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Second Order Sufficient Conditions for Optimal Control Problems with Free Final Time: The Riccati Approach [1]

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Abstract. Second order sufficient conditions (SSC) for control problems with control–state constraints and free final time are presented. Instead of deriving such SSC de initio, the control problem with free final time is tranformed into an augmented control problem with fixed final time for which well-known SSC exist. SSC are then expressed as a condition on the positive definiteness of the second variation. A convenient numerical tool for verifying this condition is based on the Riccati approach where one has to find a bounded solution of an associated Riccati equation satisfying specific boundary conditions. The augmented Riccati equations for the augmented control problem are derived and their modifications on the boundary of the control–state constraint are discussed. Two numerical examples, (1) the classical Earth-Mars orbit transfer in minimal time, (2) the Rayleigh problem in electrical engineering, demonstrate that the Riccati equation approach provides a viable numerical test of SSC.

Key Words. optimal control, control–state constraints, free final time, second order sufficient conditions, Riccati equation, Earth–Mars orbit transfer, Rayleigh problem.

AMS subject classifications. 49K15, 49K40, 65L10, 70M20, 94C99.

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1. Introduction

In the last three decades, one can find an extensive literature on second order sufficient optimality conditions (SSC) for optimal control problems with control and state constraints; cf. Refs. 2, 4, 6–10, 20, 25, 26, 31, 32, 35, and further literature cited in these papers. SSC have shown to be of fundamental importance for stability and sensitivity analysis of parametric optimal control problems; cf. e.g. Refs. 2, 5, 9, 10, 18, 19, 21, 23, 24, 33, 34.

SSC are usually expressed in terms of the positiveness of a quadratic form on a certain critical cone which is obtained through linearization of equality and inequality constraints. In general, such conditions are far too abstract to lend themselves to numerical verification. A practical test for SSC can be devised on the basis of a matrix-valued Riccati equation, Refs. 23, 25, 37. The main ideas underlying this approach are already exposed in the book of Bryson, Ho (Ref. 1) for unconstrained control problems. This test requires the construction of a bounded solution to a Riccati equation which has to satisfy additional boundary conditions. An inherent difficulty arises from the fact that the coefficients of the Riccati equation depend on the accurate solution for state, control and adjoint variables.

Most of the cited papers deal with control problems on a *fixed* time interval. Extensions of the results to problems with *free* final time or with non–fixed time intervals have been discussed in Refs. 1, 4, 12, 26, 31, 32. The method in Ref. 1, Chapter 6, uses heuristic arguments and also suffers from the drawback that, e.g., time–optimal control problems are not tractable via this approach. More precisely, the function α defined in (6.6.13) of Ref. 1 is identically zero and, hence, the quantity in (6.6.16) is not defined. A rigorous proof of the SSC in Ref. 1 may be found in Chamberland, Zeidan (Ref. 4) where also extensions of the results to control problems with mixed control–state constraints are given. However, again the time–optimal case is not covered by these conditions. A remedy for this deficiency has been proposed in Ref. 12 for unconstrained control problems. These conditions have been tested in a numerically not very challenging situation. We emphasize that a general approach for SSC on non–fixed time intervals has been developed by Osmolovskii (Refs. 26, 31, 32) but the author does not offer any practical device to test his conditions numerically.

The aim of the present paper is to develop verifiable SCC for control problems with free final time and mixed control–state constraints. Our approach is rather straightforward in the sense that it uses the well known idea (cf. e.g. Ref. 11) of reducing the *free* final time case to the *fixed* final time case by treating the free final time as an *augmented* state variable. It is not surprising that this procedure will lead to an augmented set of Riccati equations and boundary conditions. For unconstrained control problems, this derivation has been already described in Ref. 22.

The organization of the paper is as follows. In Section 2, we recall known SSC for control problems with fixed final time, Refs. 25, 37. Section 3 describes the effect of the time transformation on the augmented variables and functions of the problem. A straightforward calculation shows that the Riccati equation for the augmented problem splits into three separate parts. A salient feature of the approach is that it suffices to solve a reduced form of the Riccati equation on the boundary of the control–state constraint. The boundary conditions for the Riccati equation are worked out in some cases of practical interest. In particular, we derive additional sign conditions of the Riccati solution at the initial and final time which turn out to be crucial in the numerical test.

In sections 4 and 5, we apply the numerical methodology to two practical and challenging examples. A highly accurate numerical solution to both examples is obtained via the multiple shooting method, cf. Refs. 3, 29. The classical problem of a planar Earth–Mars orbit transfer in minimal time, cf. Refs. 14, 16, 17, is treated in section 4. The augmented Riccati equation test succeed in confirming the optimality of the numerical solution. Section 5 presents a modification of the Rayleigh problem which has been solved in Refs. 13, 23, 26 on fixed time intervals. Surprisingly, when no control constraints are imposed, the free final time problem has several local minima and one local maximum. The augmented Riccati test is capable of proving optimality for both local minima. Then the Rayleigh problem subject to control constraints is studied. We derive the reduced Riccati equations on the boundary of the constraint and compute a bounded solution which satisfies the extra boundary condition.

Let us mention two further applications and extensions. Firstly, on the basis of SSC it is rather straightforward to perform a computational *sensitivity analysis* for both examples. The numerical techniques in Refs. 21, 23, 24, 33, 34 indicate that sensitivity differentials of optimal solutions with respect to parameters can be obtained through the solution of an additional linear boundary value problem. The second extension concerns optimal control problems with *pure* state constraints to which the techniques of this paper apply as well.

2. SSC for control problems with fixed final time

We consider the following autonomous control problem (CP) subject to mixed control–state constraints: for a given final time $T > 0$ determine a control function $u \in L^\infty(0,T;\mathbb{R}^m)$ and a state function $x \in W^{1,\infty}(0,T;\mathbb{R}^n)$ that minimize the functional

$$F(x,u) = g(x(0),x(T)) + \int_0^T L(x(t),u(t))dt \tag{2.1}$$

subject to

$$\dot{x}(t) = f(x(t),u(t)) \quad \text{for a.e. } t \in [0,T]\,, \tag{2.2}$$
$$\varphi(x(0),x(T)) = 0\,, \tag{2.3}$$
$$C(x(t),u(t)) \le 0 \quad \text{for a.e. } t \in [0,T]\,. \tag{2.4}$$

It is assumed that the functions $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^r$, $0 \le r \le 2n$, and $C : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k$ are C^2-functions on appropriate open sets. In this section, the *final time* T is supposed to be specified. Further we assume that there exists a feasible pair of functions $(x_0,u_0) \in W^{1,\infty}(0,T;\mathbb{R}^n)\times L^\infty(0,T;\mathbb{R}^m)$ satisfying the constraints (2.2)–(2.4).

The first order necessary conditions for an optimal pair (x_0, u_0) are well known in the literature, cf. Refs. 11, 28. The *unconstrained* Hamiltonian function H^0 resp. the *augmented* Hamiltonian H are defined as

$$H^0(x,u,\lambda) = L(x,u) + \lambda^* f(x,u)\,, \quad H(x,u,\lambda,\mu) = H^0(x,u,\lambda) + \mu^* C(x,u), \quad (2.5)$$

where $\lambda \in \mathbb{R}^n$ denotes the adjoint variable and $\mu \in \mathbb{R}^k$ is the multiplier associated with the control–state constraint (2.4); the asterisk denotes the transpose. Henceforth, partial derivatives will often be denoted by subscripts. In the following, we shall make the hypothesis that first order conditions are satisfied in *normal form* with a non–zero cost multiplier. Hence, we assume that there exist Lagrange-multipliers (for convenience, we shall drop the lower subscript zero)

$$(\lambda, \mu, \rho) \in W^{1,\infty}(0,T;\mathbb{R}^n) \times L^\infty(0,T;\mathbb{R}^k) \times \mathbb{R}^r$$

such that the following first-order necessary conditions hold for a.e. $t \in [0,T]$,

$$\dot\lambda(t) = -H_x(x_0(t), u_0(t), \lambda(t), \mu(t))^*\,, \quad (2.6)$$

$$(-\lambda(0), \lambda(T)) = \nabla_{(x(0),x(T))}\,(g + \rho^*\varphi)(x_0(0), x_0(T))\,, \quad (2.7)$$

$$H^0(x_0(t), u_0(t), \lambda(t)) = \, min_u\,\{\, H^0(x_0(t), u, \lambda(t)) \mid C(x_0(t), u) \le 0\,\}, \quad (2.8)$$

$$\Rightarrow \quad H_u[t] = 0,\ H_{uu}[t] \ge 0 \ \text{(positive semi-definite)},$$

$$\mu(t) \ge 0 \quad \text{and} \quad \mu(t)^* C(x_0(t), u_0(t)) = 0, \quad (2.9)$$

$$H^0(x_0(t), u_0(t), \lambda(t)) \equiv \text{const.}. \quad (2.10)$$

We shall use the notation $[t]$ to denote arguments of functions at the reference solution $x_0(t), u_0(t), \lambda(t), \mu(t)$. To introduce regularity assumptions we consider for $\beta \ge 0$ the set of β–active constraints

$$I_\beta(t) := \{i \in \{1,\dots,k\} \mid C^i[t] \ge -\beta\}\,,$$

where C^i denotes the i-th component of the vector C. In particular, for $\beta = 0$ we obtain the set of active indices

$$I_0(t) = \{i \in \{1,\dots,k\} \mid C^i[t] = 0\}\,.$$

The following regularity assumption concerns *linear independence* of gradients for active constraints; cf. Refs. 18, 19, 21, 25, 37.

(A1) For some $\beta > 0$ the gradients $C^i_u[t]$ are uniformly linear independent for all $i \in I_\beta(t)$ a.e. on $[0,T]$.

Further we consider a margin $\delta \ge 0$ and define the set of indices

$$J_\delta(t) := \{i \in \{1,...k\} \mid \mu_i(t) > \delta\}, \quad j_\beta(t) := \text{card}(J_\beta(t)),$$

where μ_i denotes the i-th component. It is obvious that $J_\delta(t) \subset I_0(t)$ holds for all $\delta \geq 0$. In particular, the *strict complementarity condition* $\mu^i(t) > 0$ is valid for all indices $i \in J_\delta(t)$. It will be convenient to introduce the notation

$$C^\delta[t] = (C^i[t])_{i \in J_\delta(t)}.$$

We assume that the following *modified strict Legendre-Clebsch condition*, cf. Refs. 18, 19, 21, 25, 37, is satisfied where $|\,.\,|$ denotes the euclidean norm:

(A2) There exists $c > 0$ such that for all $t \in [0,T]$ the estimate $v^* H_{uu}[t]v \geq c|v|^2$ holds for all $v \in \mathbb{R}^m$ satisfying $C_u^\delta[t]v = 0$.

Second order sufficient conditions (SSC) can now be derived by studying the behavior of the second variation on the variational system associated with equations (2.2)–(2.4). In the sequel, we shall use the abbreviation $\varphi[0,T] = \varphi(x_0(0), x_0(T))$ and similar notations. The *variational system* of equations (2.2)–(2.4) is the set of functions $(y,v) \in W^{1,2}(0,T;\mathbb{R}^n) \times L^2(0,T;\mathbb{R}^m)$ satisfying

$$\dot{y}(t) = f_x[t]y(t) + f_u[t]v(t) \quad \text{a.e. } t \in [0,T], \tag{2.11}$$

$$D_{x(0)}\varphi[0,T]y(0) + D_{x(T)}\varphi[0,T]y(T) = 0, \tag{2.12}$$

$$C_x^i[t]y(t) + C_u^i[t]v(t) = 0 \quad \text{for all } i \in J_\delta(t), \text{ a.e. } t \in [0,T]. \tag{2.13}$$

Moreover, we introduce the function

$$G(x(0), x(T)) := g(x(0), x(T)) + \rho^* \varphi(x(0), x(T)),$$

and define the $(2n, 2n)$–matrix

$$\Gamma[0,T] := D^2_{(x(0),x(T))} G(x_0(0), x_0(T)) = \begin{pmatrix} G_{00}[0,T] & G_{0T}[0,T] \\ G_{T0}[0,T] & G_{TT}[0,T] \end{pmatrix} \tag{2.14}$$

with obvious notations $G_{00}[0,T] = D^2_{(x(0),x(0))}G[0,T]$, $G_{0T}[0,T] = D^2_{((x(0),x(T))}G[0,T]$, etc. Then the so-called *second variation* is given by the quadratic form

$$\begin{aligned} J^2(y,v) &= \frac{1}{2}\int_0^T (y(t)^*, v(t)^*) \begin{pmatrix} H_{xx}[t] & H_{xu}[t] \\ H_{ux}[t] & H_{uu}[t] \end{pmatrix} \begin{pmatrix} y(t) \\ v(t) \end{pmatrix} dt \\ &\quad + \frac{1}{2}(y(0)^*, y(T)^*)\Gamma[0,T] \begin{pmatrix} y(0) \\ y(T) \end{pmatrix}. \end{aligned} \tag{2.15}$$

The next theorem summarizes the SSC for a weak local minimum which are to be found in Refs. 21, 25, 35, 37.

Theorem (2.1) (SSC for control problems with fixed final time.)
Let (x_0, u_0) be admissible for problem (CP). Suppose that there exist multipliers $(\lambda, \mu, \rho) \in W^{1,\infty}(0,T;\mathbb{R}^n) \times L^\infty(0,T;\mathbb{R}^k) \times \mathbb{R}^r$ such that the following conditions hold:

(1) the necessary conditions (2.6)–(2.10) are satisfied;

(2) assumptions (A1) and (A2) hold;

(3) there exist $\delta > 0$ and $c > 0$ such that the quadratic form in (2.15) can be estimated from below as

$$J^2(y,v) \geq c\,[\,||y||_{1,2}^2 + ||v||_2^2\,] \tag{2.16}$$

for all variations $(y,v) \in W^{1,2}(0,T;\mathbb{R}^n) \times L^2(0,T;\mathbb{R}^m)$ satisfying the variational system (2.11)–(2.13);

(4) if u_0 is continuous then one may choose $\beta = 0$ and $\delta = 0$ in assumptions (A1) and (A2) and in condition (3).

Then for all constants $0 < c_0 < c$ with c as in (2.16) there exists $\alpha > 0$ such that

$$F(x,u) \geq F(x_0,u_0) + c_0\,[\,||x-x_0||_{1,2}^2 + ||u-u_0||_2^2\,]$$

holds for all admissible (x,u) with $||x-x_0||_{1,\infty} + ||u-u_0||_\infty \leq \alpha$. In particular, (x_0,u_0) provides a strict weak local minimum for problem (CP).

The SSC in the previous theorem usually are not suitable for a *direct numerical verfication* in practical control problems. Let us mention that for a *discretized* version of the control problem (CP) optimization techniques have been developed that allow to check the positiveness condition by computing the reduced Hessian; cf. Ref. 2. In order to obtain *verifiable sufficient conditions* for the control problem in function spaces, the SSC in Theorem 2.1 are strengthened in the following way.

Consider a symmetric matrix function $Q \in W^{1,\infty}(0,T;M_{n\times n})$. For every variation $y(t)$ satisfying the linearized state equation (2.11) we have $y(t)^*Q(t)(\dot{y}(t) - f_x[t]y(t) - f_u[t]v(t)) \equiv 0$. Adding the last identity to the second variation $J^2(y,v)$ in (2.15) and performing a partial integration we find that the definiteness condition (2.16) in Theorem 2.1 holds if the following two conditions (a) and (b) are satisfied.

Condition (a): There exists a symmetric matrix $Q \in W^{1,\infty}(0,T;M_{n\times n})$ and $c > 0$ such that

$$\begin{aligned}(y^*,v^*) &\begin{pmatrix} \dot{Q}(t) + Q(t)f_x[t] + f_x[t]^*Q(t) + H_{xx}[t] & H_{xu}[t] + Q(t)f_u[t] \\ H_{ux}[t] + f_u[t]^*Q(t) & H_{uu}[t] \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix} \\ &\geq c|(y,v)|^2 \end{aligned} \tag{2.17}$$

holds *uniformly* in $t \in [0,T]$ for all vectors $(y,v) \in \mathbb{R}^n \times \mathbb{R}^m$ with

$$C_x^i[t]y + C_u^i[t]v = 0 \quad \forall\, i \in J_\delta(t)\,. \tag{2.18}$$

Condition (b): The boundary condition

$$(\xi_0^*, \xi_1^*) \begin{pmatrix} G_{00}[0,T] + Q(0) & G_{0T}[0,T] \\ G_{T0}[0,T] & G_{TT}[0,T] - Q(T) \end{pmatrix} \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} > 0 \tag{2.19}$$

is valid for all $(\xi_0, \xi_1) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$ satisfying

$$D_{x(0)}\varphi[0,T]\xi_0 + D_{x(T)}\varphi[0,T]\xi_1 = 0\,. \tag{2.20}$$

A first consequence is that the definiteness condition (2.16) holds if the matrix in (2.17) is positive definite on the *whole* space $\mathbb{R}^n \times \mathbb{R}^m$ and if conditions (2.19) and (2.20) are satisfied. Firstly, this leads to the requirement that the strict Legendre–Clebsch condition

$$H_{uu}[t] \geq c \cdot I_m \quad \forall\, t \in [0,T]\,, \quad c > 0,$$

is valid on the *whole* interval $[0,T]$. Secondly, by evaluating the Schur complement of this matrix and using the continuous dependence of ODEs on system data, the estimate (2.16) follows from the following assumption: there exists a solution of the Riccati equation

$$\dot{Q} = -Qf_x[t] - f_x[t]^*Q - H_{xx}[t] + (H_{xu}[t] + Qf_u[t])H_{uu}[t]^{-1}(H_{xu}[t] + Qf_u[t])^*, \tag{2.21}$$

which is *bounded* on $[0,T]$ and satisfies the boundary conditions (2.19) and (2.20); cf. Ref. 25, Theorem 5.2.

However, in some applications these conditions are too strong since the Riccati equation (2.21) may fail to have a *bounded* solution; cf. the Rayleigh problem in Ref. 23. A weaker condition can be obtained by introducing the following *modified* or *reduced* Riccati equation. Recall for $\delta \geq 0$ the definition of the vector

$$C^\delta[t] = (C^i[t])_{i \in J_\delta(t)}\,, \quad j_\delta(t) = card(J_\delta(t))\,.$$

We need the pseudoinverse of the matrix $C_u^\delta[t]$ which is given by the $(m \times j_\delta(t))$–matrix

$$(C_u^\delta[t])^+ = (C_u^\delta[t]C_u^\delta[t]^*)^{-1}C_u^\delta[t]\,.$$

It follows that every $v \in \mathbb{R}^m$ can be decomposed as $v = v^\perp + v^+$ with

$$v^\perp = (C_u^\delta[t])^*(C_u^\delta[t])^+ v \,\in [Ker\,(C_u^\delta[t])]^\perp, \;\; v^+ = (I - (C_u^\delta[t])^*(C_u^\delta[t])^+)v \,\in Ker\,(C_u^\delta[t])\,.$$

For convenience we omit the time argument and define the following matrices; cf. Refs. 9, 10, 25, 37:

$$D^\delta := -(C_u^\delta)^+ C_x^\delta\,, \quad P^\delta := I - (C_u^\delta)^*(C_u^\delta)^+, \quad A^\delta := f_x + f_u D^\delta, \tag{2.22}$$

$$\mathcal{H}_{xx}^\delta := H_{xx} + H_{xu}D^\delta + (D^\delta)^* H_{ux} + (D^\delta)^* H_{uu} D^\delta, \tag{2.23}$$

$$\mathcal{H}_{xu}^\delta := H_{xu} + (D^\delta)^* H_{uu}\,, \tag{2.24}$$

$$(\mathcal{H}_{uu}^\delta)^{(-1)} := P^\delta (P^\delta H_{uu} P^\delta)^{-1} P^\delta\,. \tag{2.25}$$

Note that the matrix $(\mathcal{H}^{\delta}_{uu})^{(-1)}$ in (2.25) is well defined by virtue of assumption (A2). It follows that the estimate (2.16) holds if the matrix Riccati equation

$$\dot{Q} = -QA^{\delta} - (A^{\delta})^{*}Q - \mathcal{H}^{\delta}_{xx} + (\mathcal{H}^{\delta}_{xu} + Qf_u)(\mathcal{H}^{\delta}_{uu})^{(-1)}(\mathcal{H}^{\delta}_{xu} + Qf_u)^{*} \tag{2.26}$$

has a *bounded* solution on $[0,T]$ which satisfies the boundary conditions (2.19), (2.20).

In general it is rather tedious to elaborate this Riccati equation explicitly. To facilitate the numerical treatment in practical applications we discuss special cases in more detail. On *interior arcs* with $C[t] < 0$ we have $j_{\delta}(t) = 0$ and thus the Riccati equation (2.26) reduces to the one introduced in (2.21). Consider now a *boundary arc* with $j_{\delta}(t) = m$ where we have as many control components as active constraints. Due to assumption (A1) the pseudoinverse is given by $(C^{\delta}_u)^{+} = (C^{\delta}_u)^{-1}$ and, hence, the projection $P^{\delta} = 0$ in (2.22) vanishes. Then the matrices in (2.22)–(2.25) become

$$D^{\delta} = -(C^{\delta}_u)^{-1}C^{\delta}_x \quad A^{\delta} = f_x - f_u(C^{\delta}_u)^{-1}C^{\delta}_x\,, \quad (\mathcal{H}^{\delta}_{uu})^{(-1)} = 0\,, \tag{2.27}$$

$$\mathcal{H}^{\delta}_{xx} = H_{xx} - H_{xu}(C^{\delta}_u)^{-1}C^{\delta}_x + [(C^{\delta}_u)^{-1}C^{\delta}_x]^{*}[H_{uu}(C^{\delta}_u)^{-1}C^{\delta}_x - H_{ux}]\,, \tag{2.28}$$

and thus the Riccati equation (2.26) reduces to the *linear* ODE

$$\dot{Q} = -QA^{\delta} - (A^{\delta})^{*}Q - \mathcal{H}^{\delta}_{xx}\,. \tag{2.29}$$

Another special case arises for *pure control* constraints where we have $C_x \equiv 0$. Then (2.28) and (2.29) lead to the simple *linear* ODE

$$\dot{Q} = -Qf_x - f_x^{*}Q - H_{xx}\,. \tag{2.30}$$

The Rayleigh problem in Ref. 23 has provided an illustrative application of this approach.

Let us also evaluate the boundary conditions (2.19) and (2.20) in a special case of practical interest. Suppose that the boundary conditions are separated and that components for the initial and final state are fixed according to

$$x_k(0) = a_k \ \text{ for } \ k \in K_0 \subset \{1,...,n\}, \quad x_k(T) = b_k \ \text{ for } \ k \in K_T \subset \{1,...,n\}. \tag{2.31}$$

Denote the complements of the index sets by $K_0^c = \{1,...,n\} \setminus K_0$, $K_T^c = \{1,...,n\} \setminus K_T$. Then it is easy to see that the boundary conditions (2.19), (2.20) are satisfied if the following submatrices are positive definite:

$$[\,Q(0)\,]_{(i,j)\in K_0^c\times K_0^c} > 0\,, \quad [\,-Q(T)\,]_{(i,j)\in K_T^c\times K_T^c} > 0\,. \tag{2.32}$$

By virtue of the continuous dependence of solutions to ODEs on systems data, one of these definiteness conditions can be relaxed. E.g., it suffices to require only positive semi–definiteness,

$$[\,Q(0)\,]_{(i,j)\in K_0^c\times K_0^c} \geq 0\,.$$

This relaxation will be convenient for the numerical verfication of SSC applied to the examples in sections 4 and 5.

3. SSC for control problems with free final time

We consider again the control problem (CP) in (2.1)–(2.4) but in this section the final time will *not* be specified. The necessary conditions for problem (CP) with free final time are well known, cf. Refs. 11, 28, and extend those given in the last section.

Let H be the Hamiltonian defined in (2.5). Then it is assumed that there exist multipliers in normal form,

$$(\lambda, \mu, \rho) \in W^{1,\infty}(0,T;\mathbb{R}^n) \times L^\infty(0,T;\mathbb{R}^k) \times \mathbb{R}^r,$$

which satisfy equations (2.6)–(2.10). In addition, the following transversality condition associated with the free final time T holds,

$$H[T] = 0\,, \quad \text{i.e.,} \quad H[t] \equiv 0 \quad \forall\, t \in [0,T]\,. \tag{3.1}$$

These conditions can be obtained by transforming problem (CP) with *free* final time T into a problem $(\widetilde{\text{CP}})$ with *fixed* final time $\tilde{T} = 1$. The transformation proceeds by augmenting the state dimension and by introducing the free final time as an additional state variable. Indeed, it is this transformation that will allow us to develop SSC for the free final time case on the basis of the SSC in Theorem 2.1 for fixed final time. Define now the *new time variable* $\tau \in [0,1]$ by

$$t = T \cdot \tau\,, \quad 0 \le \tau \le 1\,. \tag{3.2}$$

We shall use the same notation $x(\tau) := x(T\tau)$ and $u(\tau) := u(T\tau)$ for the state and the control variable with respect to the new time variable τ. The *augmented* state

$$\tilde{x} := \begin{pmatrix} x \\ x_{n+1} \end{pmatrix} \in \mathbb{R}^{n+1}, \quad x_{n+1} := T\,,$$

satisfies the differential equations

$$dx/d\tau = T \cdot f(x(\tau), u(\tau))\,, \qquad dx_{n+1}/d\tau \equiv 0\,. \tag{3.3}$$

As a result of this time transformation we consider the following augmented control problem $(\widetilde{\text{CP}})$ on the fixed time interval $[0,1]$: minimize the functional

$$F(\tilde{x}, u) = F(x, T, u) = \tilde{g}(\tilde{x}(0), \tilde{x}(1)) + \int_0^1 \tilde{L}(\tilde{x}(\tau), u(\tau))\, d\tau \tag{3.4}$$

subject to

$$d\tilde{x}/d\tau = \tilde{f}(\tilde{x}(\tau), u(\tau)), \quad \text{a.e. } \tau \in [0,1]\,, \tag{3.5}$$

$$\tilde{\varphi}(\tilde{x}(0), \tilde{x}(1)) = 0\,. \tag{3.6}$$

$$\tilde{C}(\tilde{x}(\tau), u(\tau)) \le 0, \quad \text{a.e. } \tau \in [0,1]\,. \tag{3.7}$$

The functions herein are given by

$$\tilde{g}(\tilde{x}(0), \tilde{x}(1)) := g(x(0), x(1))\,, \quad \tilde{L}(\tilde{x}, u) := T \cdot L(x,u)\,, \tag{3.8}$$

$$\tilde{f}(\tilde{x}, u) := \begin{pmatrix} T \cdot f(x,u) \\ 0 \end{pmatrix}, \tag{3.9}$$

$$\tilde{\varphi}(\tilde{x}(0), \tilde{x}(1)) := \varphi(x(0), x(1))\,, \quad \tilde{C}(\tilde{x}, u) := C(x,u)\,. \tag{3.10}$$

The transformed problem $(\widetilde{\mathrm{CP}})$ on the fixed time interval $[0,1]$ falls into the category of control problems treated in the preceding section. Thus we are able to obtain SSC for the transformed problem $(\widetilde{\mathrm{CP}})$ by evaluating the SSC in Theorem 2.1 for the augmented state variable $\tilde{x} = (x, T)$.

First we relate the multipliers for problem $(\widetilde{\mathrm{CP}})$ to those of problem (CP) on the time interval $[0, T]$. The Hamiltonian for problem $(\widetilde{\mathrm{CP}})$ becomes

$$\begin{aligned} \tilde{H}(\tilde{x}, u, \tilde{\lambda}, \tilde{\mu}) &= \tilde{L}(\tilde{x}, u) + \tilde{\lambda}^* \tilde{f}(\tilde{x}, u) + \tilde{\mu}^* \tilde{C}(\tilde{x}, u) \\ &= T \cdot [\, L(x,u) + \lambda^* f(x,u)\,] + \tilde{\mu}^* C(x,u), \end{aligned} \tag{3.11}$$

where $\tilde{\lambda}^* = (\lambda^*, \lambda_{n+1}) \in \mathbb{R}^{n+1}$ denotes the augmented adjoint variable and $\tilde{\mu} \in \mathbb{R}^k$ is the multiplier for the constraint (3.7). Introducing the scaled multiplier $\mu := \tilde{\mu}/T$, the Hamiltonian $\tilde{H}$ is related to the Hamiltonian H in (2.5) as follows,

$$\tilde{H}(\tilde{x}, u, \tilde{\lambda}, \tilde{\mu}) = T \cdot [\, L(x,u) + \lambda^* f(x,u) + \mu^* C(x,u)\,] = T \cdot H(x, u, \lambda, \mu)\,. \tag{3.12}$$

According to equations (2.6)–(2.10), the multipliers for problem $(\widetilde{\mathrm{CP}})$,

$$(\tilde{\lambda}, \tilde{\mu}, \rho) \in W^{1,\infty}(0,1;\mathbb{R}^{n+1}) \times L^{\infty}(0,1;\mathbb{R}^k) \times \mathbb{R}^r\,, \quad \tilde{\lambda} = \begin{pmatrix} \lambda \\ \lambda_{n+1} \end{pmatrix},$$

satisfy the following necessary conditions for a.e. $\tau \in [0,1]$:

$$d\lambda/d\tau = -\tilde{H}_x[\tau]^* = -T \cdot H_x[\tau]^*\,, \quad d\lambda_{n+1}/d\tau = -\tilde{H}_T[\tau] = -H[\tau]\,, \tag{3.13}$$

$$(-\lambda(0), \lambda(1)) = \nabla_{(x(0),x(1))}(g + \rho^* \varphi)(x_0(0), x_0(1))\,, \tag{3.14}$$

$$\lambda_{n+1}(0) = \lambda_{n+1}(1) = 0\,, \tag{3.15}$$

$$\tilde{H}_u[\tau] = 0, \quad \tilde{H}_{uu}[\tau] \geq 0 \ \text{(positive semi–definite)}, \tag{3.16}$$

$$\tilde{\mu}(\tau) \geq 0 \quad \text{and} \quad \tilde{\mu}(\tau)^* C[\tau] = 0, \tag{3.17}$$

$$\tilde{H}[\tau] \equiv \text{const.} \quad \forall\, \tau \in [0,1]. \tag{3.18}$$

Relations (3.13), (3.15) and (3.18) immediately yield

$$0 = \tilde{H}[1] = T \cdot H[1]\,, \tag{3.19}$$

which proves the transversality condition (3.1). To check the Legendre–Clebsch condition in assumption (A2) one has to observe the scaling

$$\tilde{H}_{uu}[\tau] = T \cdot H_{uu}[\tau]\,. \tag{3.20}$$

In order to apply the SSC in Theorem 2.1 for problem $(\widetilde{\mathrm{CP}})$, we have to evaluate all terms in relations (2.11)–(2.32) for the tilde quantities defined in (3.8)–(3.10). Recalling the scaled multiplier $\mu_0 = \tilde{\mu}_0/T$ in (3.12) we obtain the transformed quantities

$$\tilde{f}_{\tilde{x}} = \begin{pmatrix} T \cdot f_x & f \\ 0 & 0 \end{pmatrix}, \quad \tilde{f}_u = \begin{pmatrix} T \cdot f_u \\ 0 \end{pmatrix}, \quad \tilde{C}_{\tilde{x}} = (C_x, 0)\,, \quad \tilde{C}_u = C_u\,, \tag{3.21}$$

$$\tilde{H}_{\tilde{x}\tilde{x}} = \begin{pmatrix} T \cdot H_{xx} & (H_x^0)^* \\ H_x^0 & 0 \end{pmatrix}, \quad \tilde{H}_{xu} = \begin{pmatrix} T \cdot H_{xu} \\ H_u^0 \end{pmatrix}, \quad \tilde{H}_{uu} = T \cdot H_{uu}\,. \tag{3.22}$$

Observe that the last relations make use of the identity $\tilde{H}_T = H^0$ which can be seen from (3.11) with H^0 denoting the unconstrained Hamiltonian. The preceding considerations lead to the following SSC for problem $(\widetilde{\mathrm{CP}})$ with free final time.

Theorem (3.1) (SSC for control problems with free final time.)
Let (x_0, T_0, u_0) be admissible for problem $(\widetilde{\mathrm{CP}})$. Suppose that there exist multipliers $(\lambda, \mu, \rho) \in W^{1,\infty}(0,1;\mathbb{R}^n) \times L^\infty(0,1;\mathbb{R}^k) \times \mathbb{R}^r$ such that the following conditions hold:

(1) the necessary conditions (3.13)–(3.19) are satisfied;

(2) assumptions (A1) and (A2) hold with respect to the time interval $[0,1]$;

(3) there exist $\delta > 0$ and $c > 0$ such that the quadratic form (2.15) expressed in terms of the transformed functions (3.21), (3.22) can be estimated from below as

$$J^2(\tilde{y}, \tilde{v}) \geq c\,[\,||\tilde{y}||_{1,2}^2 + ||\tilde{v}||_2^2\,]$$

for all variations $(\tilde{y}, \tilde{v}) \in W^{1,2}(0,1;\mathbb{R}^{n+1}) \times L^2(0,1;\mathbb{R}^m)$ which satisfy the variational system (2.11)–(2.13) on the time interval $[0,1]$.

(4) if u_0 is continuous then one may choose $\beta = 0$ and $\delta = 0$ in assumptions (A1) and (A2) and in condition (3).

Then for all constants $0 < c_0 < c$ there exists $\alpha > 0$ such that

$$F(x,T,u) \geq F(x_0,T_0,u_0) + c_0\{\,||x - x_0||_{1,2}^2 + |T - T_0|^2 + ||u - u_0||_2^2\,\}$$

holds for all admissible (x,T,u) with $||x - x_0||_{1,\infty} + |T - T_0| + ||u - u_0||_\infty \leq \alpha\,.$
In particular, (x_0, T_0, u_0) provides a strict weak local minimum for problem $(\widetilde{\mathrm{CP}})$.

It is apparent that these conditions are not very handy in practical applications. Again, we may resort to the Riccati equations and boundary conditions developed in (2.21)–(2.32). Now we consider the augmented $(n+1, n+1)$–matrix

$$\tilde{Q} = \begin{pmatrix} Q & R \\ R^* & q_T \end{pmatrix}, \tag{3.23}$$

where Q is a symmetric $n \times n$ matrix, R is an $n-$vector and q_T is a scalar. Inserting the transformed quantities (3.21) and (3.22) into the Riccati equation (2.21) for the matrix

$\tilde{Q}$, we obtain a Riccati equation for Q, a linear equation for R and a direct integration for q_T on the interval $[0,1]$; the argument τ is omitted for simplicity:

$$dQ/d\tau = T\cdot[\,-Qf_x - f_x^*Q - H_{xx} + (H_{xu}+Qf_u)(H_{uu})^{-1}(H_{xu}+Qf_u)^*\,], \quad (3.24)$$
$$dR/d\tau = -Qf - Tf_x^*R - (H_x^0)^* + (H_{xu}+Qf_u)(H_{uu})^{-1}(H_u^0+Tf_u^*R)^*, \quad (3.25)$$
$$dq_T/d\tau = -2R^*f + \frac{1}{T}\cdot(H_u^0+Tf_u^*R)(H_{uu})^{-1}(H_u^0+Tf_u^*R)^*\,. \quad (3.26)$$

Clearly, the Riccati equation (3.24) evaluated on the interval $[0,1]$ agrees with the Riccati equation (2.21) on the interval $[0,T]$. Note again that $H_u^0 \equiv 0$ holds on totally interior arcs with $C[t] < 0$. We wish to draw attention to the fact that equations (3.25) and (3.26) are *not* identical to corresponding equations in Bryson, Ho, Ref. 1, sections 6.6, 6.7, and Chamberland, Zeidan, Ref. 4, formulae (28)–(30), or Hull, Ref. 12.

The *modified* Riccati equation (2.26) can be worked out on the time interval $[0,1]$ in a similar way using the transformed quantities (3.21) and (3.22). However, since this procedure is quite cumbersome we restrict the discussion to the special case $m = j_\delta(t)$ which was considered already in (2.27)–(2.29). Upon computing the matrices in (2.27) and (2.28),

$$A^\delta = f_x - f_u(C_u^\delta)^{-1}C_x^\delta\,,$$
$$\mathcal{H}_{xx}^\delta = H_{xx} - H_{xu}(C_u^\delta)^{-1}C_x^\delta + [(C_u^\delta)^{-1}C_x^\delta]^*[H_{uu}(C_u^\delta)^{-1}C_x^\delta - H_{ux}]\,,$$

for the tilde quantities (3.21), (3.22), we recognize that the Riccati equation (2.29) splits into the following three equations:

$$dQ/d\tau = T\cdot[\,-QA^\delta - (A^\delta)^*Q - \mathcal{H}_{xx}^\delta\,], \quad (3.27)$$
$$dR/d\tau = -Qf - T\cdot(A^\delta)^*R - (H_x^0)^* - [H_u^0(C_u^\delta)^{-1}C_x^\delta]^*, \quad (3.28)$$
$$dq_T/d\tau = -2R^*f\,. \quad (3.29)$$

These formulas simplify considerably if $C_x \equiv 0$ holds, i.e., if the constraint is a *pure control* constraint. Then we get $A^\delta = f_x$ and the last equations yield

$$dQ/d\tau = T\cdot[\,-Qf_x - f_x^*Q - H_{xx}\,], \quad dR/d\tau = -Qf - T\cdot f_x^*R - (H_x^0)^*\,,$$
$$dq_T/d\tau = -2R^*f\,. \quad (3.30)$$

These equations will provide a convenient test for SSC when applied to the Rayleigh problem in section 5.

It is rather tedious to write out the boundary conditions (2.19) and (2.20) in the general case. We shall only discuss the important case that the initial and final state are fixed, i.e., $x(0) = x_0$ and $x(1) = x_1$ hold with prescribed $x_0, x_1 \in \mathbb{R}^n$. In this situation, the positive definiteness condition (2.32) evaluated for the augmented matrix $\tilde{Q}$ reduces to the following boundary conditions:

$$q_T(0) > 0 \quad \text{and} \quad q_T(1) < 0 \, . \tag{3.31}$$

These conditions constitute extra conditions for the free final time case and will turn out to be crucial for the numerical examples discussed in the next two sections. Note that we may relax one of these conditions, e.g., the initial condition, to $q_T(0) \geq 0$ by virtue of the continuous dependence of ODE solutions on initial data.

4. Planar Earth–Mars transfer with minimal flight time

Rocket flights in an inverse square law field have been studied extensively in the literature; see, e.g., Refs. 14–17, 27, 30. We consider the classical Earth–Mars orbit transfer with minimal tranfer time. The state variables are r: distance of the vehicle to the sun, w: radial component of the velocity, v: horizontal component of the velocity, m: mass of the vehicle. The control variable is ψ: angle of the thrust vector with respect to local horizon. The thrust is always at its maximal value β_{max} since the final time is minimized. All variables are scaled according to the dynamic model treated in Oberle, Taubert, Ref. 30.

The optimal control problem is to minimize the final time

$$F(x,T,u) = T = \int_0^T 1 \, dt \tag{4.1}$$

subject to the equations of motion

$$\begin{aligned} dr/dt &= w, \\ dw/dt &= \frac{v^2}{r} - \frac{1}{r^2} + \beta_{\max} \frac{c}{m} \sin\psi, \\ dv/dt &= -\frac{w\,v}{r} + \beta_{\max} \frac{c}{m} \cos\psi, \\ dm/dt &= -\beta_{\max}\,, \end{aligned} \tag{4.2}$$

and the boundary conditions for the initial and final state

$$\begin{array}{llll} r(0) = 1.0, & w(0) = 0.0, & v(0) = 1.0, & m(0) = 1.0, \\ r(T) = 1.525, & w(T) = 0.0, & v(T) = 1.0/\sqrt{r(T)}. & \end{array} \tag{4.3}$$

The constants are given by $c = 1.872$ and $\beta_{max} = 0.075$. The underlying physical data of the vehicle are given as follows. Initial mass: $m_0 = 679.78$ kg, (maximal) thrust $= 0.56493$ N, (constant) equivalent exit velocity: $v_c = 55809$ m/s.

4.1. The boundary value problem

Since there is no control constraint in this problem, the Hamiltonian (3.12) is given by

$$\begin{aligned} \tilde{H} \; &= \; T \cdot [\; 1 \,+\, \lambda_r\, w \,+\, \lambda_w \left(\frac{v^2}{r} \,-\, \frac{1}{r^2} \,+\, \beta_{\max} \frac{c}{m} \sin\psi \right) \\ &\quad + \lambda_v \left(- \frac{w\,v}{r} \,+\, \beta_{\max} \frac{c}{m} \cos\psi \right) \,-\, \lambda_m\, \beta_{max} \;] \end{aligned} \tag{4.4}$$

Let us evaluate the first order optimality conditions (3.13)–(3.16) and omit for convenience the subscript zero referring to the optimal solution. The *optimal control* ψ is derived from conditions (3.16) as

$$\sin\psi \;=\; -\frac{\lambda_w}{\sqrt{\lambda_w^2 \,+\, \lambda_v^2}}\;, \qquad \cos\psi \;=\; -\frac{\lambda_v}{\sqrt{\lambda_w^2 \,+\, \lambda_v^2}}\,. \tag{4.5}$$

The *adjoint equations* on the normalized time interval $[0, 1]$ are given by

$$\begin{aligned} d\lambda_r/d\tau \;&=\; T \cdot [\, \lambda_w \left(\frac{v^2}{r^2} \,-\, \frac{2}{r^3} \right) \,-\, \lambda_v \frac{wv}{r^2} \,]\,, \\ d\lambda_w/d\tau \;&=\; T \cdot [-\, \lambda_r \,+\, \lambda_v\, \frac{v}{r}\,]\,, \\ d\lambda_v/d\tau \;&=\; T \cdot [-\, \lambda_w\, \frac{2v}{r} \,+\, \lambda_v\, \frac{w}{r}\,]\,, \\ d\lambda_m/d\tau \;&=\; T \cdot [-\, \frac{\beta_{\max}\, c}{m^2} \sqrt{\lambda_w^2 \,+\, \lambda_v^2}\,]\,. \end{aligned} \tag{4.6}$$

The transversality conditions (3.14) and (3.19) yield

$$\begin{aligned} \lambda_m(1) \;&=\; 0, \\ H[1] \;&=\; 1 \,-\, \frac{\beta_{\max}\, c}{m(1)} \sqrt{\lambda_w(1)^2 \,+\, \lambda_v(1)^2} \;=\; 0\,. \end{aligned} \tag{4.7}$$

After transforming the state equations (4.2) to the normalized time interval $[0, 1]$ according to (3.3) we obtain a two-point boundary value problem (BVP) consisting of equations (4.2)–(4.8). This BVP can be further simplified by eliminating the variables $m(\tau)$ and $\lambda_m(\tau)$. The variable $m(\tau)$ is substituted according to

$$m(\tau) \;=\; 1.0 \,-\, \beta_{\max}\, T \cdot \tau \tag{4.8}$$

and the variable λ_m can be dropped since it that does not enter into the first three equations in (4.6). The reduced BVP then comprises the six ODEs with respect to the variables r, w, v, λ_r, λ_w, and λ_v on the fixed time interval $[0,1]$ and the trivial equation $dT/d\tau \equiv 0$ in view of (3.3). The corresponding boundary conditions are the six boundary conditions (4.3) with respect to r, w, and v and the Hamiltonian boundary condition in (4.7).

Once a solution of this BVP has been determined, the adjoint variable λ_m can be obtained through an integration of the last equation in (4.6),

$$\lambda_m(\tau) \;=\; T \cdot \int_\tau^1 \frac{\beta_{\max}\, c}{m(\tau)^2}\, \sqrt{\lambda_w(\tau)^2 \,+\, \lambda_v(\tau)^2}\, d\tau\,.$$

Alternatively, λ_m can be eliminated from the condition $H \equiv 0$.

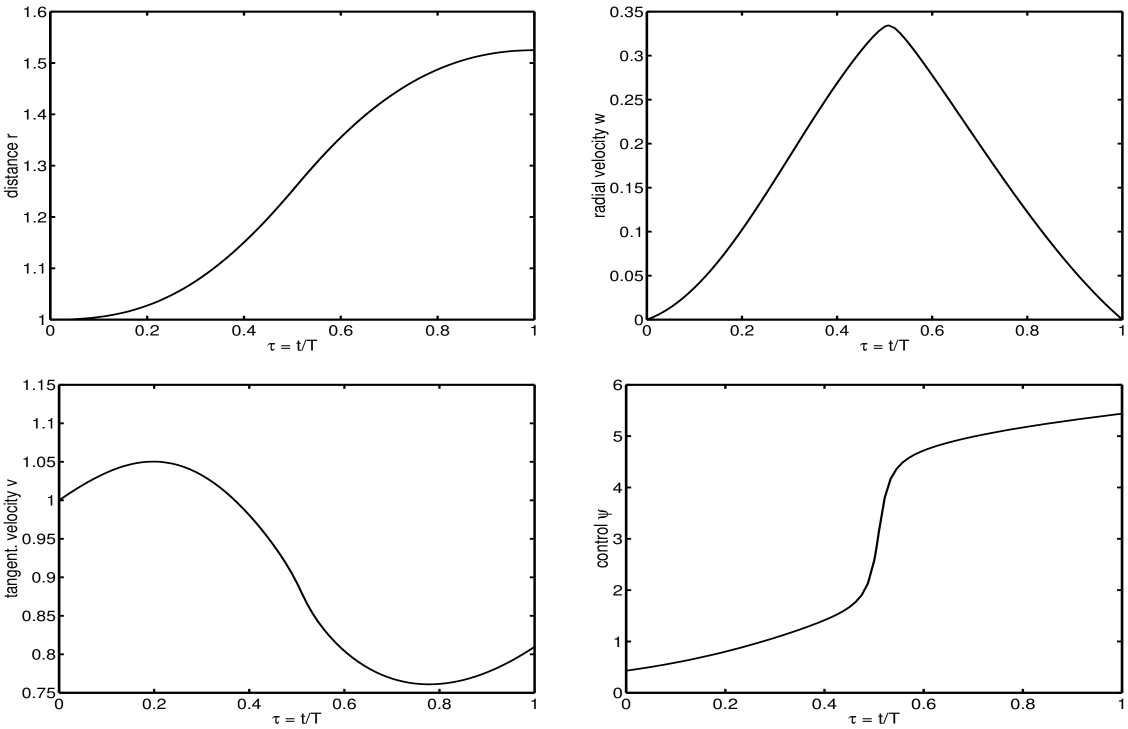


Figure 1: Earth-Mars Transfer: State variables r, w, v and control variable ψ .

The code BNDSCO in Oberle and Grimm, Ref. 29, provides the following initial and final values for the adjoint variables and the final time:

$$\begin{aligned}
\lambda_r(0) &= -0.52729\,67236 \times 10^1, & \lambda_r(1) &= -0.37511\,95452 \times 10^1,\\
\lambda_w(0) &= -0.26088\,76037 \times 10^1, & \lambda_w(1) &= 0.40004\,02548 \times 10^1,\\
\lambda_v(0) &= -0.56884\,53434 \times 10^1, & \lambda_v(1) &= -0.35509\,23985 \times 10^1,\\
T &= 0.33199\,21219 \times 10^1. & &
\end{aligned} \tag{4.9}$$

Figure 1 displays the corresponding state variables r, w, v and control variable ψ while Figure 2 shows the adjoint variables.

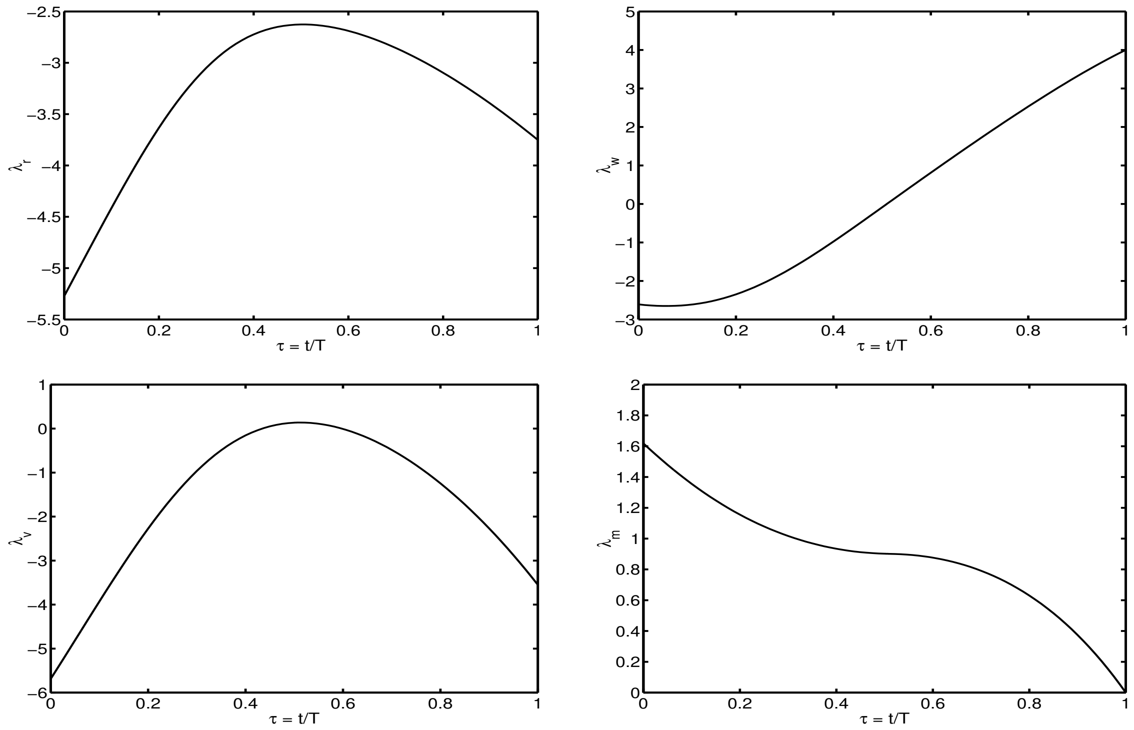


Figure 2: Earth-Mars Transfer: Adjoint variables λ_r, λ_w, λ_v, λ_m .

4.2. Second order sufficient conditions

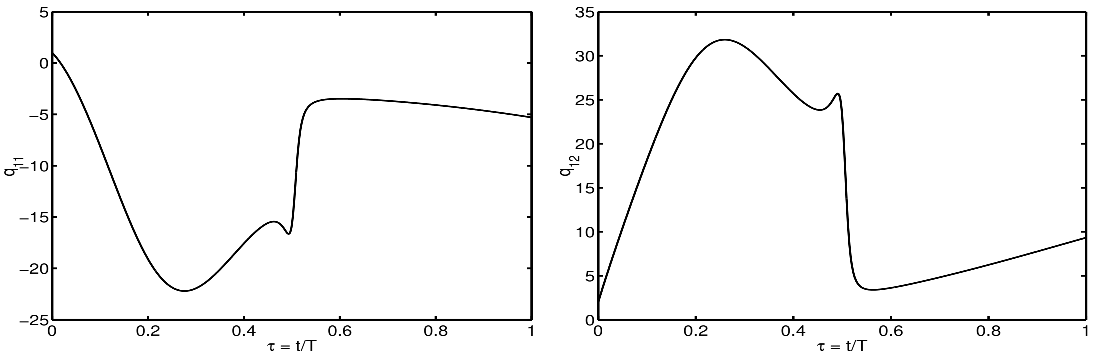
Let us first check the strict Legendre–Clebsch condition in assumption (A2) taking into account the scaling $\tilde{H}_{\psi\,\psi}[\tau] = T \cdot H_{\psi\,\psi}[\tau]$ in (3.20). We obtain

$$H_{\psi\,\psi} \;=\; \frac{\beta_{\max}\, c}{m}\sqrt{\lambda_w^2 \;+\; \lambda_v^2}$$

and find

$$\min\,\{\tilde{H}_{\psi\,\psi}[\tau] \mid \tau \in [\,0,1]\,\} \;=\; \tilde{H}_{\psi\,\psi}[\tau_0] \;= 0.07390\,1871 \;> 0, \quad \tau_0 = 0.50935\,25818\,,$$

which verifies the strict Legendre-Clebsch condition (A2). To prove the SSC in Theorem 3.1 it remains to show that the Riccati equations (3.24)–(3.26) possesses a bounded solution such that the sign conditions (3.31) hold.



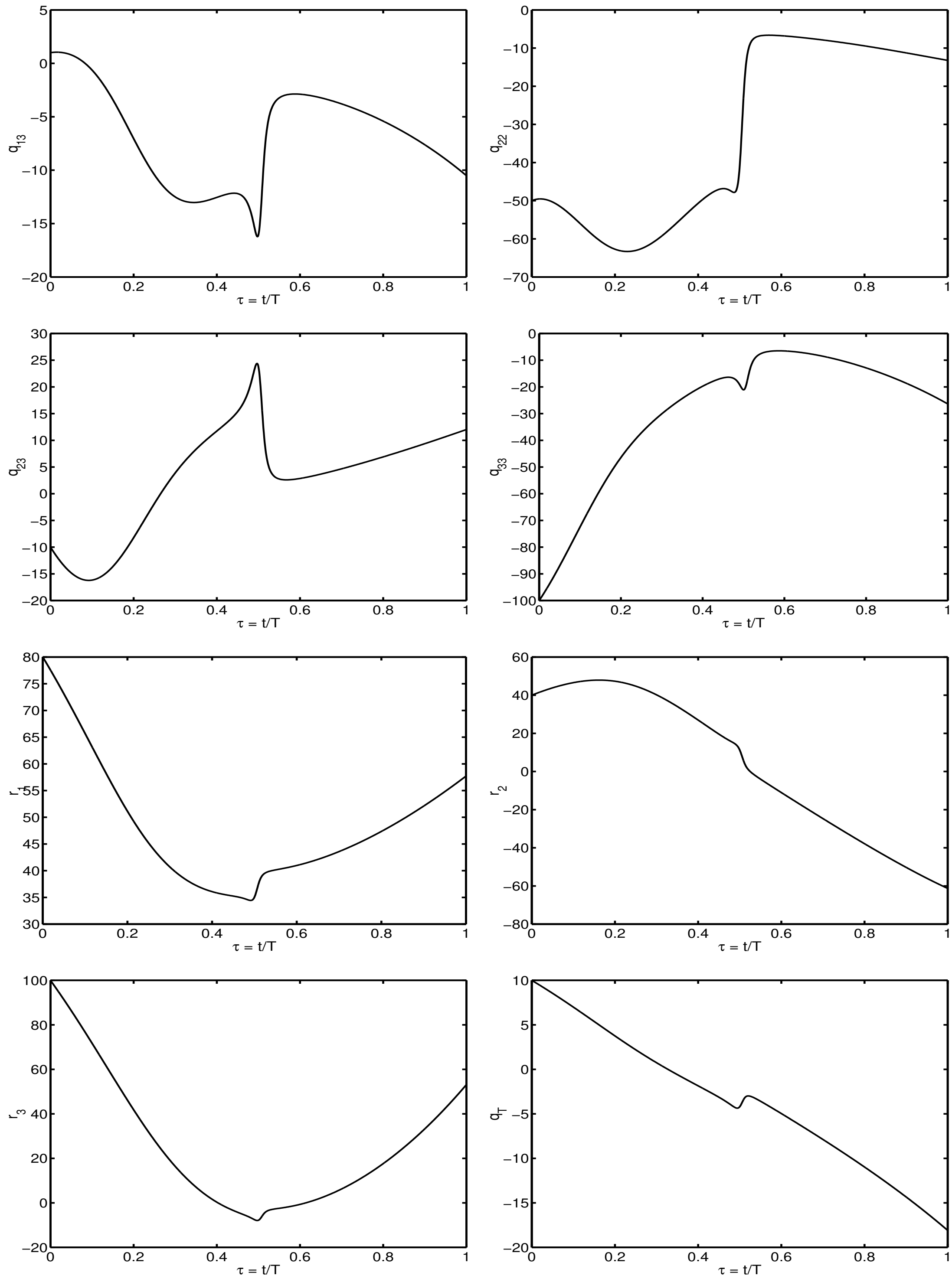


Figure 3: Solutions of the Riccati equations (3:24)–(3.26)

The reader is reminded that we have eliminated the state variable m so that the remaining state variables r, w, v have fixed final values. The symmetric Riccati matrix $\tilde{Q}$ in (3.23) is given in the form

$$\tilde{Q} = \begin{pmatrix} Q & R \\ R^* & q \end{pmatrix} = \left(\begin{array}{ccc|c} q_{11} & q_{12} & q_{13} & r_1 \\ q_{12} & q_{22} & q_{23} & r_2 \\ q_{13} & q_{23} & q_{33} & r_3 \\ \hline r_1 & r_2 & r_3 & q_T \end{array}\right)$$

We refrain from writing down the Riccati equations (3.24)–(3.26) explicitly. The evaluation is rather tedious but can be simplified with the help of symbolic computations offered for example in the package MAPLE. It should be noted that the coefficients of the Riccati equation are functions of the nominal trajectory characterized by (4.9). We merely indicate how to find appropriate initial values for $\tilde{Q}(0)$ such that the sign conditions (3.31) hold:

$$q_T(0) > 0, \quad q_T(1) < 0.$$

We succeeded using a rather heuristic optimization technique. Starting with initial estimates for $Q(0)$ which allowed integration of (3.24)–(3.26) on $[0,1]$, we changed iteratively one component of $Q(0)$ in order to minimize $q_T(1)$. Changing the indices of components in a cyclic way, we got the following initial values:

$$\begin{array}{rclrclrcl} q_{11}(0) & = & 1.0, & q_{12}(0) & = & 2.0, & q_{13}(0) & = & 1.0 \\ q_{22}(0) & = & -50.0, & q_{23}(0) & = & -10.0, & q_{33}(0) & = & -100.0 \\ r_1(0) & = & 80.0, & r_2(0) & = & 40.0, & r_3(0) & = & 100.0, \\ q_T(0) & = & 10.0 > 0. & & & & & & \end{array}$$

For these data, a solution of the Riccati equations was found to exist on the whole interval $[0,1]$ with final value $q_T(1) = -18.090\,44002 < 0$. Hence, all assumptions for Theorem 3.1 are verified and we draw the conclusion that the trajectory characterized by (4.9) provides a weak local minimum for problem (4.1)–(4.3). The component functions of $\tilde{Q}[\tau]$ are shown in Figure 3.

5. Control of current in a tunneldiode oscillator

The following Rayleigh problem has been treated in Refs. 13, 36 as a *fixed* final time control problem. SSC and sensitivity analysis for this model have been discussed in Maurer, Augustin, Ref. 23. In this section, we investigate a slightly modified problem with *free* final time. Figure 5.1 displays an electric circuit (tunneldiode oscillator) where L denotes inductivity, C capacity, R resistance, I electric current and where D is a diode. The state variables are the electric current $x_1(t) = I(t)$ at time $t \in [\,0, T\,]$ and $x_2(t) := \dot{x}_1(t)$. The control $u(t)$ is a suitable transformation of the voltage V_0 at the generator.

With an additional parameter $c \geq 0$ the Rayleigh problem with *free* final time is defined as follows: minimize the functional

$$F_c(x,T,u) = c \cdot T + \int_0^T (\, u(t)^2 + x_1(t)^2 \,)\, dt \;=\; \int_0^T (\, c + u(t)^2 + x_1(t)^2 \,)\, dt \qquad (5.1)$$

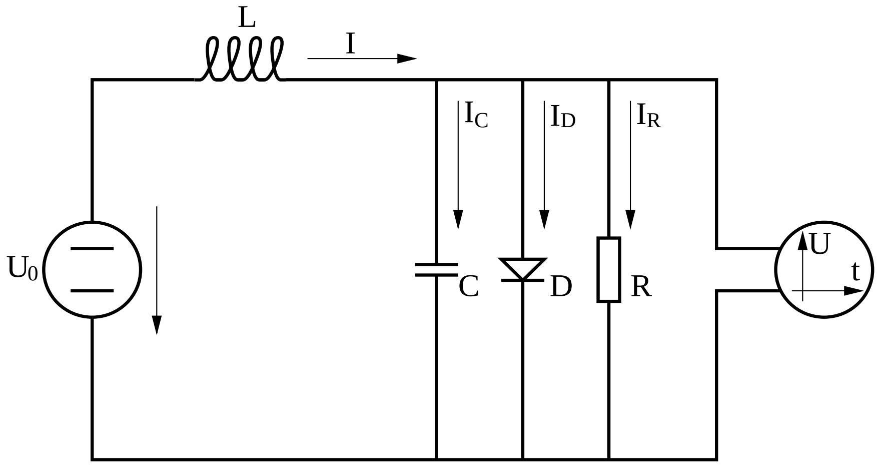


Figure 4: Tunneldiode oscillator, $x_1(t) = I(t)$.

subject to

$$\dot{x}_1(t) = x_2(t)\,,\ \dot{x}_2(t) = -x_1(t) + x_2(t)\,(\,1.4 - 0.14\,x_2(t)^2\,) + 4\,u(t)\,, \tag{5.2}$$

$$x_1(0) = x_2(0) = -5\,, \quad x_1(T) = x_2(T) = 0\,, \tag{5.3}$$

$$|\,u(t)\,| \le 1 \quad \text{for} \ \ t \in [0, T\,]\,. \tag{5.4}$$

The solution of this problem with final time $T = 4.5$ specified and $c = 0$ may be found in Ref. 23.. In the following, we denote by $F_c(T)$ the optimal value of the control problem (5.1)–(5.4) for *fixed* final time T. The behavior of this function gives insight into the behavior of optimal solutions for *free* final time.

5.1. Unconstrained optimal solutions

We consider the unconstrained problem with control constraint (5.4) deleted. After applying the time transformation (3.2) the unconstrained Hamiltonian in (3.12) becomes

$$\tilde{H}^0(\tilde{x}, u, \tilde{\lambda}) = T \cdot [\, c + u^2 + x_1^2 + \lambda_1 x_2 + \lambda_2\,(-x_1 + x_2\,(1.4 - 0.14x_2^2) + 4u)\,] \tag{5.5}$$

Henceforth, we omit the lower index zero to denote the optimal solution. The control is computed from the equation $\tilde{H}^0_u[\tau] = 0$ which yields

$$u(\tau) = -2\lambda_2(\tau)\,.$$

The transformed state equations (3.3) and adjoint equations (3.13) lead to the following ODEs in $[0, 1]$:

$$\begin{aligned} dx_1/d\tau &= T \cdot x_2, \\ dx_2/d\tau &= T \cdot (-x_1 + 1.4x_2 - 0.14x_2^3 - 8\lambda_2), \\ d\lambda_1/d\tau &= T \cdot (-2x_1 + \lambda_2), \\ d\lambda_2/d\tau &= T \cdot (-\lambda_1 - 1.4\lambda_2 + 0.42x_2^2\lambda_2). \end{aligned} \tag{5.6}$$

To compute the optimal solution of the control problem with *fixed* final time we resolve the BVP which comprises (5.6) and the boundary conditions

$$x_1(0) = x_2(0) = -5, \quad x_1(1) = x_2(1) = 0\,. \tag{5.7}$$

The BVP is solvable only for final times $T > T^* > 0$, where T^* is a suitable final time. The optimal value function $F_c(T)$ is shown in Figure 5.2 for $T > T^* = 2$ and two different values $c = 1/16$ and $c = 0$. Observe that for $c = 1/16$ the graph of $F_c(T)$ in Figure 5.2 depicts two local minima and one local maximum with respect to the final time T whereas no clear minimum can be discerned for $c = 0$.

For *free* final time T, the transversality condition (3.19) and the boundary conditions (5.7) yield $0 = \tilde{H}^0[1] = T \cdot (c + u(1)^2 + 4\lambda_2(1)u(1)) = T \cdot (c - 4\lambda_2(1)^2)$ from which we get the boundary condition

$$\lambda_2(1)^2 = c/4\,. \tag{5.8}$$

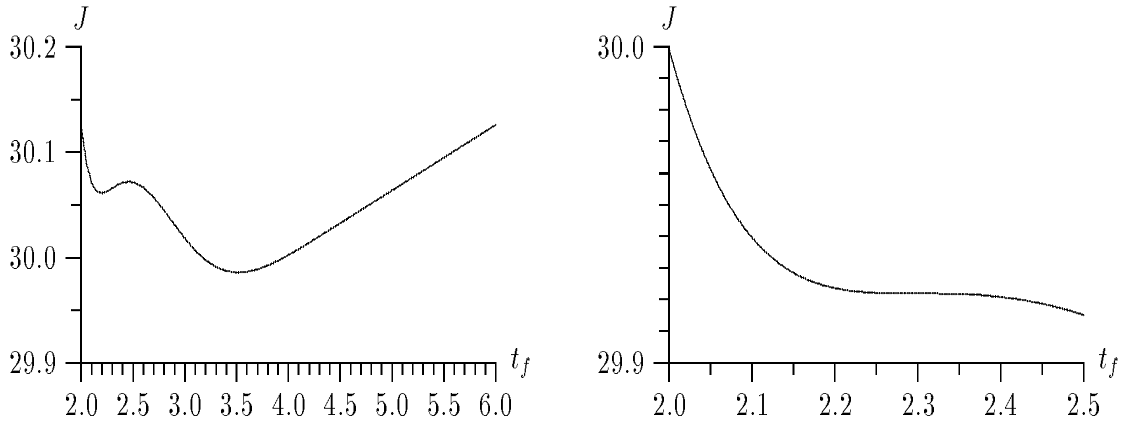


Figure 5: Graph of $F_c(T)$ for $c = 1/16$ and $c = 0$

Now we solve the BVP (5.6)–(5.8) for the cases $c = 1/16$ and $c = 0$ using again the code BNDSCO in Ref. 29.

Case $c = 1/16$: We find three solutions:

Solution 1:
$$\begin{aligned} T &= 2.19460\,79912, & F_c(T) &= 30.06097\,62322,\\ \lambda_1(0) &= -9.01234\,54748, & \lambda_1(1) &= 0.97693\,36044,\\ \lambda_2(0) &= -2.67606\,29500, & \lambda_2(1) &= 0.125. \end{aligned}$$

Solution 2:
$$\begin{aligned} T &= 2.46029\,38602, & F_c(T) &= 30.07173\,02593,\\ \lambda_1(0) &= -9.01228\,20002, & \lambda_1(1) &= 0.95904\,74639,\\ \lambda_2(0) &= -2.67605\,43511, & \lambda_2(1) &= -0.125. \end{aligned}$$

Solution 3:
$$\begin{aligned} T &= 3.51535\,36980, & F_c(T) &= 29.98534\,49252,\\ \lambda_1(0) &= -9.01085\,93855, & \lambda_1(1) &= 0.15146\,20116,\\ \lambda_2(0) &= -2.67586\,16249, & \lambda_2(1) &= -0.125. \end{aligned}$$

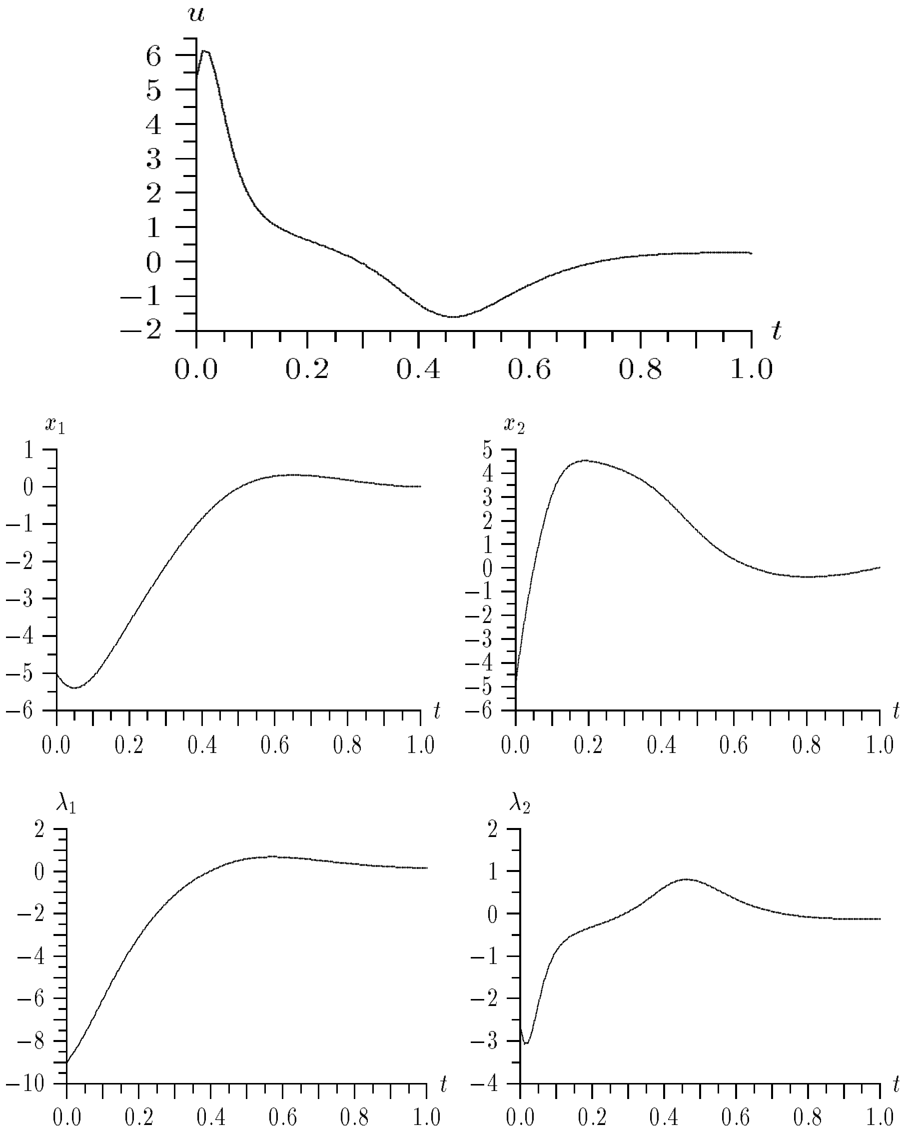


Figure 6: Optimal control, state and adjoint variables for $c = 1/16$ and $T = 3.5153536980$.

It is obvious that these three solutions correspond to the two local minima and one local maximum shown in Figure 5. Note that $\lambda_2(1)$ changes sign when passing from solution 1 to solutions 2 and 3.

Now let us show that *solution 3* provides indeed a local minimum. The respective optimal control, state and the adjoint variables are displayed in Figure 6. The Legendre–Clebsch condition (A2) trivially holds in view of $\tilde{H}^0_{uu}[\tau] \equiv 2 \cdot T > 0$. Next, we verify that the Riccati equations (3.24)–(3.26) have a bounded solution such that the boundary conditions (3.31) hold in relaxed form $q_T(0) \geq 0$ and $q_T(1) < 0$. The matrix (3.23) becomes

$$\tilde{Q} = \begin{pmatrix} Q & R \\ R^* & q \end{pmatrix} =: \left(\begin{array}{cc|c} q_1 & q_2 & r_1 \\ q_2 & q_4 & r_2 \\ \hline r_1 & r_2 & q_T \end{array}\right), \tag{5.9}$$

for which we evaluate the Riccati equations (3.24)–(3.26) as

$$
\begin{aligned}
dq_1/d\tau &= T\cdot[\,2q_2 - 2 + 8q_2^2\,], \\
dq_2/d\tau &= T\cdot[\,-q_1 - (1.4 - 0.42x_2^2)q_2 + q_4 + 8q_2q_4\,], \\
dq_4/d\tau &= T\cdot[\,-2(q_2 + (1.4 - 0.42x_2^2)q_4) + 0.84x_2\lambda_2 + 8q_4^2\,], \\
dr_1/d\tau &= -q_1x_2 - q_2(-x_1 + 1.4x_2 - 0.14x_2^3 - 8\lambda_2) + Tr_2 \\
&\quad -2x_1 + \lambda_2 + 8Tq_2r_2\,, \\
dr_2/d\tau &= -q_2x_2 - q_4(-x_1 + 1.4x_2 - 0.14x_2^3 - 8\lambda_2) - Tr_1 \\
&\quad -T\cdot(1.4 - 0.42x_2^2)r_2 - \lambda_1 - \lambda_2(1.4 - 0.42x_2^2) + 8Tq_4r_2\,, \\
dq_T/d\tau &= -2[\,r_1x_2 + r_2(-x_1 + 1.4x_2 - 0.14x_2^3 - 8\lambda_2)\,] + 8Tr_2^2\,.
\end{aligned}
\tag{5.10}
$$

It suffices to find a bounded solution of these Riccati equations satisfying $q_T(0) = 0$. After several trials we were successful with the initial values

$$
\begin{array}{lll}
q_1(0) = 2.00684\,76891, & q_2(0) = 0.47018\,97048, & q_4(0) = -0.35197\,44265, \\
r_1(0) = 0, & r_2(0) = 0, & q_T(0) = 0
\end{array}
$$

for which we get the final values

$$
\begin{array}{lll}
q_1(1) = 0 & q_2(1) = 0, & q_4(1) = 0, \\
r_1(1) = -0.125 = \lambda_2(1), & r_2(1) = 0.02353\,79884, & q_T(1) = -r_2(1) < 0.
\end{array}
$$

and the bound $||\widetilde{Q}(\tau)||_\infty \le 3$ for all $\tau \in [0,1]$.

Hence, Theorem 3.1 asserts that solution 3 is indeed a weak local minimum. We mention that the sufficient conditions in Hull, Ref. 12, can also be checked numerically for solution 3.

In a similar way we can test the optimality of *solution 1* with $T = 2.1946079912$. We obtain a bounded solution of the Riccati equation for initial values

$$
\begin{array}{lll}
q_1(0) = 1.59068\,73787, & q_2(0) = 0.33016\,84322, & q_4(0) = -0.39917\,54076, \\
r_1(0) = 0, & r_2(0) = 0, & q_T(0) = 0,
\end{array}
$$

and final values

$$
\begin{array}{lll}
q_1(1) = 0 & q_2(1) = 0, & q_4(1) = 0, \\
r_1(1) = 0.125 = \lambda_2(1), & r_2(1) = -1.15193\,36046, & q_T(1) = r_2(1) < 0.
\end{array}
$$

These values yield the bound $||\widetilde{Q}(\tau)||_\infty \le 5$ for all $\tau \in [0,1]$.

The situation is different for *solution 2* which provides a *local maximum* with respect to the final time T. All initial values $q_T(0) \ge 0$ that we tested produce a solution of the Riccati equation with $q_T(1) \ge 0$. Though this test does not exclude optimality of the solution, it is a rather strong indication for non–optimality. Thus we may draw the conclusion that solution 2 behaves like a *saddle point solution* which is a local minimum with respect to control for every fixed time but a local maximum with respect to final time.

Case $c = 0$ **:** Here the situation is more complicated since Figure 5.2 does not indicate a distinctive local minimum. The code BNDSCO of Ref.29 provides e.g. the following two solutions:

$$\textbf{Solution 1:} \quad \begin{aligned} T &= 2.29903\,95815, & F_c(T) &= 29.9218\,12616, \\ \lambda_1(0) &= -9.00409\,78999, & \lambda_1(1) &= 1.00813\,43679, \\ \lambda_2(0) &= -2.67325\,14651, & \lambda_2(1) &= 0. \end{aligned}$$

$$\textbf{Solution 2:} \quad \begin{aligned} T &= 4.50237\,87337, & F_c(T) &= 29.75107\,51464, \\ \lambda_1(0) &= -9.00247\,06599, & \lambda_1(1) &= -0.0044561\,06307, \\ \lambda_2(0) &= -2.67303\,08344, & \lambda_2(1) &= 0. \end{aligned}$$

Solving the Riccati equation (5.10), e.g., for the final time $T = 2.2990395815$ we find that the initial value $q_T(0) = 0$ produces the final value $q_T(1) = 0$ in all tested cases. Thus the sign conditions (3.29) can not be verified and the Riccati test is not able to detect whether solution 1 is a local minimum.

5.2. Constrained optimal solutions

Now we consider solutions satisfying in addition the control constraint (5.4),

$$-1 \le u(\tau) \le 1 \quad \text{for all} \quad \tau \in [0,1]\,.$$

The augmented Hamiltonain (3.12) becomes in view of (5.5):

$$\begin{aligned} \tilde{H}(\tilde{x}, u, \tilde{\lambda}, \tilde{\mu}) &= \tilde{H}^0(\tilde{x}, u, \tilde{\lambda}) + \tilde{\mu}_1(-u-1) + \tilde{\mu}_2(u-1) \\ &= T \cdot [\, c + u^2 + x_1^2 + \lambda_1 x_2 + \lambda_2\,(-x_1 + x_2\,(1.4 - 0.14x_2^2) + 4u \\ &\qquad + \mu_1(-u-1) + \mu_2(u-1)\,]\,, \end{aligned} \tag{5.11}$$

where $\mu_i := \tilde{\mu}_i/T$, $i = 1,2$, are the scaled multipliers. The state and adjoint equations agree with those given in (5.6). Again we get the control law $u(\tau) = -2\lambda_2(\tau)$ on interior arcs $|u(\tau)| < 1$. The *unconstrained* control $u(\tau)$ depicted in Figure 6 suggests that the *constrained* control has one boundary arc with $u(\tau) \equiv 1$ and one boundary arc with $u(\tau) \equiv -1$. Thus we may assume the following solution structure of the optimal control:

$$u(\tau) = \left\{ \begin{array}{rcl} 1 & , & 0 \le \tau \le \tau_1 \\ -2\lambda_2(\tau) & , & \tau_1 \le \tau \le \tau_2 \\ -1 & , & \tau_2 \le \tau \le \tau_3 \\ -2\lambda_2(\tau) & , & \tau_3 \le \tau \le 1 \end{array} \right\}, \tag{5.12}$$

The junction points τ_1, τ_2, τ_3 are implicitly determined through the conditions that the control is *continuous* at these points. This leads to the junction conditions

$$\lambda_2(\tau_1) = -0.5, \quad \lambda_2(\tau_2) = 0.5, \quad \lambda_2(\tau_3) = -0.5\,. \tag{5.13}$$

Hence, on the interval $[0,1]$ we have to solve the multipoint BVP which is composed by the state and adjoint equations (5.6) with control substituted from (5.12) as well as the boundary and junction conditions (5.7) and (5.13).

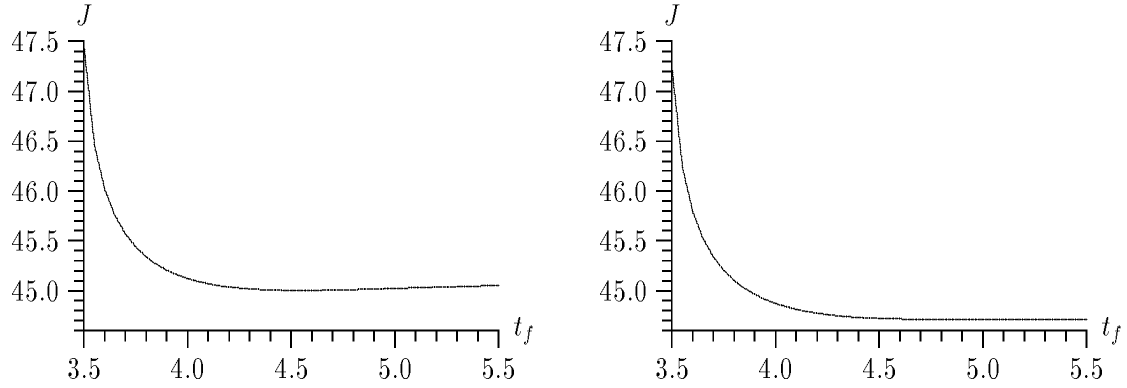


Figure 7: Graph of $F_c(T)$ for control constraint $|u| \le 1$: cases $c = 1/16$ and $c = 0$.

The optimal value function $\tilde{F}_c(T)$ for the constrained problem is depicted in Figure 7 for the values $c = 1/16$ and $c = 0$. A distinctive minimum can only be detected in case $c = 1/16$.

Case $c = 1/16$**:** Again we use the code BNDSCO in [29] and obtain:

$$\begin{array}{rclrcl} T &=& 4.54230\,98018, & \tau_1 &=& 0.22932\,06694, \\ \tau_2 &=& 0.37589\,55717, & \tau_3 &=& 0.63465\,63122, \\ \lambda_1(0) &=& -12.70813\,77440, & \lambda_1(1) &=& 0.02860\,41331, \\ \lambda_2(0) &=& -4.59503\,53190, & \lambda_2(1) &=& -0.125, \\ F_c(T) &=& 44.71797\,06589. & & & \end{array}$$

The corresponding optimal control, state and adjoint variables are shown in Figure 8. In order to check the sufficient conditions in Theorem 3.1 we try to find a *bounded* solution of the Riccati equations (5.10) on the *interior arcs* $[\tau_1, \tau_2]$ and $[\tau_3, 1]$ and the *modified* Riccati equations (3.30) on the *boundary arcs* $[0, \tau_1]$ and $[\tau_2, \tau_3]$. The modified Riccati equations yield the following *linear* equations:

$$\begin{array}{rcl} dq_1/d\tau &=& T \cdot 2(q_2 - 1)\,, \\ dq_2/d\tau &=& T \cdot [\, -q_1 - (1.4 - 0.42x_2^2)q_2 + q_4\,]\,, \\ dq_4/d\tau &=& T \cdot [\, -2(q_2 + (1.4 - 0.42x_2^2)q_4) + 0.84x_2\lambda_2\,]\,, \\ dr_1/d\tau &=& -q_1x_2 - q_2(-x_1 + 1.4x_2 - 0.14x_2^3 + 4u) + Tr_2 - 2x_1 + \lambda_2\,, \\ dr_2/d\tau &=& -q_2x_2 - q_4(-x_1 + 1.4x_2 - 0.14x_2^3 + 4u) - Tr_1 \\ && -T \cdot (1.4 - 0.42x_2^2)r_2 - \lambda_1 - \lambda_2(1.4 - 0.42x_2^2)\,, \\ dq_T/d\tau &=& -2\,[\, r_1x_2 + r_2(-x_1 + 1.4x_2 - 0.14x_2^3 + 4u)\,]\,. \end{array} \qquad (5.14)$$

We wish to find a solution satisfying the terminal values $q_1(1) = q_2(1) = q_4(1) = 0$. A bounded solution of the Riccati equations then is obtained for the initial values

$$r_1(0) = 0, \quad r_2(0) = 0, \quad q_T(0) = 0,$$

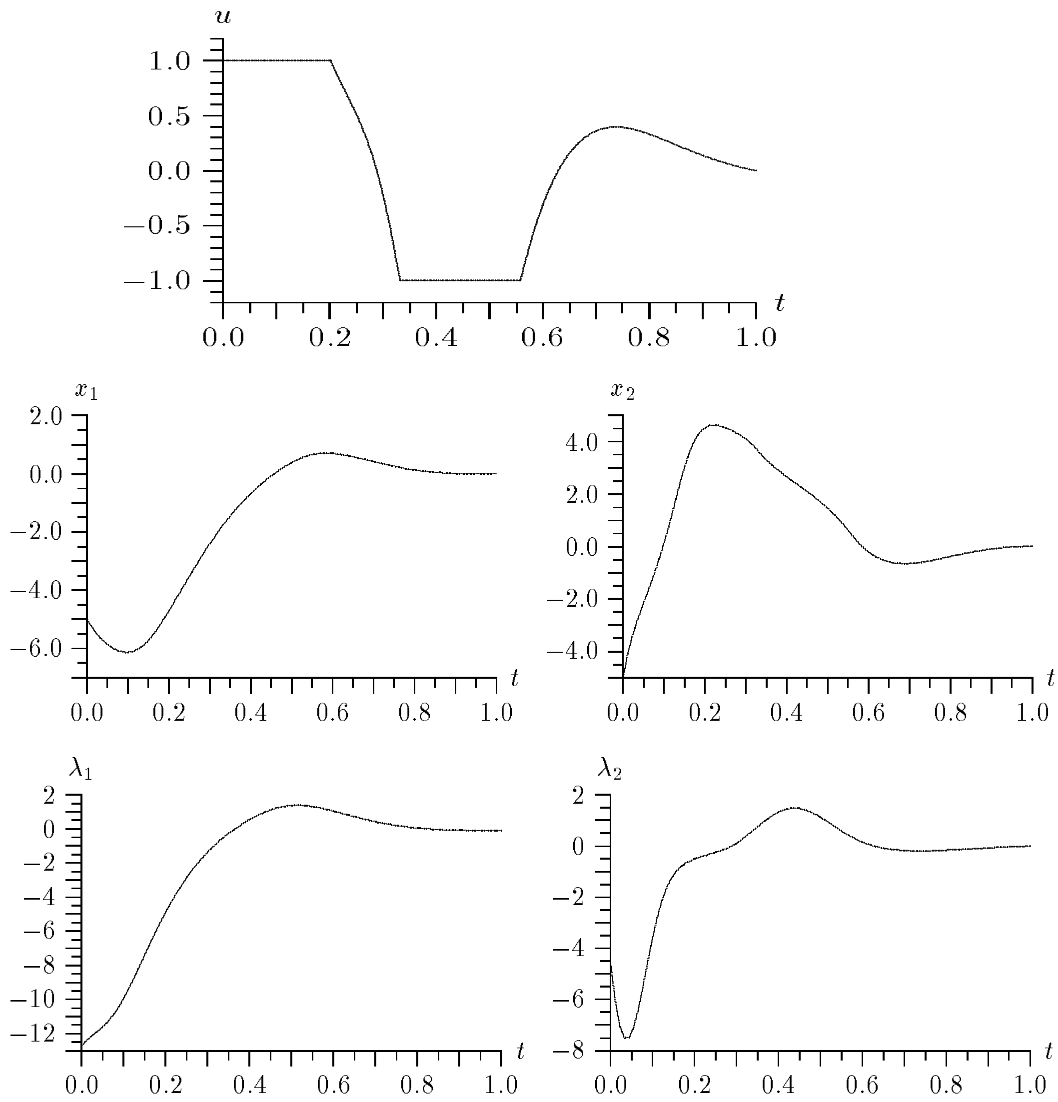


Figure 8: Optimal control, state and adjoint variables for $c = 1/16$ and $T = 4.5423098018$.

which produce the desired terminal values for q_1, q_2, q_4 and

$$r_1(1) = -0.125 = \lambda_2(1), \quad r_2(1) = 0.146395866, \quad q_T(1) = -r_2(1) < 0\,.$$

For these values we get the bound $||\widetilde{Q}(\tau)||_\infty \le 4 \quad \forall\ \tau \in [0,1]$. It is interesting to note that it was not possible to obtain a *bounded solution* of the Riccati equation (5.10) on the *whole interval.* Thus the Riccati test developed in section 3 in its weaker form considerably facilitates the numerical check of SSC.

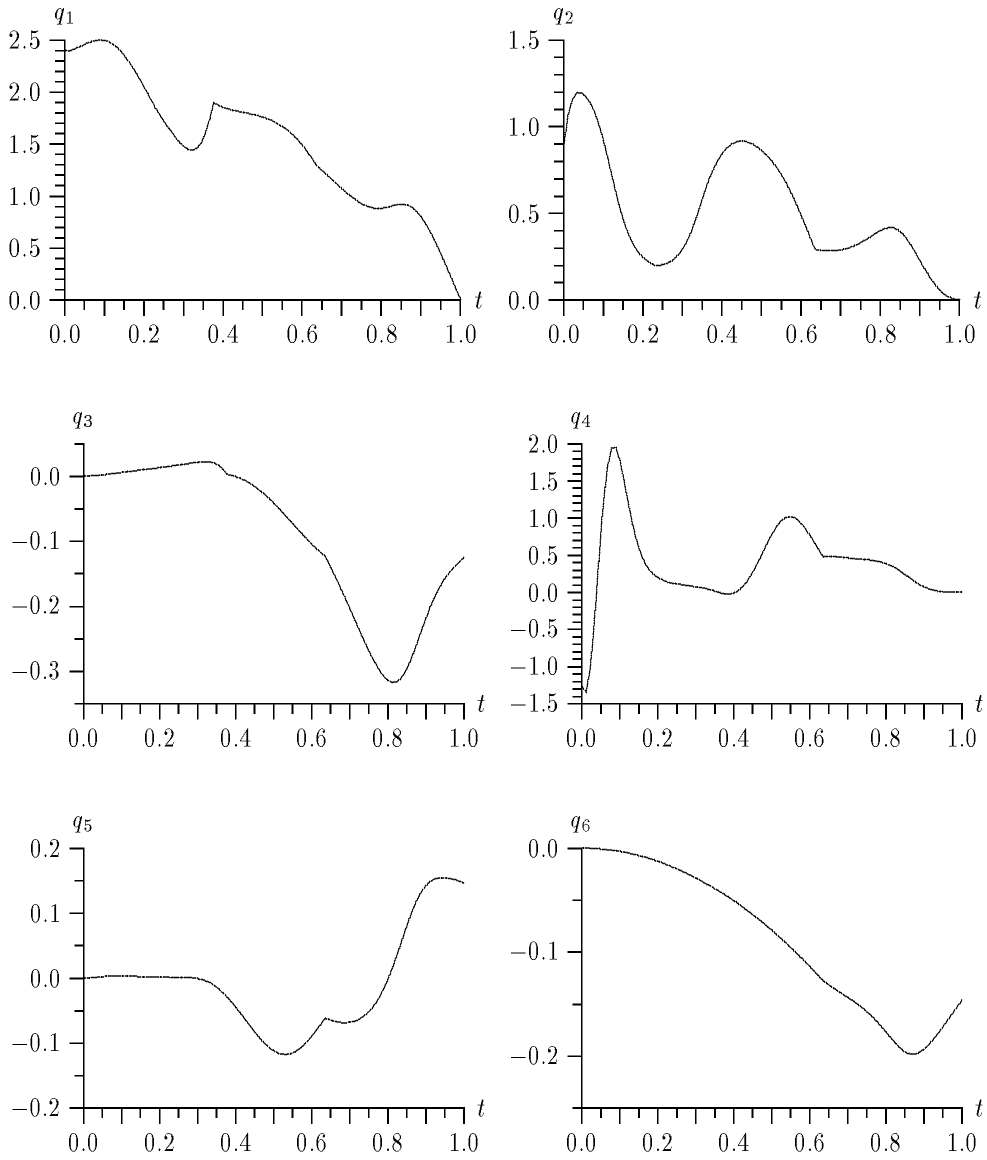


Figure 9: Solutions $q_1, q_2, q_4, r_1, r_2, q_T$ of Riccati equations (5.10) and (5.14) for $c = 1/16$ and $T = 4.5423098018$.

Case $c = 0$**:** The code BNDSCO yields the solution

$$\begin{array}{rclcrcl} T &=& 5.15173\,31990, && \tau_1 &=& 0.20192\,87957, \\ \tau_2 &=& 0.33186\,51046, && \tau_3 &=& 0.55797\,03824, \\ \lambda_1(0) &=& -12.70087\,48310, && \lambda_1(1) &=& 0.09649\,19382, \\ \lambda_2(0) &=& -4.58996\,79054, && \lambda_2(1) &=& 0, \\ \tilde{F}_{c=0}(T) &=& 44.70866\,79043 &&&& \end{array}$$

Evaluating the Riccati equations (5.10) and (5.14) along this specific solution, we were not able to find a bounded solution satisfying the sign conditions $q_T(0) \geq 0$ and $q_T(1) < 0$. This fact confirms our impression gained from Figure 7 that no local minimum can be identified for $c = 0$.

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