Spectrum of invariant differential operators for the supersymmetric pair $(\mathfrak{gl}_{m|2n}, \mathfrak{osp}_{m|2n})$

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Quantum CMS System

Calogero–Moser–Sutherland operators

$$\mathscr{L} = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i<j} \frac{k(k-1)}{\sin^2(x_i - x_j)} \quad , \quad k = 1, \frac{1}{2}.$$

$$\mathscr{L} = -\sum_{i=1}^{N} \left(x_i \frac{\partial}{\partial x_i} \right)^2 + k(k-1) \sum_{1 \le i \neq j \le n} \left(\frac{x_i x_j}{x_i - x_j} \right) \frac{\partial}{\partial x_i}.$$

- Radial part of Laplacian corrected by the Euler operator.

Olshanetsky–Perelomov (1980)

$$\mathscr{L} = -\Delta + \sum_{\alpha \in \Phi^+} \frac{m_\alpha(m_\alpha + 2m_{2\alpha} + 1)(\alpha, \alpha)}{\sin^2(\alpha, x)}.$$

- Eigenstates are expressible as **Jack symmetric functions**.

$$\lambda := (\lambda_1, \ldots, \lambda_n),\ \lambda_1 \ge \cdots \ge \lambda_n \ge 0 \ \Rightarrow\ \mathsf{J}_\lambda = \mathsf{m}_\lambda + \text{lower order terms}$$

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multiplicity-free spaces and the Capelli basis

Polynomial coefficient differential operators

- $\mathscr{P}(W)$, $\mathrm{GL}(W) \times \mathscr{P}(W) \to \mathscr{P}(W)$, $g \cdot p(w) := p(g^{-1} \cdot w)$.
- $\mathscr{D}(W)$ $\partial_{w_1} \cdots \partial_{w_r}$

  $$\partial_v p(w) := \lim_{t \to 0} \left(\frac{1}{t} p(w + tv) - p(w) \right)$$

- $\mathscr{D}(W) \cong \mathcal{S}(W)$ as $\mathrm{GL}(W)$-modules.
- $\mathscr{P}(W) \otimes \mathscr{D}(W) \cong \mathscr{P}\mathscr{D}(W)$, $p \otimes \partial_v \mapsto p\partial_v$

  $g \in \mathrm{GL}(W)$, $D \in \mathscr{P}\mathscr{D}(W) \Rightarrow (g \cdot D)p := g \cdot \big(D(g^{-1} \cdot p)\big)$

- **Problem.** Study $\mathscr{P}\mathscr{D}(W)^G$ for a reductive subgroup $G \subseteq \mathrm{GL}(W)$.

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$$\mathcal{S}(W) \cong \bigoplus_{\lambda\in\mathcal{I}_W} V_\lambda \Rightarrow \mathscr{P}(W) \cong \bigoplus_{\lambda\in\mathcal{I}_W} V_\lambda^*.$$

$$\mathscr{P}\mathscr{D}(W) \cong \mathscr{P}(W)\otimes\mathcal{S}(W) \cong \bigoplus_{\lambda,\mu\in\mathcal{I}_W} V_\lambda^*\otimes V_\mu \cong \bigoplus_{\lambda,\mu\in\mathcal{I}_W} \mathrm{Hom}(V_\mu^*, V_\lambda^*)$$

$$\mathrm{Hom}_G(V_\mu^*, V_\lambda^*) := \begin{cases} \mathbb{C} & \text{if } \lambda=\mu, \\ \{0\} & \text{if } \lambda\neq\mu. \end{cases} \qquad D_\lambda \leftrightarrow \mathbf{1}\in\mathrm{Hom}_G(V_\lambda^*, V_\lambda^*)$$

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- $\lambda,\mu\in\mathcal{I}_W \Rightarrow\; D_\lambda : V_\mu^* \to V_\mu^*$ acts by $\mathbf{c}_\lambda(\mu)\in\mathbb{C}$.
- **Problem (Kostant).** Give an explicit description of $\mathbf{c}_\lambda(\mu)$.

The spectrum $\mathbf{c}_\lambda(\mu)$

Example

$V = \mathbb{C}^n$, $W = \mathcal{S}^2(V)$, $G = \mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$, $K = \mathrm{O}(V) \cong \mathrm{O}_n(\mathbb{C})$.

- $\mathscr{P}(W) \cong \bigoplus_{\lambda} V_\lambda^*$ where $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$.
- $\boldsymbol{\lambda}$ = h.w. of $V_\lambda^* = (-2\lambda_n)\varepsilon_1 + \cdots + (-2\lambda_1)\varepsilon_n$.
- Every $V_\lambda^* \subset \mathscr{P}(W)$ contains a K-invariant vector $0 \neq \mathbf{z}_\lambda \in V_\lambda^*$.
- $w_\circ \in W$ a K-invariant vector $\rightsquigarrow$ $\iota : G/K \hookrightarrow W$, $g \mapsto g \cdot w_\circ$.
- $\mathfrak{a} := \{\mathrm{diag}(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in \mathbb{C}\} \subset \mathfrak{gl}_n \cong \mathfrak{gl}(V)$.

$$\mathfrak{a} \hookrightarrow \mathfrak{gl}(V) \xrightarrow{d\iota} W \xrightarrow{\mathbf{z}_\lambda} \mathbb{C} \quad \rightsquigarrow \quad \boxed{J_\lambda := \mathbf{z}_\lambda\big|_{\mathfrak{a}} \in \mathscr{P}(\mathfrak{a})}$$

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- $\boldsymbol{\lambda}$ = h.w. of $V_\lambda^* = (-2\lambda_n)\varepsilon_1 + \cdots + (-2\lambda_1)\varepsilon_n$.
- Every $V_\lambda^* \subset \mathscr{P}(W)$ contains a K-invariant vector $0 \neq \mathbf{z}_\lambda \in V_\lambda^*$.
- $w_\circ \in W$ a K-invariant vector $\rightsquigarrow$ $\iota : G/K \hookrightarrow W$, $g \mapsto g \cdot w_\circ$.
- $\mathfrak{a} := \{\mathrm{diag}(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in \mathbb{C}\} \subset \mathfrak{gl}_n \cong \mathfrak{gl}(V)$.

$\mathfrak{a} \hookrightarrow \mathfrak{gl}(V) \xrightarrow{d\iota} W \xrightarrow{\mathbf{z}_\lambda} \mathbb{C}$ $\rightsquigarrow$ $\boxed{\mathrm{J}_\lambda := \mathbf{z}_\lambda\big|_{\mathfrak{a}} \in \mathscr{P}(\mathfrak{a})}$

The spectrum $\mathbf{c}_\lambda(\mu)$

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Theorem (Sahi '94, Knop–Sahi '96)

Fix $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$.

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Other examples: Hermitian symmetric pairs – $\mathrm{GL}(V) \times \mathrm{GL}(V)/\mathrm{GL}(V)$ leads to factorial Schur functions (Biedenharn, Louck, Okounkov, Olshanskii).

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Deformed CMS operators

The deformation $\mathscr{L}_{m,n,\theta}$ (Sergeev–Veselov, 2005)

$$\mathscr{L}_{m,n,\theta} = -\sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} + \theta \sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2} + \sum_{1\le i<j\le m} \frac{2\theta(\theta-1)}{\sin^2(x_i - x_j)}$$

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θ-supersymmetric functions

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Shifted super Jack polynomials

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Shifted super Jack polynomials

The algebra $\Lambda^\natural_{m,n,\theta}$ and the polynomials $\mathsf{sJ}^\star_\lambda$

Let $\Lambda^\natural_{m,n,\theta} \subset \mathbb{C}[x_1,\ldots,x_m,y_1,\ldots,y_n]^{S_m\times S_n}$ be defined as follows:

$$f \in \Lambda^\natural_{m,n,\theta} \;\text{ iff }\; f\left(x_i+\frac{1}{2}, y_j-\frac{1}{2}\right) = f\left(x_i-\frac{1}{2}, y_j+\frac{1}{2}\right) \text{ on the hyperplane } x_i+\theta y_j = 0.$$

Set

$$\varphi^\natural : \Lambda_\theta \to \Lambda^\natural_{m,n,\theta} \quad , \quad \varphi^\natural(f)(\mathbf{p},\mathbf{q}) := f(\mathsf{F}^{-1}(\mathbf{p},\mathbf{q}))$$

where the map

$$\mathsf{F} : \{(\lambda_1,\ldots,\lambda_m,\mu_1,\ldots,\mu_n)\} \to \mathbb{C}^{m+n}$$

is given by "Frobenius coordinates":

$$\begin{cases} p_i = \lambda_i - \theta(i-\frac{1}{2}) - \frac{1}{2}(n-\theta m) & 1\le i\le m, \\ q_j = \mu'_j - \theta^{-1}(j-\frac{1}{2}) + \frac{1}{2}(\theta^{-1}n+m) & 1\le j\le n. \end{cases}$$

$\mathsf{sJ}^\star_\lambda := \varphi^\natural(\mathsf{J}^\star_\lambda)$ **shifted super Jack polynomials**

$$\lambda = (\lambda_1,\lambda_2,\ldots) \text{ such that } \lambda_{m+1} > n \;\Rightarrow\; \mathsf{sJ}^\star_\lambda = 0.$$

Shifted super Jack polynomials

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Generalities on Lie superalgebras

- SVec: symmetric monoidal category SVec of $\mathbb{Z}/2$-graded vector spaces.

$$\mathbb{Z}/2 = \{\overline{0}, \overline{1}\} \qquad V \in \mathsf{obj}_{\mathsf{SVec}} \rightsquigarrow V = V_{\overline{0}} \oplus V_{\overline{1}}.$$

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$$V \otimes W \to W \otimes V \ , \ v \otimes w \mapsto (-1)^{|v|\cdot|w|} w \otimes v.$$

$$\mathcal{S}(V) \cong \mathcal{S}(V_{\overline{0}}) \otimes \Lambda(V_{\overline{1}}) \ , \ \mathscr{P}(V) = \mathcal{S}(V^*)$$

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The Lie superalgebra $\mathfrak{gl}_{m|n}$

Root system

- $V = \mathbb{C}^{m|n} \rightsquigarrow \mathfrak{gl}_{m|n} = \mathrm{End}(\mathbb{C}^{m|n})$.
- $\mathfrak{g} = \mathfrak{gl}_{m|n} \Rightarrow \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where

$$\mathfrak{n}^{\pm} = \bigoplus_{\alpha \in \Phi^{\pm}} \mathfrak{g}_\alpha \qquad \text{for} \quad \Phi^{\pm} = \Phi_{\bar{0}}^{\pm} \cup \Phi_{\bar{1}}^{\pm}.$$

$$\Phi_{\bar{0}}^{+} = \{\varepsilon_i - \varepsilon_j \ : \ 1 \le i < j \le m \ \text{ or } \ m+1 \le i < j \le m+n\},$$

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Invariant form

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \Rightarrow \mathrm{str}(X) = \mathrm{tr}(A) - \mathrm{tr}(D).$$

$X, Y \in \mathfrak{gl}_{m|n} \Rightarrow \kappa(X,Y) = \mathrm{str}(XY)$ is a nondegenerate invariant form.

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Highest weight modules of $\mathfrak{gl}_{m|n}$

- Every irreducible finite dimensional representation of $\mathfrak{gl}_{m|n}$ is a highest weight module V_λ where

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satisfies

$$\lambda_i \in \mathbb{Z} \;\text{ and }\; \lambda_1 \geq \cdots \geq \lambda_m \text{ and } \lambda_{m+1} \geq \cdots \geq \lambda_{m+n}.$$

Signed S_d-action on $V^{\otimes d}$

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_d) = (-1)^{\epsilon(\sigma^{-1}; v_1, \ldots, v_d)} v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(d)}$$

where $\epsilon(\sigma; v_1, \ldots, v_d) = \sum_{\substack{1\leq r<s\leq d \\ \sigma(r)>\sigma(s)}} |v_\sigma(r)| \cdot |v_\sigma(s)|$.

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The Lie superalgebra $\mathfrak{gl}_{m|n}$

Schur–Weyl duality

- (m,n)**-hook diagram**: a Young diagram $\mathbf{D} = (\flat_1, \flat_2, \ldots)$ that satisfies $\flat_{m+1} \leq n$.
- $\mathrm{H}(m,n,d) = \{\, (m,n)\text{-hook diagrams of size } d \,\}$.
- Recall: $V = \mathbb{C}^{m|n}$.
- **(Sergeev '84, Berele–Regev '87)** As $\mathfrak{gl}_{m|n} \times S_d$-module,

$$V^{\otimes d} \cong \bigoplus_{\mathbf{D} \in \mathrm{H}(m,n,d)} V_{\mathbf{D}} \otimes U_{\mathbf{D}}$$

$$\text{h.w. of } V_{\mathbf{D}} = \flat_1\varepsilon_1 + \cdots + \flat_m\varepsilon_m + \langle \flat_1' - m\rangle\varepsilon_{m+1} + \cdots + \langle \flat_n' - m\rangle\varepsilon_{m+n}$$

where $\langle \flat_i' - m\rangle := \max\{\flat_i' - m, 0\}$.

- $\in \mathrm{H}(2,3,16) \rightsquigarrow 7\varepsilon_1 + 5\varepsilon_2 + 2\varepsilon_3 + \varepsilon_4 + \varepsilon_5$

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The Lie superalgebra $\mathfrak{gl}_{m|n}$

Schur–Weyl duality

- (m,n)**-hook diagram**: a Young diagram $\mathbf{D} = (\flat_1, \flat_2, \ldots)$ that satisfies $\flat_{m+1} \leq n$.
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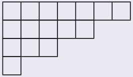
-  $\in \mathrm{H}(2,3,16) \rightsquigarrow 7\varepsilon_1 + 5\varepsilon_2 + 2\varepsilon_3 + \varepsilon_4 + \varepsilon_5$

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Proposition (Sahi–S.)

For every $\mathbf{D} \in \mathrm{H}(m, 2n, d)$, the irreducible $\mathfrak{g}$-module $V_{2\lambda}^* \subset \mathscr{P}(W)$ contains a unique (up to scalar) $\mathfrak{k}$-fixed vector $0 \neq \mathbf{z}_{\mathbf{D}}$. (In fact $\mathbf{z}_{\mathbf{D}}\big|_{\mathfrak{a}} = \mathsf{sJ}_{\mathbf{D}}$ for $\theta = \frac{1}{2}$.)

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Decomposing $\mathcal{S}(\mathcal{S}^2(W))$ where $W = \mathcal{S}^2(V)$

Theorem (Brini–Huang–Teolis '92, Cheng–Wang '01) As a $\mathfrak{gl}_{m|n}$-module,

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The Capelli basis

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- Up to the Frobenius coordinates, $\mathbf{c}_{\boldsymbol{\lambda}} = \mathsf{sJ}^\star_\lambda$ for $\theta = \frac{1}{2}$.

$$\begin{cases} p_i = \flat_i - \theta(i - \frac{1}{2}) - \frac{1}{2}(n - \theta m) & 1 \leq i \leq m, \\ q_j = \flat'_j - \theta^{-1}(j - \frac{1}{2}) + \frac{1}{2}(\theta^{-1} n + m) & 1 \leq j \leq n. \end{cases}$$

The Capelli problem

- Action of $\mathfrak{gl}_{m|n}$ on $\mathscr{P}(W)$:

$$E_{i,j} \in \mathfrak{gl}_{m|n} \rightsquigarrow \sum_r (-1)^{|i|+|i|\cdot|j|} y_{r,i} \partial_{r,j} \in \mathscr{P}\mathscr{D}(W).$$

- $\rho : \mathfrak{gl}_{m|n} \to \mathscr{P}\mathscr{D}(W) \quad \rightsquigarrow \quad \rho : \mathbf{U}(\mathfrak{gl}_{m|n}) \to \mathscr{P}\mathscr{D}(W).$
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Idea of proof

- $\mathfrak{gl}_{m|n}$-invariants of $\mathscr{P}(W) \otimes \mathcal{S}(W)$ are spanned by

$$\mathbf{t}_\sigma = \sum_{t_1,\dots,t_d} (-1)^{\check{\epsilon}(\sigma;t_1,\dots,t_d)} \varphi_{t_{\sigma(1)},t_1} \cdots \varphi_{t_{\sigma(d)},t_d} \quad , \quad \sigma \in S_d.$$

- $\mathbf{t}_\sigma$ depends only on the conjugacy class of $\sigma \in S_d$.
- $\sigma = \sigma_1 \dots \sigma_\ell,\ \ \sigma_k = (d_k + 1, \dots, d_{k+1}) \quad \Rightarrow \quad \mathbf{t}_\sigma = \mathbf{t}_{\sigma_1} \cdots \mathbf{t}_{\sigma_\ell}.$
  $0 = d_1 < \cdots < d_{\ell+1} = d.$
- $\mathbf{E}_{i,j} = (-1)^{|i|\cdot|j|} E_{i,j}$, $Z_d = \mathrm{str}(\mathbf{E}^d) \in \mathbf{U}(\mathfrak{gl}_{m|n})$.
- **Gelfand elements** : $Z_d \in \mathbf{Z}^d(\mathfrak{gl}_{m|n})$.
- $\mathscr{P}(W) \otimes \mathcal{S}(W) \xrightarrow{\mathsf{m}} \mathscr{P}\mathscr{D}(W)$, $p \otimes \partial \mapsto p\partial$.

$$\sigma = (1, \dots, d) \quad \Rightarrow \quad \mathrm{ord}\big(\rho(Z_d) - \mathsf{m}(\mathbf{t}_\sigma)\big) < d.$$

Thank you!