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Reihe A	Preprints
Reihe B	Berichte
Reihe C	Mathematische Modelle und Simulation
Reihe D	Elektrische Netzwerke und Bauelemente
Reihe E	Scientific Computing
Reihe F	Computational Fluid Dynamics and Data Analysis

The spectrum of the Frobenius–Perron operator and its discretization for circle diffeomorphisms

Wernt Hotzel[*], Nicole Lehmkuhl[†] and Bodo Werner[‡]

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Abstract

To investigate the dynamical behaviour of a discrete dynamical system given by a map f, it is nowadays a standard method to look at the discretization of the Frobenius–Perron operator $\mathcal{P}_f$ w. r. t. a box–partition of the state space resulting in a transition matrix $P_{f,N}$ of a finite Markov chain. We are interested in information about the dynamics of f obtained from the spectrum of $P_{f,N}$, especially for circle diffeomorphisms. Therefore the spectra of $\mathcal{P}_f$ and $P_{f,N}$ are investigated.

Key words: Frobenius–Perron operator, transition matrix, Markov chain, dynamical system, spectrum, circle map

AMS Mathematics Subject Classification: 58F08, 58F11, 58F27

1 Introduction

Throughout the whole paper we assume that X is a n–dimensional C^1–manifold, equipped with the Lebesgue measure μ on the σ–algebra of Borel sets in X. Let $f\colon X \to X$ be a diffeomorphism. The Frobenius–Perron operator (FPO) $\mathcal{P}_f$ is a linear operator, either implicitely and continuously defined on $L^1(X)$ by

$$\int_A \mathcal{P}_f h \, d\mu = \int_{f^{-1}(A)} h \, d\mu \quad \text{for all} \quad \mu\text{-measurable } A \subset X, \tag{1.1}$$

or defined on the measure space $\mathcal{M}$ of X w.r.t. the Borel algebra by

$$(\mathcal{P}_f(\nu))(A) := \nu(f^{-1}(A)) \quad \text{for all} \quad \mu\text{-measurable } A \subset X. \tag{1.2}$$

If in (1.2) ν is absolutely continuous w. r. t. to the measure μ, then (1.2) is equivalent to (1.1).

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Now consider a *partition* $\mathcal{B}$ of X into boxes (cells) B_i, $i = 1, \ldots, N$. Let $\mathcal{H}_N$ be the space of step functions w.r.t. this partition and $\Pi_N \colon L^1(X) \to \mathcal{H}_N$ the projection defined by averaging,

$$(\Pi_N K)(x) = \frac{1}{\mu(B_i)} \int_{B_i} K(y)\, d\mu, \quad \text{for } x \in B_i \text{ and } K \in L^1(X).$$

Then $\mathcal{P}_{f,N} := \Pi_N \mathcal{P}_f|_{\mathcal{H}_N}$ is a discrete version of $\mathcal{P}_f$.

It is easy to see that $\mathcal{P}_{f,N}$ is represented by the $N \times N$ matrix $P_{f,N} = (\mu_{ij})$ w.r.t. the canonical basis of $\mathcal{H}_N$, where

$$\mu_{ij} = \frac{\mu(f^{-1}(B_i) \cap B_j)}{\mu(B_j)}$$

is the transition probability for a point in B_j being randomely chosen w.r.t. the measure μ to be mapped into B_i by f.

The stochastic matrix $P_{f,N}$ defines a finite homogenous Markov chain, particularely a linear, discrete dynamical system on $\mathbb{R}^N$. Its stochasticity allows to restrict to discrete measures $H \in \mathbb{R}^N$ being nonnegative and satisfying $\|H\|_1 = 1$.

It is a standard approach to analyze the dynamical behaviour of f by means of $P_{f,N}$ (see Li [18], Lasota–Mackey [17], Kreuzer [16], where $P_{f,N}$ is associated with the generalized cell mapping, Dellnitz–Junge [8], [7], Dellnitz, Hohmann, Junge, Rump [6], Guder, Dellnitz, Kreuzer [13]).

The general question is: what information about f can be obtained from $P_{f,N}$? For instance, nonnegative fixed points of $P_{f,N}$ (the Frobenius–Perron eigenvectors) are candidates for the approximation of the invariant measures of f (*Ulam's method*).

It appears to be useful to extend the notions *ergodic* and *mixing* of maps f to the discrete case: we explain in terms of $P_{f,N}$ what we understand under f being mixing or ergodic w.r.t. the partition $\mathcal{B}$ (Definition 2.1). These properties can be linked with the theory of nonnegative matrices and finite Markov chains (Berman–Plemmons[2]): ergodicity w.r.t. $\mathcal{B}$ is characterized by irreducibility of $P_{f,N}$, f is mixing w.r.t. $\mathcal{B}$ if and only if $P_{f,N}$ is primitive, and f is ergodic, but not mixing w.r.t. $\mathcal{B}$ if and only if $P_{f,N}$ is q-cyclic with $q > 1$ (Prop. 2.2).

Now on one hand, q-cyclicity of $P_{f,N}$ is characterized by its spectrum (Prop. 2.3), on the other hand it is obviously related to a q-periodic behaviour of f (Dellnitz–Junge[8]). This is the starting point of our interest in the spectrum of $P_{f,N}$ and as a possible limit case, in the spectrum σ_f of P_f itself (Sec. 3). Here P_f is considered as a map on $L^1(X)$ – in contrast to the analysis of Keller[15], [3] which is based on the space $B_V[a, b]$ of functions of bounded variations and on piecewise monotonic maps f.

For diffeomorphisms f, it can be easily seen that σ_f is a subset of the unit circle $\mathbb{C}_1^*$ in $\mathbb{C}$. In the special case that f has a stable hyperbolic periodic orbit, we state that $\sigma_f = \mathbb{C}_1^*$ (Thm. 3.5).

Finally we turn to $X = S^1$ and to orientation preserving circle maps f. Here we have a very special, but interesting situation, which is investigated in

NICOLAISEN–WERNER [21] (see also extensions in Sec. 4). Such a map f is *ergodic* if and only if the rotation number $\varrho(f)$ is irrational, and it is never *mixing*.

The case that $f = r_\omega$ is a rotation about an angle ω, can be completely analysed: Here $\sigma_\omega := \sigma_f = \mathbb{C}_1^*$ with infinite many eigenvalues for $\omega \notin \mathbb{Q}$, while for $\omega = p/q$ the spectrum σ_ω consists of q eigenvalues – the q-th unit roots (Prop. 3.6).

If f is still conjugate to a rotation, the situation is the same. Hence, by Denjoy's Theorem, $\varrho(f) = \omega \notin \mathbb{Q}$ and f being sufficiently smooth imply that $\sigma_f = \sigma_\omega = \mathbb{C}_1^*$ (Thm. 3.8).

But, if $\varrho(f) = \omega = p/q \in \mathbb{Q}$, we cannot expect such conjugacy, but instead phase locking where f has stable, hyperbolic q-periodic orbits. Then our general Theorem 3.5 applies, and yields again $\sigma_f = \mathbb{C}_1^*$. Moreover, the spectrum contains no eigenvalues.

This last result shows that the discretization of circle maps f resulting in $P_{f,N}$ is connected with some qualitative change in the structure of the spectrum: The existence of stable, hyperbolic q-periodic orbits of f yields *reducible* $P_{f,N}$ for sufficiently large N ([21]) and hence a q-symmetric spectrum with exactly q eigenvalues on the unit circle (Cor. 4.3).

The discretization is connected with another qualitative change: If $\varrho(f) \notin \mathbb{Q}$, $P_{f,N}$ is always irreducible, even primitive, e.g. f is mixing w.r.t. $\mathcal{B}$ though f is not (Thm. 4.4). In general, also the rational case $\varrho(f) \in \mathbb{Q}$ is connected with mixing f w.r.t. $\mathcal{B}$, if the attractivity of a periodic orbit is too weak to be resolved by the partition such that $P_{f,N}$ is reducible.

In the last section we perform numerical experiments computing the spectra of $P_{f,N}$, where f is a member of the Arnol'd family. It turnes out that the spectra show an obvious q-symmetry for parameters near Arnol'd tongues with denominator q. Hence we can state for these maps that – varying one parameter – a close phase-locking with rotation number $\varrho(f) = p/q$ is somehow indicated by the q-symmetry of the spectrum.

2 Discrete Ergodicity

Ergodicity of f is connected with an ergodic measure on X, which is a special invariant Borel measure ν such that any f–invariant Borel set A (i.e. $f^{-1}(A) = A$) satisfies $\nu(A) \in \{0, 1\}$. The importance of ergodic measures lies in the *Birkhoff ergodic theorem* (time averages converge against space average). Ergodic maps can have the stronger property *mixing*. We do not go into details, but refer to LASOTA–MACKEY[17] or the survey book of REITMANN[19]. We mention only that invariant measures of f which are absolutely continuous w.r.t. the Lebesgue measure are fixed points of the FPO $\mathcal{P}_f$. In the spirit of LASOTA–MACKEY[17] the properties *ergodic* and *mixing* can be transferred to $P_{f,N}$:

Let $\mathcal{D} := \{H \in \mathbb{R}^N, H \geq 0, \|H\|_1 = 1\}$ contain all discrete densities.

Definition 2.1. *We call f*

- **ergodic w.r.t. the partition** $\mathcal{B}$ *if the following Ergodic Theorem holds: There exists (a unique) $H_0 \in \mathcal{D}$ with $H_0 \gg 0$ such that*

$$\frac{1}{n}\sum_{k=0}^{n-1} P_{f,N}^k H \to H_0 \quad (n \to \infty) \quad \textit{for all} \quad H \in \mathcal{D}. \tag{2.1}$$

- **mixing w.r.t. the partition** $\mathcal{B}$ *if there exists (a unique) $H_0 \in \mathcal{D}$ with $H_0 \gg 0$*

$$P_{f,N}^n H \to H_0 \quad (n \to \infty) \quad \textit{for all} \quad H \in \mathcal{D}. \tag{2.2}$$

2.1 Frobenius–Perron theory for nonnegative, stochastic matrices

It is of no surprise that there is a link between Def. 2.1 and certain statements in the Frobenius–Perron theory of nonnegative, stochastic matrices (see Berman–Plemmons[2]). The link can be stated as follows

Proposition 2.2. *f is*

- ergodic w.r.t. $\mathcal{B}$ *if and only if $P_{f,N}$ is irreducible.*
- mixing w.r.t. $\mathcal{B}$ *if and only if $P_{f,N}$ is primitive (all eigenvalues $\neq 1$ have modulus less than 1).*
- *ergodic, but not mixing w.r.t. $\mathcal{B}$ if and only if $P_{f,N}$ is q-cyclic with $q > 1$, e.g. if $P_{f,N}$ is cogradient to ($m_0 := m_q$)*

$$P := P_1 = \begin{pmatrix} 0 & & \dots & 0 & C_1 \\ C_2 & & & & 0 \\ 0 & C_3 & & & \\ \vdots & & \ddots & & \\ 0 & & \dots & C_q & 0 \end{pmatrix}, \; C_j \in I\!R^{m_j \times m_{j-1}}, \; j = 1, \dots, q. \tag{2.3}$$

It is interesting to note that irreducibilty and primitivity depend only on the directed graph $G_{f,N}$ associated with $P_{f,N}$.

For our investigations it is crucial to point out that cyclicity can be exploited by the eigenvalues of P (see Thm. (2.20) in Ch. 2 in Berman-Plemmons[2], the second half of the famous Perron-Frobenius Theorem):

Proposition 2.3. *An irreducible, stochastic matrix P is q-cyclic if and only if one of the following (equivalent) conditions holds with q being maximal:*

- *There are q different eigenvalues of P of modulus one.*

- *There are q symmetrically distributed algebraically simple eigenvalues $e^{i2\pi\frac{k}{q}}$, $k = 0, 1, \ldots q-1$ of P.*
- *The whole spectrum of P is invariant under the rotation about the angle $2\pi/q$.*

2.2 Finite markov chains

It is very natural to look at propositions 2.2, 2.3 in the context of random walks in finite homogenous Markov chains, see BERMAN–PLEMMONS[2], KREUZER[16], CHUNG[5].

The vertices of the directed graph $G_{f,N}$ can be subdivided into equivalence classes using the communication relation, e.g. $i \sim j$ if and only if vertex i has access to j and vice versa. The classes can be either *recurrent* or *transient.* The last ones form the *transient group.* This structure is reflected by the standard form, given by

$$P_{f,N} = \begin{pmatrix} P_1 & 0 & \ldots & 0 & T_1 \\ 0 & P_2 & & & T_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & P_k & T_k \\ 0 & 0 & \ldots & 0 & Q \end{pmatrix}. \tag{2.4}$$

Here P_j is a column stochastic irreducible matrix associated with the jth recurrent class, $j = 1, \ldots k$. Q is a quadratic nonnegative matrix associated with the transient group.

$\lambda = 1$ is an algebraically simple eigenvalue of P_j with a unique positive eigenvector x^j. The eigenvalues of the substochastic matrix Q all have modulus less than one.

If an irreducible matrix associated with a recurrent class is q-cyclic as in (2.3), than the recurrent class can be subdivided into q disjoint subsets D_j of vertices (of size m_j), such that C_j describes the transition from D_{j-1} to D_j. q equals the greatest common divisor of all lengths of the circuits in the directed graph associated with P. Hence q-cyclicity as in Prop. 2.3 is always associated with *periodicity* of period q in the framework of random walks and also of the original dynamical system.

Using the notions in [2] the finite Markov chain associated with $P_{f,N}$ is ergodic (respectively regular) if and only if f is ergodic (respectively mixing) w.r.t. $\mathcal{B}$. Hence an ergodic, but not regular finite Markov chain has a q-periodic behaviour.

3 Spectrum of $\mathcal{P}_f$ for diffeomorphisms f

We now investigate the spectrum of the continuous FPO $\mathcal{P}_f$ as a possible limit case of the spectrum of its discretization $P_{f,N}$. We first state some general properties of the spectrum of the FPO before we examine circle diffeomorphisms.

Proposition 3.1. *Let $f : X \to X$ be a diffeomorphism. Then $\sigma(\mathcal{P}_f) \subset \mathbb{C}_1^* := \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.*

Proof. Follows from $\|\mathcal{P}_f h\| = \|h\|$ for all $h \in L^1$. $\square$

Remark 3.2. *For other assumptions on f which leads to $\sigma(\mathcal{P}_f) \subset \mathbb{C}_1^*$ see [10].*

Proposition 3.3. *Let f_1 and f_2 be C^1-conjugate diffeomorphism on X with conjugacy g (hence $g \circ f_1 = f_2 \circ g$). Then, on $L^1(X)$ one has*

$$\mathcal{P}_g \circ \mathcal{P}_{f_1} = \mathcal{P}_{f_2} \circ \mathcal{P}_g.$$

If f is a diffeomorphism on X we have

$$(\mathcal{P}_f)^{-1} = \mathcal{P}_{f^{-1}}.$$

Proof. Follows easily by using the implicit definition (1.1). $\square$

The following theorem is the basis of our study of the spectrum for circle diffeomorphisms with rational rotation number, but is also of interest by itself.

Lemma 3.4. *If there exists a measurable subset I_0 of X with positive measure fullfilling the property*

$$I_0 \cap f^k(I_0) = \emptyset \textit{ for all } k \geq 1, \tag{3.1}$$

the spectrum of $\mathcal{P}_f$ consists of the whole unit circle in the complex plane – $\sigma(\mathcal{P}_f) = \mathbb{C}_1^*$.

Proof. It is sufficient to show that any λ which satisfies $\lambda^n = 1$ for some $n \in \mathbb{N}$ is in the spectrum of $\mathcal{P}_f$. This is the case if there is $w \in L^1$ such that

$$\mathcal{P}_f h - \lambda h = w \tag{3.2}$$

is not solvable for $h \in L^1$. We will show that $w = \chi_{I_0}$, the characteristic function of I_0 in (3.1), will do the job.

By definition of $\mathcal{P}_f$ we have — setting $g := f^{-1}$ — for any measurable J

$$\int_J \mathcal{P}_f h \, d\mu \; = \int_{g(J)} h \, d\mu \quad \text{for } h \in L^1 \tag{3.3}$$

and — using (3.2) —

$$\int_{g(J)} h \, d\mu = \lambda \int_J h \, d\mu + \int_J w \, d\mu. \tag{3.4}$$

Hence, since $w = \chi_{I_0}$,

$$\int_{g(J)} h \, d\mu = \lambda \int_J h \, d\mu, \quad \text{if } J \cap I_0 = \emptyset. \tag{3.5}$$

From (3.1) it follows that (3.5) holds for $J = g^k(I_0)$ and also for $J = g^{-k}(I_0)$ $(k \geq 1)$. By induction we obtain for $k \geq 1$

$$\begin{aligned} \int_{g^{k+1}(I_0)} h\,d\mu &= \lambda^k \int_{g(I_0)} h\,d\mu, \\ \lambda^{-k} \int_{I_0} h\,d\mu &= \int_{g^{-k}(I_0)} h\,d\mu. \end{aligned} \tag{3.6}$$

Now we take $k = mn$, $m \in \mathbb{N}$, such that $\lambda^k = 1 = \lambda^{-k}$. We get from (3.6)

$$\begin{aligned} \int_{g^{mn+1}(I_0)} h\,d\mu &= \int_{g(I_0)} h\,d\mu \quad \text{and} \\ \int_{I_0} h\,d\mu &= \int_{g^{-mn}(I_0)} h\,d\mu. \end{aligned} \tag{3.7}$$

From our assumption (3.1) we obtain pairwisely disjoint $g^{mn+1}(I_0)$ if m runs through $\mathbb{N}$. We set

$$B_+ = \bigcup_{m \in \mathbb{N}} g^{mn+1}(I_0) \quad \text{and} \quad B_- = \bigcup_{m \in \mathbb{N}} g^{-mn}(I_0)\,. \tag{3.8}$$

Hence

$$\int_{B_+} h\,d\mu = \sum_{m=1}^{\infty} \int_{g^{mn+1}(I_0)} h\,d\mu = \sum_{m=1}^{\infty} \int_{g(I_0)} h\,d\mu. \tag{3.9}$$

from where

$$\int_{g(I_0)} h\,d\mu = 0. \tag{3.10}$$

Replacing B_+ by B_- and using the same argument we get in the same way

$$\int_{I_0} h\,d\mu = 0. \tag{3.11}$$

Setting $J = I_0$ in (3.4) and plugging (3.10) and (3.11) into (3.4) we obtain $\int_{I_0} w\,d\mu = 0$ in contradiction to $w = \chi_{I_0}$. □

We can find a sufficient condition for assumption (3.1) of lemma 3.4.

Theorem 3.5. *If $f\colon X \to X$ is a diffeomorphism, which has an asymptotically stable, hyperbolical q-periodic orbit, we have $\sigma(\mathcal{P}_f) = \mathbb{C}_1^*$.*

Proof. To apply Lemma 3.4 show that there is an open set I_0 satisfying (3.1). Let $z_0, \ldots z_{q-1}$ be our periodic-orbit. Because of the stability assumption we may choose disjoint neighbourhoods U_i of z_i, $i = 0, \ldots q-1$ such that $f(U_i) \subset U_{i+1}$ and $f(U_{q-1}) \subset U_0$. If we choose $I_0 \subset U_0$, it is sufficient to show that $f^{kq}(I_0) \cap I_0 = \emptyset$ for all $k > 0$. Using a suitable chart of X we may assume that

$U_0 \subset \mathbb{R}^n$ and that f^q is a C^1–map from U_0 into U_0. As the periodic orbit is attractive and hyperbolic, we can find a neighbourhood $U \subset U_0$ from z_0 and a norm such that

$$\|f^q(x) - f^q(y)\| \leq L \, \|x - y\| \quad \text{for all } x, y \in U$$

with $L < 1$. Now let $x_0 \neq z_0$ be in U. Then there is a $\delta > 0$ with

$$\|f^k(x_0) - x_0\| > 2\delta \quad \text{for all } k > 0.$$

We now set $I_0 := U_\delta(x_0) = \{x : \|x - x_0\| < \delta\}$. Then for all $\xi \in I_0$ and all $k > 0$ we have

$$2\delta < \|f^{qk}(x_0) - x_0\| \leq \|f^{qk}(x_0) - f^{qk}(\xi)\| + \|f^{qk}(\xi) - x_0\| \leq L^k\delta + \|f^{qk}(\xi) - x_0\|.$$

So we can conclude that

$$\|f^{qk}(\xi) - x_0\| > \delta$$

and it is clear that (3.1) holds. □

3.1 Circle diffeomorphisms

We assume that the basic theory of circle maps as in Guckenheimer–Holmes [12], Arnol'd[1] or Devaney[9] is familiar to the reader.

Let us now apply the last results to circle diffeomorphisms. We first treat simple rotations. In studying general circle diffeomorphisms we have to distinguish between those with irrational and those with rational rotation number.

It will be helpful to use the explicit form of the FPO for circle diffeomorphisms

$$(\mathcal{P}_f h)(x) = \frac{h(f^{-1}(x))}{f'(f^{-1}(x))}. \tag{3.12}$$

3.1.1 Rotations

For the rotations $f := r_\omega : S^1 \to S^1$, $x \mapsto x + \omega$ the continuous FPO is defined by (3.12)

$$(\mathcal{P}_\omega h)(x) := h(x - \omega), \quad h \in L^1(S^1).$$

$\omega \mapsto \mathcal{P}_\omega$ defines a linear, abelian representation of the circle group S^1 on $L^1(S^1)$. Consider the linear isomorphism

$$T : L^1(S^1) \to L^1(S^1), \quad h \mapsto Th(x) := e^{i2\pi x} \cdot h(x).$$

Then

$$\mathcal{P}_\omega \circ T = e^{-i2\pi\omega} \cdot T \circ \mathcal{P}_\omega. \tag{3.13}$$

It follows that $\mathcal{P}_\omega$ and $e^{i2\pi\omega}\mathcal{P}_\omega$ are similar and hence any eigenvalue λ of $\mathcal{P}_\omega$ is accompanied by a numerable set $e^{i2\pi k\omega} \cdot \lambda$, $k \in \mathbb{Z}$, of further eigenvalues. The corresponding eigenfunctions are transformed by T^k. Since $h_0(x) \equiv 1$ satisfies $\mathcal{P}_\omega h_0 = h_0$ we have

Proposition 3.6. *Consider the rotations $f := r_\omega : S^1 \to S^1$, $x \mapsto x + \omega$, and the corresponding FPO $\mathcal{P}_\omega$.*

- *For each $k \in \mathbb{Z}$ the function $h_k(x) := e^{i2\pi kx}$ is an eigenfunction of $\mathcal{P}_\omega$ with eigenvalue $\lambda_k := e^{i2\pi k\omega}$.*
- *For rational $\omega = \frac{p}{q} \in \mathbb{Q}$ the spectrum $\sigma(\mathcal{P}_\omega)$ consists of q eigenvalues, namely the q unit roots $\lambda_k = e^{i2\pi\frac{k}{q}}, k = 0, 1, 2, ..., q-1$.*
- *For irrational $\omega \notin \mathbb{Q}$, $\sigma(\mathcal{P}_\omega) = \mathbb{C}_1^*$ with a dense set of eigenvalues $\lambda_k := e^{i2\pi k\omega}$, $k \in \mathbb{Z}$.*

Proof. Use $\|\mathcal{P}_\omega h\| = \|h\|$ for all $h \in L^1(S^1)$ and as a consequence of (3.13) that for λ in the resolvent set $\varrho(\mathcal{P}_\omega)$ also $\lambda \cdot e^{i2\pi k\omega} \in \varrho(\mathcal{P}_\omega)$. Because of $\sigma(\mathcal{P}_\omega) \subset \sigma(\mathcal{P}_\omega^q)^{1/q}$, for rational rotation number the spectrum consists only of the q unit roots. $\square$

Remark 3.7. *In this special case of a simple rotation one can decide whether r_ω is ergodic or mixing: r_ω is ergodic if and only if $\omega \notin \mathbb{Q}$. r_ω is never mixing.*

3.1.2 Irrational rotation numbers

What can be proved for general circle maps f about the spectrum of $\mathcal{P}_f$? The answer depends on the rotation number $\omega := \varrho(f)$. We assume that f is a C^2–diffeomorphism such that Denjoy's theorem holds.

The proof follows Chenciner–Iooss [4]. If $\omega \notin \mathbb{Q}$, the Denjoy theorem guarantees the existence of an homeomorphism $g : S^1 \to S^1$ such that $f = g^{-1} \circ r_\omega \circ g$ or

$$g(f(x)) = g(x) + \omega.$$

Let $\lambda \in \sigma(\mathcal{P}_f)$. Then $\mathcal{P}_f - \lambda I$ does not have a continuous inverse. This is still true if we multiply $\mathcal{P}_f - \lambda I$ by $e^{2\pi ikg(\cdot)}$. Having in mind that

$$e^{2\pi ikg(f(x))} = e^{2\pi ikr_\omega(g(x))} = e^{2\pi ik\omega} e^{2\pi ikg(x)},$$

we get

$$\begin{aligned}
e^{2\pi ikg(x)}\big(\mathcal{P}_f h(x) - \lambda h(x)\big) &= \frac{1}{|f'(f^{-1}(x))|}\, e^{2\pi ikg(x)}\, h\big(f^{-1}(x)\big) \\
&\qquad - e^{-2\pi ik\omega}\,\lambda\, e^{2\pi ikg(f(x))}\, h(x) \\
&= \frac{1}{|f'(f^{-1}(x))|}\, \tilde{h}\big(f^{-1}(x)\big) - e^{-2\pi ik\omega}\,\lambda\, \tilde{h}(x) \\
&= \mathcal{P}_f \tilde{h}(x) - e^{-2\pi ik\omega}\,\lambda\, \tilde{h}(x)
\end{aligned}$$

$$\text{with} \quad \tilde{h}(x) = e^{2\pi ikg(f(x))}\, h(x)$$

Now we conclude that with λ, also $e^{-2\pi ik\omega}\lambda$, $k \in \mathbb{Z}$, are spectral values of $\mathcal{P}_f$. Since $\sigma(\mathcal{P}_f) \subset \mathbb{C}_1^*$ is not empty and closed, and ω is irrational we have $\sigma(\mathcal{P}_f) = \mathbb{C}_1^*$.

If the conjugacy g is absolutely continuous, $h_0 := g'$ is a fixed point of $\mathcal{P}_f$, $\lambda = 1$ is an eigenvalue and $e^{-2\pi ik\omega}, k \in \mathbb{Z}$ also. We have shown

Theorem 3.8. *Let f be a C^2–circle diffeomorphism with $\omega := \varrho(f) \notin Q$. The spectrum of $\mathcal{P}_f$ is the whole unit circle $\mathbb{C}_1^*$. Moreover for each $k \in \mathbb{Z}$, the FPO $\mathcal{P}_f$ has the eigenvalue $e^{i2\pi k\omega}$ with eigenfunction $h_k(x) := e^{2\pi ikg(x)}$ if the Denjoy conjugacy is absolutely continuous.*

3.1.3 Rational rotation numbers

The case of rational $\varrho(f) = \frac{p}{q}$ is more difficult, since in general the Denjoy theorem does not hold. Instead we expect phase-locking, e.g. the existence of an asymptotically stable, hyperbolical q-periodic orbit $x_1, \ldots, x_q$. It is easy to see that each q-th unit root $\lambda_k := e^{i2\pi k/q}, k = 0, 1, 2, ...q-1$, is an eigenvalue of $\mathcal{P}_f$ considered as an operator on the measure space with an *eigenmeasure* $\nu = \sum_j h_j \delta_{x_j}$, where $h_j = e^{i2\pi jk/q}$ and δ_x is the Dirac measure at point x. But treating $\mathcal{P}_f$ as an operator on $L^1(S^1)$ Lemma 3.4 implies that $\sigma(\mathcal{P}_f) = \mathbb{C}_1^*$. Additionally we can show that there are no eigenvalues:

Theorem 3.9. *Let f be a circle diffeomorphism with $\omega := \varrho(f) = \frac{p}{q} \in Q$. Assume that there is an asymptotically stable, hyperbolical q-periodic orbit. Then the spectrum of $\mathcal{P}_f$ is the whole unit circle $\mathbb{C}_1^*$. Moreover, if every periodic point is isolated, there are no eigenvalues.*

Proof. We have only to show that there are no eigenvalues. Assume there is a $h \in L^1$ satisfying $\mathcal{P}_f h = \lambda h$. Then $\mathcal{P}_f|h| = |\lambda|\,|h| = |h|$ and $|h|$ is an invariant density.

Now for every point on S^1, which is not periodic, we can find a neighbourhood I_0 satisfying (3.1). Proceeding the same way as in the proof of Lemma 3.4 we obtain $\int_{I_0} |h|\, d\mu = 0$. But, in contradiction to our assumption, this implies $|h| \equiv 0$. □

4 Spectrum of $P_{f,N}$ for circle diffeomorphisms

Now the aim is to investigate the eigenvalues of $P_{f,N}$ and to relate them to the spectrum of $\mathcal{P}_f$.

4.1 Rotations

Let $\mathcal{B}_N$ be an equidistant partition of S^1 into $B_j := [\varphi_{j-1}, \varphi_j), \varphi_{j-1} := \frac{j-1}{N}, j = 1, ..., N$. As above we start with $f = r_\omega$ being a rotation about an angle ω.

Denote the transition matrix of r_ω by $P_{\omega,N}$. Its structure comes out very simply by setting

$$\omega = \frac{L-1+\mu}{N}, \quad L \in \{1, ..., N\}, \mu \in [0,1).$$

Observe that L and μ are uniquely defined and depend on ω and N. Then $P_{\omega,N}$ is a circulant band matrix with a rotating band of width two (entries $1-\mu$ and μ) starting in column L in the first row. Observe that $\mu = 0$ holds only for rational $\omega = \frac{p}{q}$ and for N being multiples of q.

The eigenvalues and eigenvectors of $P_{\omega,N}$ can be easily obtained:

Theorem 4.1. *The eigenvalues of* $P_{\omega,N}$ *are given by*

$$\lambda_{k,N} := (1-\mu)e^{-i2\pi k \frac{L-1}{N}} + \mu e^{-i2\pi k \frac{L}{N}}, k = 0, 1, ..., N-1 \tag{4.1}$$

with corresponding eigenvectors $u^k \in I\!R^N$,

$$u_j^k := e^{i2\pi k \frac{j-1}{N}}, j = 1, 2, ..., N. \tag{4.2}$$

Proof. We have

$$P_{\omega,N} = (1-\mu)P_{\frac{L-1}{N},N} + \mu P_{\frac{L}{N},N}$$

is a convex combination of two permutation matrices having the same eigenvectors u^k given by (4.2). $\square$

Now there is a clear relation between the eigenvalues

$$\lambda_k = e^{-i2\pi k\omega} = e^{-i2\pi k(L-1)/N} e^{-i2\pi k\mu/N} \tag{4.3}$$

and their corresponding eigenfunctions $h_k(x) = e^{-i2\pi kx}$ of the continuous FPO and the eigenvalues $\lambda_{k,N}$ of the discrete FPO $P_{\omega,N}$ for $k = 0, 1, ..., N-1$ and their corresponding eigenvectors u^k:

We have $u_j^k = h_k(\varphi_{j-1}), j = 1, ..., N$.

Concerning the eigenvalues we distinguish the two cases, $\mu = 0$ and $\mu > 0$.

a) Let $\mu = 0$: (This holds only for rational $\omega = p/q$ and $N = rq$ with $r \in I\!N$).

Here the spectrum of $P_{\omega,N}$ and of $\mathcal{P}_\omega$ coincides, the geometrical and algebraical multiplicity of λ_k as an eigenvalue of $P_{\omega,N}$ is r. There exist r periodic classes of $P_{\omega,N}$ of period and of size q and no other classes. The rotation r_ω is ergodic w.r.t. to $\mathcal{B}$ if and only if $P_{\omega,N}$ is N-cyclic (or equivalently $r = 1$ or $N = q$), but never mixing (unless $N = q = 1$).

The bandwidth inclusion in [21] yields

$$\varrho(l) = \frac{L-1}{N} = \omega = \varrho(r) = \frac{L-1}{N}.$$

b) Let $\mu > 0$:

Then $P_{\omega,N}$ is primitive, and hence f_ω is mixing w.r.t. $\mathcal{B}$. The bandwidth inclusion in [21] yields

$$\varrho(l) = \frac{L-1}{N} < \omega < \varrho(r) = \frac{L}{N}.$$

Comparing (4.1) and (4.3) we obtain for fixed k

$$|\lambda_k - \lambda_{k,N}| = O\left(\frac{1}{N}\right).$$

As a consequence, we remark that discretization leads to a kind of *regularization* in the sense that it might increase the strength of mixing: For non-ergodic r_ω, only for very special N being multiples of the denominator of the rotation number ω, we have also non–ergodicity w. r. t. $\mathcal{B}$. In general ($\mu > 0$) we will get a mixing r_ω w. r. t. $\mathcal{B}$ though r_ω is never mixing. These two cases are seperated by the case that $\omega = p/q$ and $N = q$, where we have an ergodic, but not a mixing r_ω w. r. t. the partition $\mathcal{B}$.

4.2 Rational rotation numbers

Consider first the case of a rational rotation number $\varrho(f) = p/q$. Only for rational rotation numbers we can get reducible $P_{f,N}$. And this really happens under mild conditions for sufficient large N ([21]) – but N may be incredible large. Moreover, in this reducible case, $\varrho(f) = p/q$ can be simply determined from the *bandwidth maps* of $P_{f,N}$.

Now we prove

Theorem 4.2. *Let $P_{f,N}$ be reducible. Then all recurrent classes are q-periodic with the same q. All transient classes, which consist of more than one state are also q-periodic. There is $p \in \mathbb{N}$ such that $\varrho(f) = p/q$ and p, q are coprime.*

Proof. Follows from the proof of Thm. 3.1 in [21] where q disjoint intervals $I_j \subset S^1$ are constructed which are cyclically mapped into each other. Since all period orbits of f have the same period (the denominator of the rotation number), the period q of each recurrent class is the same. For reasons of monotonicity the same holds for every cycle of the directed graph. □

Corollary 4.3. *Let $P_{f,N}$ be reducible and $\varrho(f) = p/q$. Then there are q eigenvalues $e^{i2\pi j/q}, j = 0, 1, ..., q-1$ of $P_{f,N}$. These are the only eigenvalues of modulus 1. Moreover the whole spectrum is invariant under a rotation by $2\pi/q$.*

4.3 Irrational rotation numbers

If $\varrho(f) \notin \mathbb{Q}$, then $P_{f,N}$ is always irreducible ([21]). We can show slightly more

Theorem 4.4. *Let $\varrho(f)$ be irrational. Then $P_{f,N}$ is primitive, e.g. f is mixing w. r. t. $\mathcal{B}$.*

Proof. Assume that $\lambda = 1$ is not the only eigenvalue of modulus 1. Then there is a q-periodic recurrent class with $q > 1$. Since $P_{f,N}$ is irreducible, it must be q-cyclic. But then it is easy to show that f must have a q-periodic orbit contradicting the irrationality of $\varrho(f)$. $\square$

5 Numerical experiments

We consider the Arnol'd family

$$f(x) = x + \lambda + \epsilon \sin(2\pi x) \bmod 1$$

of standard maps. We will look at three different situations close to Arnol'd tongues with rotation numbers $1/3$ and $1/5$, see also the numerical experiments in [21].

Example 1.1: $\epsilon = 0.05$.
Here $\varrho(f) = 1/3$ for $\lambda \in J := [0.335018, 0.336091]$.

Example 1.2: $\epsilon = 0.05$.
Here $\varrho(f) = 1/5$ for $\lambda \in J := [0.2053020, 0.2053485]$.

The following figures show the spectra for different values of ω und four subsequent N. The values for N are printed in the center of the images.

5.1 Example 1.1

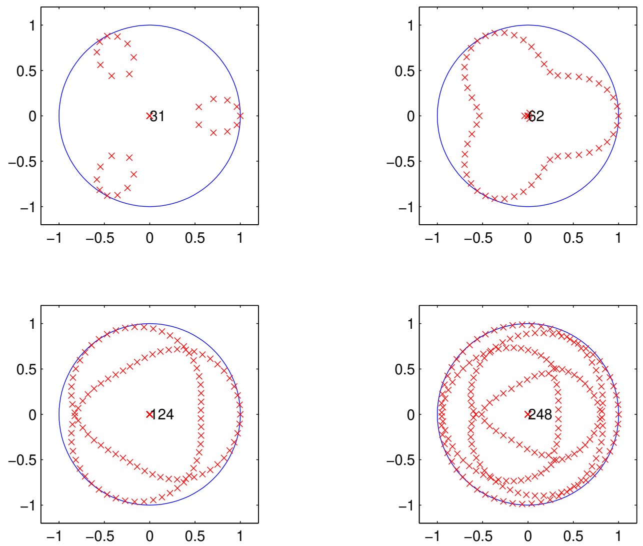

Figure 1: $\epsilon = 0.05, \lambda = 0.33$.

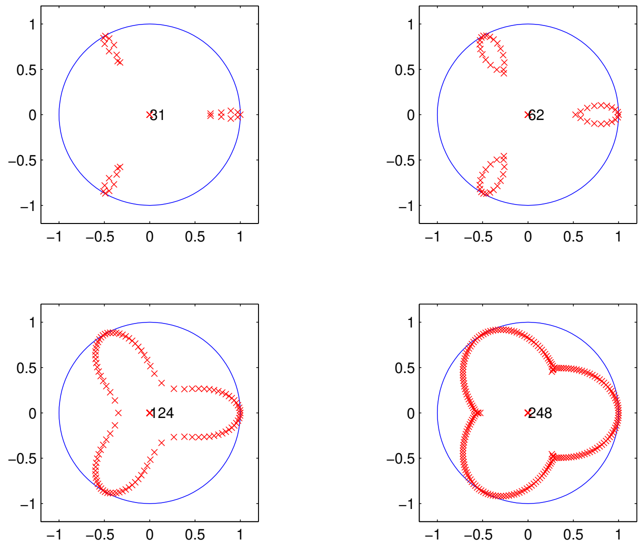

Figure 2: $\epsilon = 0.05, \lambda = 0.334$.

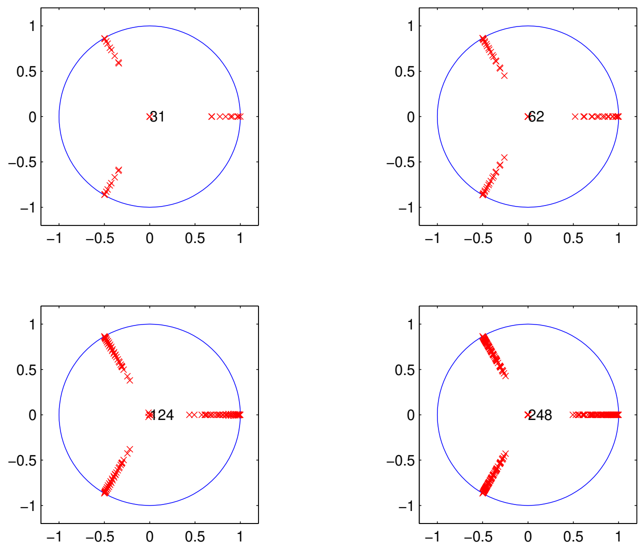

Figure 3: $\epsilon = 0.05, \lambda = 0.3355$ (interior point of the phase locking interval)

5.2 Example 1.2

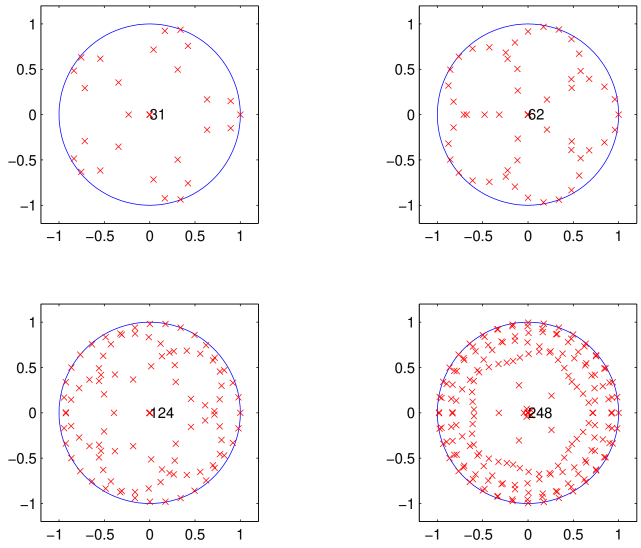

Figure 4: $\epsilon = 0.05, \lambda = 0.2$.

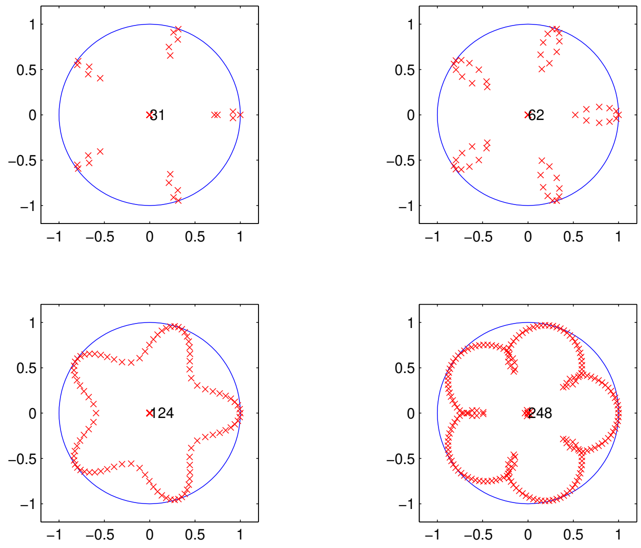

Figure 5: $\epsilon = 0.05, \lambda = 0.204$.

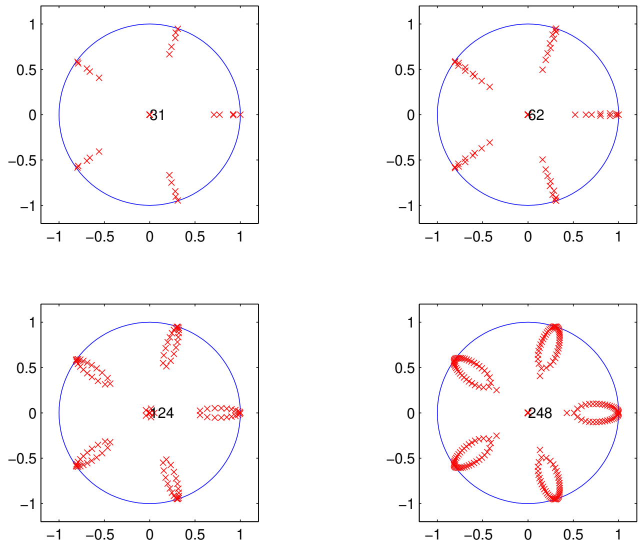

Figure 6: $\epsilon = 0.05, \lambda = 0.205$.

References

[1] Arnol'd, V.I. (1983). *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer, Berlin.

[2] Berman, A., Plemmons, R.J. (1979). *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, London.

[3] Blank, M., Keller, G. (1998). Random perturbations of chaotic dynamical systems: stability of the spectrum. *Nonlinearity* **11**, 1351–1364.

[4] Chenciner, A., Iooss, G. (1979). Bifurcations de tores invariants. *Arch. Rational Mech. Anal.* **69**, 109–198.

[5] Chung, K.L. (1960). *Markov Chains with Stationary Transition Probabilities*, Springer, Berlin.

[6] Dellnitz, M., Hohmann, A., Junge, O., Rumpf, M. (1997). Exploring invariant sets and invariant measures. *Chaos* **7**, No.2, 221–228.

[7] Dellnitz, M., Junge, O. (1998). An adaptive subdivision techniques for the approximation of attractors and invariant measures. *Computation and Visualization in Science*, **1**, No. 2, 63–68.

[8] Dellnitz, M., Junge, O. (1999). On the approximation of complicated dynamical behavior. *SIAM J. Numer. Anal.* **36**, No.2, 491–515.

[9] Devaney, R. L. (1987). *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley, Redwood City.

[10] Ding, J., Du, Q., Li, T.Y. (1994). The spectral analysis of Frobenius–Perron operators. *J. Math. Anal. Appl.* **184**, No. 2, 285–301.

[11] Duff, I.S., Reid, J.K. (1978). An implementation of Tarjan's algorithm for the block triangularization of a matrix. *ACM Trans. Math. Softw.* **4**, 137–147.

[12] Guckenheimer, J., Holmes, P. (1983). *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields*, Springer, Berlin.

[13] Guder, R., Dellnitz, M., Kreuzer, E. (1997). An adaptive method for the approximation of the generalized cell mapping. *Chaos, Solitons and Fractals* **8**(4), 525–534.

[14] Huppert, B. (1990). *Angewandte Lineare Algebra*, Walter de Gruyter, Berlin.

[15] Keller, G. (1984). On the rate of convergence to equilibrium in one-dimensional systems. *Commun. Math. Phys.* **96**, 181–193.

[16] Kreuzer, E. (1987). *Numerische Untersuchung nichtlinearer dynamischer Systeme*, Springer, Berlin.

[17] Lasota, A., Mackey, C. (1994). *Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics*, Springer, Berlin.

[18] Li, T.Y. (1976). Finite approximation for the Frobenius–Perron operator. A solution to Ulam's conjecture. *J. Approx. Theory* **17**, 177–186.

[19] Reitmann, V. (1996). *Reguläre und chaotische Dynamik*, Teubner, Leipzig.

[20] Walters, P. (1975). *Ergodic Theory. Introductory Lectures*, Lecture Notes in Math. 458, Springer, New York.

[21] Werner, B., Nicolaisen, N. (1998). Discretization of circle maps. *Z. Angew. Math. Phys.* **49**, No.6, 869–895.