* β*, then these belong to adjacent directions, i.e.* $\alpha \equiv \beta \pm 1 \pmod 8$*,*
3. *If there are two different code numbers, one of them is singular, i.e. it never occurs twice in succession.*

A digital 8-curve having the properties of theorem 4 is not necessarily a digital line segment.

For digital lines (with rational slope) the following theorem holds:

Theorem 5. *Let* $\Gamma \subseteq \mathbb{Z}^2$ *be a digital line. Then the following three assertions are equivalent:*

1. *The chain code of* Γ *fulfills the conditions of theorem 4 and is periodic.*
2. *There exists a line with rational slope so that* Γ *is the* Δ'_4 *discretization of this line.*
3. *All lines generating* Γ *by discretization have the same (rational) slope.*

For digital lines which are characterized as in theorem 5 the axiom of parallelity is not necessarily true. There do exist pairs of parallel lines which are different but not disjoint. The intersection of such parallel lines is not necessarily a digital line. Moreover, the intersection of two lines can contain more than one point.

Freeman [7] proposed as a heuristic criterion that the singular direction should be distributed as regularly as possible in the chain code. Hübler, Klette and Voss [11] made this heuristic statement precise by giving a syntactic characterization of digital lines and digital line segments. Assume that we are given a sequence $F = \{a_j\}_{j\in\mathbb{Z}}$ of code numbers, $0 \leq a_j \leq 7$. An element a_k of this sequence is *singular* whenever $a_{k-1} \neq a_k$ and $a_k \neq a_{k+1}$, otherwise a_k is *regular*. A *regular piece* of F is a finite subsequence of maximal length of successive elements of F with mutually equal code numbers.

Let $\mathcal{F} = \{\{a_k\}_{k=-\infty}^{\infty}\}$ be the set of all sequences of integers ($a_k \in \mathbb{N}$ for all k). We introduce a *reduction operator* $R : \mathcal{F} \longrightarrow \mathcal{F}$ by means of the following procedure:

– Delete all singular elements in F which are adjacent to two regular pieces.
– Replace all regular pieces of F by their lengths.
– Leave all other elements of F unchanged.

The sequence F is said to possess the *Freeman property* if

$\mathbf{F}_1$ There exists a number $a \in \mathbb{N}$ such that $a_j \in \{a, a+1\}$ for all j.
$\mathbf{F}_2$ Whenever F contains both numbers a and $a+1$ then at least one of them occurs only singularly.

The sequence F is said to possess the *HKV (Hübler, Klette, Voss) property* if each reduced sequence $R^k(F)$, $k = 0, 1, 2 \cdots$ has the Freeman property.

We now introduce a convexity concept. A digital set $S_\Delta \subseteq \mathbb{Z}^2$ is convex in the sense of Minsky and Papert or *MP-convex* if for any two points x and y in S_Δ and $z \in [x, y] \cap \mathbb{Z}^2$ then $z \in S_\Delta$ [20].

Theorem 6. *Any digital* 8*-curve which is unbounded in both directions and MP-convex has the HKV-property.*

Proof. The proof of this theorem is rather long and technical but straightforward. We give only a sketch of it. First it is shown that the chain code of an MP-convex curve has the Freeman property. This can be done by straightforward enumerative discussion of all possible cases.

Then it is shown that each reduced code sequence has the Freeman property. In order to do this it is first shown that the Freeman code of an MP-convex curve consists of only two different regular pieces of size k and $k+1$. It is easily seen that one of these regular pieces occurs singularly.

The converse of theorem 6 is also true [10]:

Theorem 7. *Any digital* 8*-curve which is unbounded in both directions and has the HKV-property is MP-convex.*

Proof. One basic trick for proving this theorem is due to Hübler. Without loss of generality we can assume that the chain code F for the digital curve S_Δ has only code numbers 0 and 1. If the reduced sequence $R(F)$ has elements k and $k+1$ and if $k+1$ is singular then the linear mapping of the (real) plane with matrix

$$M = \begin{pmatrix} k+1 & 1 \\ 1 & 0 \end{pmatrix}$$

maps the vectors $(1, 0)$ and $(1, 1)$ on the elements $(k+1, 1)$ and $(k+2, 1)$, respectively. The mapping $S_\Delta \mapsto MS_\Delta$, if interpreted as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$, is invertible and $M^{-1}R(S_\Delta)$ can be interpreted as the chain code of a digital curve.

The remainder of the proof is rather straightforward.

For digital line segments the HKV-property must be modified in order to cope with boundary effects. For sake of completeness we indicate this modification. For details the reader is referred to the original paper of Hübler, Klette and Voss [11].

Denote by $\mathcal{F}_0$ the set of all finite sequences of integers. As above we define singular and regular elements in a sequence and also the Freeman property for sequences $F \in \mathcal{F}_0$.

The *modified reduction operator* $R' : \mathcal{F}_0 \longrightarrow \mathcal{F}_0$ is defined as follows: All maximal subsequences of successive regular elements of F which are between two singular elements are replaced by their lengths and all other elements are deleted. If F contains no singular elements, F is replaced by its length.

For a sequence $F \in \mathcal{F}_0$ with the Freeman property we denote by $\ell(F)$ the number of regular elements preceeding the first singular element in F and by $r(F)$ the number of regular elements following the last singular element in F.

The modified HKV-property then looks as follows:

HKV$_1$ All reduced sequences ${R'}^k(F)$, $k = 0, 1, 2, \cdots$ have the Freeman property (or are empty).

HKV$_2$ If ${R'}^k(F)$, $k = 1, 2, \cdots$, consists of identical elements a or of two different elements a and $a+1$ then $\ell({R'}^{k-1}(F)) \leq a+1$ and $r({R'}^{k-1}(F)) \leq a+1$.

HKV$_3$ If ${R'}^k(F)$, $k = 1, 2, \cdots$, contains two elements a and $a+1$ such that a $(\leq a+1)$ is nonsingular, then

${R'}^k(F)$ starts with a if $\ell({R'}^{k-1}(F)) = a+1$ and
${R'}^k(F)$ ends with a if $r({R'}^{k-1}(F)) = a+1$.

The following theorem characterizes digital lines by the HKV-property. The theorem was first stated by Hübler, Klette and Voss [11]. The first proof of the theorem was given by Wu [32] (see also [10]).

Theorem 8. *A (finite) 8-curve S_Δ is a digital line segment if and only if it has the modified HKV-property.*

The relevance of the theorems of this section lies in the fact that the modified HKV-property can be verified by a syntactic parsing of the chain code of a given digital curve. This process obviously can be performed in time proportional to the length of the curve.

6 Arithmetic Characterization of Digital Line Segments

An alternative definition of a digital line is due to J.-P. Reveilles [4]. Given numbers $a, b \in \mathbb{Z}$ with $b \neq 0$ such that the greatest common divisor of a and b is 1 and a number $\mu \in \mathbb{Z}$, then a digital line is the set

$$\mathcal{D}(a, b, \mu) = \left\{ (x, y) \in \mathbb{Z}^2 \mid \mu \leq ax - by < \mu + \max(|a|, |b|) \right\}. \qquad (1)$$

This definition is an immediate consequence of the modified grid-intersection-discretization of a real line. To a (nontrivial) digital line segment S_Δ one can associate two lines in $\mathbb{R}^2$, namely the line

$$\{(\xi,\eta) \in \mathbb{R}^2 \mid a\xi - b\eta = \mu\}$$

and

$$\{(\xi,\eta) \in \mathbb{R}^2 \mid a\xi - b\eta = \max_{(x,y)\in S_\Delta} ax - by\}.$$

The strip bounded by these two lines contains the digital line segment.

If an 8-curve is to be tested for linearity, a system of Diophantine inequalities has to be solved, since for each point (x, y) on the curve an inequality of the form (1) yields a condition for the integer variables μ, a, b. The solution of such a system is straigtforward and easy and can be done in time proportional to the length of the curve.

7 Digital Lines and Translations

Hübler [8–10] proposed a theory of digital lines which is based on translations. First one can state:

Observation *The chain code of a digital line is periodic if and only if there exists a nontrivial translation of $\mathbb{Z}^2$ which leaves the digital line fixed.*

This leads immediately to the following definition: Given a translation $T : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$. A digital line in the sense of Hübler is any set of the form $\{x_0 + T^n x \mid n \in \mathbb{Z}\}$ with fixed vectors $x_0, x \in \mathbb{Z}^2$. We assume that the translation is *irreducible* which means that there is no other translation $T' : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ such that $T = (T')^k$ with $k > 1$.

By means of this concept one gets digital lines which are not necessarily 8-curves but have attractive properties.

- For any two different points x and y in $\mathbb{Z}^2$ there is exactly one digital line containing these points.
- If two lines are parallel then they either coincide or else they are disjoint.
- Two lines are parallel if and only if there is a translation mapping one of them into the other.
- Two lines are parallel if and only if each translation leaving one of them fixed also leaves the other fixed.
- Two lines which are not parallel intersect in at most one point.

A *digital line segment* is a finite subset of a digital line containing consecutive points of the line.

The main advantage of Hübler's definition is its potential for generalizations to $\mathbb{Z}^d$.

Any digital Δ'_4-line as defined above always contains a line in the sense of Hübler if it is generated by a line having rational slope.

8 Digital Convexity

There exist different definitions of digital convexity in the literature. We list some of the most common definitions. Let S_Δ be a digital set in $\mathbb{Z}^2$ (or $\mathbb{Z}^d$, respectively).

The first definition of digital convexity was stated in 1969 by Minsky and Papert [20] (see above; MP-convexity). The main reason for introducing this concept was to give an example of a predicate which has parallel order 3 (i.e. it can be verified by looking at triples of points in a digital set) and which is not 'local'.

MP-convexity For x and y in S_Δ and $z \in [x,y] \cap \mathbb{Z}^2 \Longrightarrow z \in S_\Delta$.
H-convexity The convex hull of S_Δ is the set

$$\mathrm{conv} S_\Delta = \left\{ \sum_{j=1}^{n} \lambda_j x_j \mid \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0 \text{ and } x_j \in S_\Delta \right\}.$$

S_Δ is H-convex if $S_\Delta = \mathrm{conv} S_\Delta \cap \mathbb{Z}^d$.

D-convexity S_Δ is *digital convex* or D-convex if for $x, y \in S_\Delta$ the Δ_4'-line segment joining x and y belongs to S_Δ.

DH-convexity S_Δ is *digital convex in the sense of Hübler* or DH-convex if for $x, y \in S_\Delta$ the line segment (according to Hübler) joining x and y belongs to S_Δ.

Remark 3. If one considers irregular digital spaces $X_\Delta \subseteq \mathbb{R}^d$ then the definitions of MP- and H-convexity carry over without any difficulty. However, if no three different points of X_Δ are collinear, each set $S_\Delta \subseteq X_\Delta$ is MP-convex. On the other hand, D- and DH-convexity depend on a geometry of lines in X_Δ which is compatible with lines in $\mathbb{R}^d$.
In figure 2 examples are given for different convexity definitions.

All these definitions have in common that the different convex sets have some of the properties which are known from ordinary convex sets. For example, the intersection of two MP- (H-, DH-) convex sets is always an MP- (H-, DH-) convex set. However, as can be seen by the conterexample in figure 3, the intersection of two D-convex sets is not always a D-convex set.

Let S_Δ be a set which is convex according to one of the definitions above. An *interior point* of S_Δ is a point $x \in S_\Delta$ such that the direct neighbors $N_0(x)$, $N_2(x)$, $N_4(x)$ and $N_6(x)$ all belong to S_Δ. The *interior* of S_Δ is the set of all interior points. It can easily be shown that the interior of a convex digital set is also convex.

We state some assertions on the mutual relations of these convexity concepts.

Theorem 9. *A digital set is DH-convex if and only if it is MP-convex.*

The proof of this theorem is obvious.

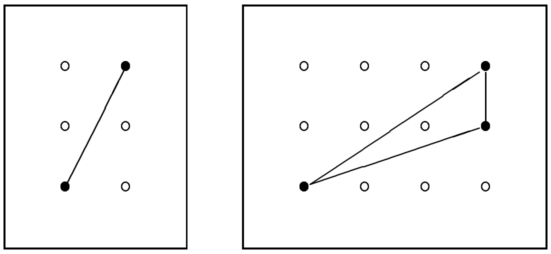


Fig. 2. Examples of convex sets. The digital set (•) in the left picture is MP-convex and H-convex but not 8-connected, hence not D-convex. The digital set in the right picture is MP-convex but not H-convex

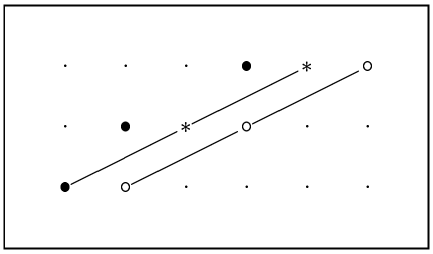


Fig. 3. Two D-convex sets whose intersection is not D-convex. The origin of the coordinate system is at the leftmost point •. The real line segment joining points $(0, 0)$ and $(4, 2)$ generates the digital Δ'_4-line segment consisting of all points • and $*$. Similarly, the line segment joining points $(1, 0)$ and $(5, 2)$ generates the digital line segment consisting of points $\circ$ and $*$. The intersection of both digital line segments (points $*$) is not 8–connected, hence not D-convex.

Theorem 10. *A digital set has the chord property if and only if it is* 8*-connected and H-convex.*

Theorem 11. *Any MP-convex and* 8*-connected digital set is D-convex.*

The proof of this theorem is very similar to the proof of theorem 6.

The relations of all these convexity definitions are illustrated in Figure 4. It is an interesting fact that under the assumption of 8-connectedness all these different concepts coincide. In the sequel we will always assume that the sets under consideration are 8-connected so that there is no need to distinguish different convexity concepts.

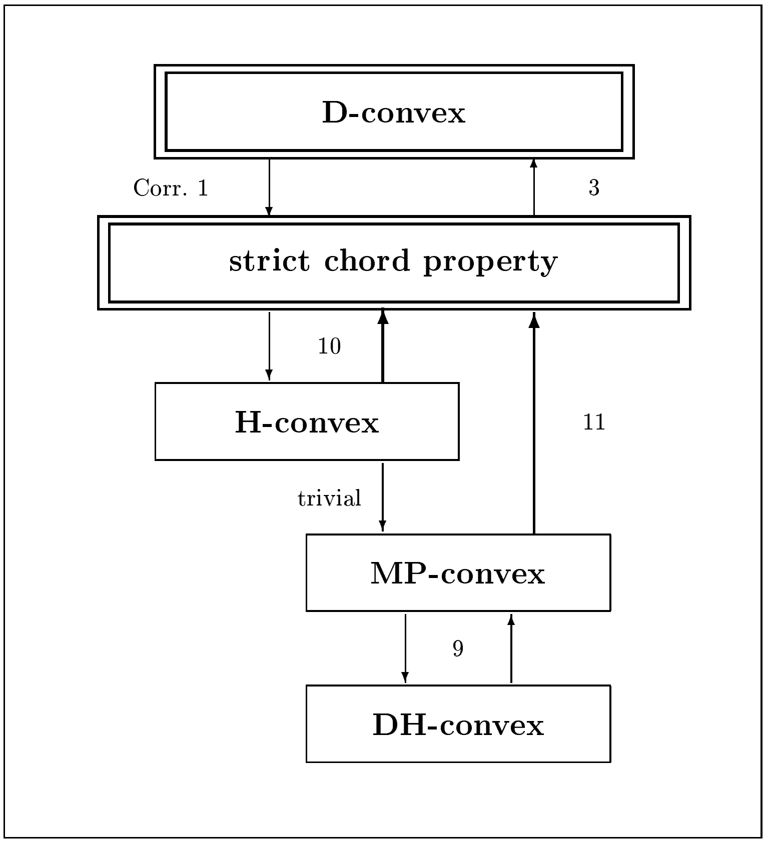


Fig. 4. Relations of different convexity definitions. Double frames indicate convexity definitions which imply 8-connectivity. Bold arrows mean that the corresponding relation holds under the assumption of 8-connectivity. Numbers at the arrows denote the theorems where the corresponding assertions are formulated

9 Convexity Verification

Given a finite set $S \subseteq \mathbb{R}^2$ with n elements, the convex hull of S can be determined in time $O(n \log n)$ [21].

The situation is more favourable if convexity of a polygonal set P in $\mathbb{R}^2$ is investigated. Such a set can be understood as an ordered set of boundary vertices such that each pair of two consecutive vertices is joined by a line segment. Convexity of a polygonal set in $\mathbb{R}^2$ can be verified in a time which is proportional to the number of vertices of the polygon. By moving along the oriented boundary of the polygonal set in such a way that the interior of the set is always on the left-hand side, one determines for each vertex the direction change of the adjacent line segments. The set S_Δ is convex if and only if at each vertex these line segments form a left turn.

Given a (bounded) digital set $S_\Delta \in \mathbb{Z}^2$, it is very easy to determine its boundary points (which are all points in S_Δ having at least one 4-neighbor not in S_Δ). In $\mathbb{Z}^2$ the set of boundary points of a simply connected digital set (i.e. a digital set which is 8-connected and its complement is 4-connected) is a closed 8-curve. If consecutive points on the digital boundary curve are joined by line segments, a closed polygonal curve is obtained. Thus, to each (simply connected) digital set in $S_\Delta \subseteq \mathbb{Z}^2$ a polygonal set $P(S_\Delta)$ can be associated in a canonical way such that $S_\Delta = \mathbb{Z}^2 \cap P(S_\Delta)$.

Unfortunately, if S_Δ is convex in $\mathbb{Z}^2$, $P(S_\Delta)$ is not necessarily convex in $\mathbb{R}^2$. It was noted by Hübler, Klette and Voss [11] that for a digital set in $\mathbb{Z}^2$ convexity can be verified by traversing the boundary along digital line segments and monitoring the direction changes at each intersection point of two consecutive digital lines. This yields a linear time algorithm for convexity verification in $\mathbb{Z}^2$ (see also [14]).

The approach of Debled-Rennesson and Reveillès [4] also leads to a linear time algorithm for detecting convexity.

The situation is more complex in higher dimensions. First of all, the complexity assertions for constructing convex hulls in the continuous space are no longer true, in dimensions ≥ 3 the picture is more complicated. Ronse [22, 23] was able to generalize the chord property to arbitrary dimensions. However, for algorithmic purposes the chord property is not very helpful. Hübler's approach is independent of the dimension [10], however, it is also not immediately suitable for convexity detection. Debled-Rennesson [3] was able to carry over the main ideas of the approach proposed by her and by Reveillés [4] to three dimensions.

10 Tietze's Theorem

In continuous convexity theory (see e.g. [29]) the boundary points of a convex set are classified by means of separation arguments. It is a remarkable fact that such arguments play virtually no role in digital convexity theory. Only in a more recent paper of Latecki and Rosenfeld digital supportedness was investigated [17].

Assume that $\mathbb{R}^d$ is equipped with Euclidean geometry, i. e. to each pair of vectors $x = (\xi_1, \xi_2, \cdots, \xi_d)$ and $y = (\eta_1, \eta_2, \cdots \eta_d)$ the *inner product*

$$\langle x, y \rangle = \sum_{j=1}^{d} \xi_j \eta_j$$

is assigned. Given a nonzero vector $x^* \in \mathbb{R}^d$ and a real number α, a *hyperplane* is a set

$$[x^* : \alpha] = \left\{ x \in \mathbb{R}^d \mid \langle x, x^* \rangle = \alpha \right\}.$$

The following Characterization Theorem holds

Theorem 12. *A digital set $S_\Delta \subseteq \mathbb{Z}^d$ is digital convex if and only if for any point $x \in \mathbb{Z}^d \setminus S_\Delta$ there exists a hyperplane $[x^* : \alpha]$ such that $\langle x, x^* \rangle = \alpha$ and $\langle y, x^* \rangle > \alpha$ for all $y \in S_\Delta$.*

This theorem is an obvious consequence of the Separation Theorem for Convex Sets [29]. A digital variant of the theorem may be found in [17]. A point of a set $S \subseteq \mathbb{R}^d$ is termed an *exposed point* of S if there exists a hyperplane $[x^* : \alpha]$ such that $\langle x, x^* \rangle = \alpha$ and $\langle y, x^* \rangle \geq \alpha$ for all $y \in S$. Using this concept, the assertion of the following theorem is also known from convexity theory [29]:

Theorem 13. *If each boundary point of a digital set is an exposed point then the set is (digital) convex.*

In 1929 H. Tietze [28] proved a remarkable theorem. A set $S \subseteq \mathbb{R}^d$ is termed *weakly supported locally* if for each boundary point x of S there exists an (open) neighborhood $U(x)$ and a hyperplane $[x^* : \alpha]$ such that $\langle x, x^* \rangle = \alpha$ and

$$\langle y, x^* \rangle < \alpha \text{ for } y \in U(x),\ y \neq x \quad \Longrightarrow \quad y \notin S.$$

Tietze's theorem now is [29, Theorem 4.10].

Theorem 14. *Let S be an open connected set in $\mathbb{R}^d$. If each point of the boundary of S is one at which S is weakly supported locally, then S is convex.*

The assertion of Tietze's theorem means that in $\mathbb{R}^d$ each set can be tested for convexity by looking only locally at boundary points. It is surprising that in Tietze's theorem one needs topological assumptions (S is required to be open). It would be very attractive to translate this theorem into the digital world since it would allow to test any digital set for convexity in a time proportional to the number of its boundary points. Unfortunately, however, Tietze's theorem does not hold in general in digital spaces.

For a point $x \in \mathbb{Z}^d$ denote by $\mathcal{N}_8(x)$ the set consisting of x together with its 8-neighbors. First we define: Given a set $S_\Delta \subseteq \mathbb{Z}^d$. A point $x \in S_\Delta$ is a *locally exposed point* of S_Δ if there exists a hyperplane $[x^* : \alpha]$ such that $\langle x, x^* \rangle = \alpha$ and $\langle y, x^* \rangle \geq \alpha$ for all $y \in (S_\Delta \cap \mathcal{N}_8(x))$.

A set $S_\Delta \subseteq \mathbb{Z}^d$ is said to meet the *interior point condition* at point $x \in S_\Delta$ if the digital set $(\mathcal{N}_8(x) \cap S) \setminus \{x\}$ is 4-connected and contains at least two points.

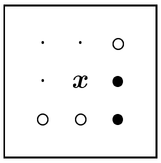
Now the following theorem is true:

Theorem 15. *Let $S_\Delta \subseteq \mathbb{Z}^d$ be a digital set and let x be a boundary point in S_Δ which is locally exposed. Assume further that the interior point condition does not hold at x.*

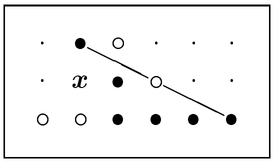
Then there exists a digital set S'_Δ with $\mathcal{N}_8(x) \cap S_\Delta = \mathcal{N}_8(x) \cap S'_\Delta$ such that each boundary point of S'_Δ is a locally exposed point and S'_Δ is not convex.

The meaning of this theorem is that without the interior point condition nothing can be said about convexity of a set only by looking at neighborhoods. The theorem can be proved by a rather lengthy discussion of all possible cases. We treat only one of these cases.

Since x was assumed to be a boundary point, one of the direct neighbors of x, say $N_6(x)$, must not belong to S_Δ. We consider the case that also one of the indirect neighbors, say $N_1(x)$, is not in S_Δ. Furthermore we assume that $N_0(x)$ and $N_7(x)$ are in S_Δ. By convexity of S_Δ is $N_5(x) \notin S_\Delta$. Therefore we have the following situation ($\bullet$ denotes a point in S_Δ, $\circ$ a point not in S_Δ and $\cdot$ denotes a point whose status with respect to S_Δ is left open):



If the interior point condition does not hold, one of the remaining points in $\mathcal{N}_8(x)$ belongs to S_Δ. If e.g. $N_2(x)$ were in S_Δ, then we choose S'_Δ as follows (all points which are not marked $\bullet$ are assumed not to belong to S'_Δ):



Obviously all points of S'_Δ are locally exposed points and S'_Δ is not convex.

The situation becomes less pessimistic if one considers digital sets fulfilling the interior point condition. Each of the eight neighbors of a point either belongs to S_Δ or not, hence there are $2^8 = 256$ different possible neighborhood configurations. These configurations can be obtained from 51 generating configurations by rotations and reflections. If one is interested in configurations fulfilling the interior point condition, there remain 8 generating configurations (these are exactly the "simple strict boundary point-configurations" of [5, p. 160]). 7 of these configurations are digitally convex and 6 of them are configurations of locally exposed points. These latter are given in Figure 5.

There remains one configuration which fulfills the interior point condition and is convex but not locally exposed:

$$\begin{array}{ccc} \circ & \circ & \bullet \\ \circ & \mathbf{3} & \bullet \\ \circ & \circ & \circ \end{array} \qquad \begin{array}{ccc} \circ & \bullet & \bullet \\ \circ & \mathbf{7} & \bullet \\ \circ & \circ & \circ \end{array} \qquad \begin{array}{ccc} \bullet & \bullet & \bullet \\ \circ & \mathbf{14} & \circ \\ \circ & \circ & \circ \end{array}$$

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \circ & \mathbf{15} & \bullet \\ \circ & \circ & \circ \end{array} \qquad \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \mathbf{31} & \bullet \\ \circ & \circ & \circ \end{array} \qquad \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \mathbf{62} & \circ \\ \bullet & \circ & \circ \end{array}$$

Fig. 5. Configurations of exposed points. $\bullet$ denotes points in S_Δ and $\circ$ denotes points not in S_Δ. The number in the center of each configuration is the configuration number $\#(\mathcal{N}_8) = \sum_{j=0}^{7} c_j \cdot 2^j$ where $c_j = 1$ for points in S_Δ and $c_j = 0$ for points not in S_Δ. It is assumed that the center point belongs to S_Δ

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \mathbf{63} & \bullet \\ \bullet & \circ & \circ \end{array}$$

We formulate an encouraging result sharpening the assertion of theorem 13:

Theorem 16. *Given a simply connected digital set S_Δ such that all boundary points of S_Δ fulfill the interior point condition and are locally exposed.*
Then S_Δ is convex.

Proof. Assume that S_Δ is simply connected and that all of its boundary points fulfill the interior point condition. Then the polygonal set $P(S_\Delta)$ in $\mathbb{R}^2$ associated canonically to S_Δ as above is "regular", i.e. the (topological) closure of its interior is the set itself (This can be seen easily by inspection of the neighborhood configurations in Figure 5). So, $P(S_\Delta)$ fulfills all conditions of Tietze's theorem, which completes the proof.

The relevance of the latter theorem is rather limited since digitally convex sets rarely have only locally exposed boundary points, which means that configuration **63** above (as well as its rotated or reflected versions) occurs quite often in realistic digital sets. If this happens, neighborhood information alone is not sufficient for testing convexity. However, it is possible to monitor the occurence of critical configurations and to remedy the situation. One possibility, of course, for treating this situation is to segment the boundary into digital line segments in the manner of Hübler, Klette and Voss [11] or of Debled-Rennesson and Reveillès [4]. These approaches, however, are not local in the sense of Tietze's theorem.

The problems encountered in connection with application of Tietze's principle are sketched here by means of a special application. For retrieving shapes from a pictoral database it is necessary to simplify the contours of the objects under consideration. This can be done by removing "irrelevant" points from the contour [16]. Such a procedure has the advantage that the original data structure (a linearly ordered list of points in $\mathbb{Z}^2$) is not changed. Moreover, this approach also has a favourable stability behaviour [15]. A first step in the method could be to replace the polygonal set $P(S_\Delta)$ obtained canonically from the chain code by a set $Q(S_\Delta)$ having the properties:

1. $Q(S_\Delta)$ is a polygonal set obtained from $P(S_\Delta)$ by deleting vertices,
2. $S_\Delta = Q(S_\Delta) \cap \mathbb{Z}^2$.
3. $Q(S_\Delta)$ has the same "convexity behaviour" of the boundary as S_Δ.

It can be shown that such a set $Q(S_\Delta)$ cannot be found in a canonic way from S_Δ. Of course, there is some freedom of choice in interpreting the third requirement which is not mathematically precise.

In Figure 6 a specific digital set is shown. The set was reduced by deleting points from the boundary as long as this is possible in a unique way under the first two conditions indicated above.

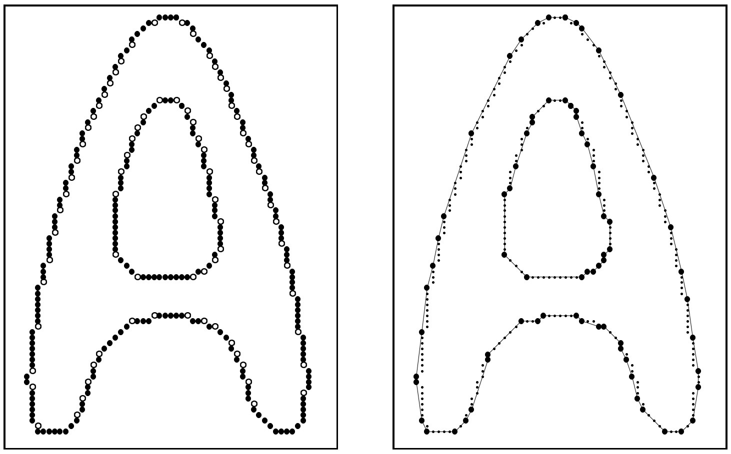
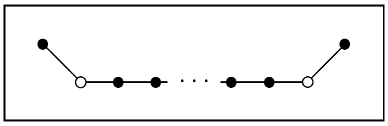


Fig. 6. Construction of an enclosing polyhedral set. In the left picture the boundary of a digital set is shown. Points having neighborhood configuration **63** (up to rotation and reflection) are indicated by $\circ$. In the right picture the enclosing polyhedral set after detection of digital lines is given. The original boundary contains 297 points, the reduced polygonal set has 75 vertices

Further reduction under the conditions stated above is no longer possible in a well-defined way. The problem is that a part of the boundary may have the following form (notations as in Figure 6, left image):



It is assumed here that the interior of the set is below the curve. It is possible to delete one of the critical points $\circ$, but this is not an invariant action. There are other pragmatic possibilities to cope with this situation, e.g. one can introduce additional points not in $\mathbb{Z}^2$ thus relaxing the first two conditions above. Another pragmatic approach which is based on a "relevance measure" associated to the vertices of the polygon is given in [16].

11 Conclusions

The concepts of digital lines and digital convexity were introduced thirty years ago in a pragmatic way by means of the grid-intersection discretization. It turned out that these concepts share a number of properties with lines and convex sets in continuous spaces. It could be shown [10] that these pragmatic concepts can be embedded into a formal axiomatic theory. Moreover, different definitions of digital convexity coincide if in addition 8–connectedness is required. Thus, at present digital lines and digital convexity are well understood theoretically and the different approaches to the subject can be unified within the framework of a nice theory.

Due to the practical relevance of the subject, a number of algorithmic methods were developed to verify convexity of digital sets. They can be classified as syntactic [11], arithmetic [4] and number-theoretic characterizations [30]. Typically, these methods allow to verify convexity in a time proportional to the length of the boundary of the set under investigation.

There do exist differences between digital and continuous convexity. In contrast to continuous sets, it is not possible to verify digital convexity by local inspections only.

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