Transgression of gauge group cocycles

Locally smooth 3-cocycles, gerbes, category of CAR representations

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Motivation from gauge theory

L^2 condition on the curvature form of a Yang-Mills connection: The connection form at infinity in $\mathbb{R}^n$ is a pure gauge mod terms of order $1/r^{n/2+\epsilon}$. Denote by $\mathcal{G}_n$ the group of smooth based maps $S^n \to G$. Up to homotopy,the moduli space $\mathcal{A}/\mathcal{G}_n$ is then parametrized by $Map(S^{n-1}, G)$. Up to homotopy, the bundle $\mathcal{G}_n \to \mathcal{A} \to \mathcal{A}/\mathcal{G}_n$ is then the bundle

$$\mathcal{G}_n \to P \to Map(S^{n-1}, G)$$

where P is contractible and $\mathcal{G}_0$ acts freely on P; restricting everything to based maps we can take P as the group of paths $f(t)$ in $\mathcal{G}_{n-1}$ with $f(0) = id$ and we get the fibration

$$\mathcal{G}_n \to P_n \to \mathcal{G}_{n-1}.$$

Motivation from gauge theory

In particular, for $n = 1$ we have $\mathcal{G}_1 \to P_1 \to G$ a fibaration over the finite dimensional group G, the fiber $\mathcal{G}_1 = \Omega G$ the based loop group.

When G is simple compact Lie group ΩG has up to isomorphism a unique central extension $\hat{\Omega}_k G$ for each **level** $k \in \mathbb{Z}$. The extension can be given as a **locally smooth** 2-cocycle $c_2 : \Omega G \times \Omega G \to S^1$. This cocycle is obtained from a class $\omega_3 \in H^3(G, \mathbf{Z})$ which corresponds to a Lie algebra cohomology class in $H^3(\mathfrak{g})$.

So one can ask whether there is a corresponding cocycle in third group cohomology of G. The answer is yes if one considers again the locally smooth cohomology. About the meaning of the 3-cocycle later.....

Relation to the BRS complex

Anomalies in quantized gauge theory can be computed from the BRS double complex. It starts from an even form $\omega^{2n,0}$ which is a characteristic class of a vector bundle over the physical space-time M. Locally, we have $\omega^{2n,0} = d\omega^{2n-1,0}$ where $\omega^{2n-1,0}$ is a Chern-Simons form. One continues

$$\delta\omega^{2n-1,0} = d\omega^{2n-2,1}$$

where δ is the coboundary operator in Lie algebra cohomology, here the Lie algebra is the algebra of infinitesimal gauge transformations. Next

$$\delta\omega^{2n-2,1} = d\omega^{2n-3,2}$$

and so on; the second index is the Lie algebra cohomology degree. In particular $\omega^{2n-2,1}$ is the (infinitesimal) gauge anomaly and $\omega^{2n-3,2}$ is the commutator anomaly (in space dimension $2n-3$).
Here we want to address the same problem on the level of locally smooth group cocycles.

The 3-cocycle: categorical representation

C an abelian category, G a group
$g \in G$, F_g a functor in C
$i_{g,h} : F_g \circ F_h \to F_{gh}$ an isomorphism
$i_{g,hk} \circ i_{h,k}$ and $i_{gh,k} \circ i_{g,h}$ isomorphisms $F_g \circ F_h \circ F_k \to F_{ghk}$
They are not necessarily equal; one can have a *central extension*
$i_{g,hk} \circ i_{h,k} = \alpha(g,h,k) i_{gh,k} \circ i_{g,h}$ with $\alpha(g,h,k) \in \mathbf{C}^\times$ a 3-cocycle

3-cocycles

Let $\mathcal{B}$ be an associative algebra and G a group. Assume that we have a group homomorphism $s : G \to \text{Out}(\mathcal{B})$ where $\text{Out}(\mathcal{B})$ is the group of outer automorphims of $\mathcal{B}$, that is, $\text{Out}(\mathcal{B}) = \text{Aut}(\mathcal{B})/\text{In}(\mathcal{B})$, all automorphims modulo the normal subgroup of inner automorphisms.

If one chooses any lift $\tilde{s} : G \to \text{Aut}(\mathcal{B})$ then we can write

$$\tilde{s}(g)\tilde{s}(g') = \sigma(g, g') \cdot \tilde{s}(gg')$$

for some $\sigma(g, g') \in \text{In}(\mathcal{B})$. From the definition follows immediately the cocycle property

$$\sigma(g, g')\sigma(gg', g'') = [\tilde{s}(g)\sigma(g', g'')\tilde{s}(g)^{-1}]\sigma(g, g'g'')$$

Prolongation by central extension

Let next H be any central extension of $\mathrm{In}(\mathcal{B})$ by an abelian group a. That is, we have an exact sequence of groups,

$$1 \to a \to H \to \mathrm{In}(\mathcal{B}) \to 1.$$

Let $\hat{\sigma}$ be a lift of the map $\sigma : G \times G \to \mathrm{In}(\mathcal{B})$ to a map $\hat{\sigma} : G \times G \to H$ (by a choice of section $\mathrm{In}(\mathcal{B}) \to H$). We have then

$$\hat{\sigma}(g, g')\hat{\sigma}(gg', g'') = [\tilde{s}(g)\hat{\sigma}(g', g'')\tilde{s}(g)^{-1}]$$
$$\times\hat{\sigma}(g, g'g'') \cdot \alpha(g, g', g'') \text{ for all } g, g', g'' \in G$$

where $\alpha : G \times G \times G \to a$.

The 3-cocycle condition

Here the action of the outer automorphism $s(g)$ on $\hat{\sigma}(*)$ is defined by $s(g)\hat{\sigma}(*)s(g)^{-1} =$ the lift of $s(g)\sigma(*)s(g)^{-1} \in \mathrm{In}(\mathcal{B})$ to an element in H. One can show that α is a 3-cocycle

$$\alpha(g_2, g_3, g_4)\alpha(g_1 g_2, g_3, g_4)^{-1}\alpha(g_1, g_2 g_3, g_4) \\ \times\alpha(g_1, g_2, g_3 g_4)^{-1}\alpha(g_1, g_2, g_3) = 1.$$

A QFT example

Remark If we work in the category of topological groups (or Lie groups) the lifts above are in general discontinuous; normally, we can require continuity (or smoothness) only in an open neighborhood of the unit element.

Next we construct an example from quantum field theory. Let G be a compact simply connected Lie group and P the space of smooth paths $f : [0, 1] \to G$ with initial point $f(0) = e$, the neutral element, and quasiperiodicity condition $f^{-1}df$ a smooth function.

P is a group under point-wise multiplication but it is also a principal ΩG bundle over G. Here $\Omega G \subset P$ is the loop group with $f(0) = f(1) = e$ and $\pi : P \to G$ is the projection to the end point $f(1)$. Fix an unitary representation ρ of G in $\mathbf{C}^N$ and denote $H = L^2(S^1, \mathbf{C}^N)$.

CAR representations

For each polarization $H = H_- \oplus H_+$ we have a vacuum representation of the CAR algebra $\mathcal{B}(H)$ in a Hilbert space $\mathcal{F}(H_+)$. Denote by $\mathcal{C}$ the category of these representations. Denote by $a(v), a^*(v)$ the generators of $\mathcal{B}(H)$ corresponding to a vector $v \in H$,

$$a^*(u)a(v) + a(v)a^*(u) = 2 < v, u >$$

and all the other anticommutators equal to zero.

Outer automorhisms

Any element $f \in P$ defines a unique automorphism of $\mathcal{B}(H)$ with $\phi_f(a^*(v)) = a^*(f \cdot v)$, where $f \cdot v$ is the function on the circle defined by $\rho(f(x))v(x)$. These automorphims are in general not inner except when f is periodic.

We have now a map $s : G \to \mathrm{Aut}(\mathcal{B})/\mathrm{In}(\mathcal{B})$ given by $g \mapsto F(g)$ where $F(g)$ is an arbitrary smooth quasiperiodic function on $[0, 1]$ such that $F(g)(1) = g$.

Any two such functions $F(g), F'(g)$ differ by an element σ of ΩG, $F(g)(x) = F'(g)(x)\sigma(x)$. Now σ is an inner automorphism through a projective representation of the loop group ΩG in $\mathcal{F}(H_+)$.

3-cocycle

In an open neighborhood U of the neutral element e in G we can fix in a smooth way for any $g \in U$ a path $F(g)$ with $F(g)(0) = e$ and $F(g)(1) = g$.

Of course, for a connected group G we can make this choice globally on G but then the dependence of the path $F(g)$ would not be a continuous function of the end point. For a pair $g_1, g_2 \in G$ we have

$$\sigma(g_1, g_2)F(g_1 g_2) = F(g_1)F(g_2)$$

with $\sigma(g_1, g_1) \in \Omega G$.

LG valued 2-cocycle

For a triple of elements g_1, g_2, g_3 we have now

$$\begin{aligned} F(g_1)F(g_2)F(g_3) &= \sigma(g_1, g_2)F(g_1g_2)F(g_3) \\ &= \sigma(g_1, g_2)\sigma(g_1g_2, g_3)F(g_1g_2g_3). \end{aligned}$$

In the same way,

$$\begin{aligned} F(g_1)F(g_2)F(g_3) &= F(g_1)\sigma(g_2, g_3)F(g_2g_3) \\ &= [g_1\sigma(g_2, g_3)g_1^{-1}]F(g_1)F(g_2g_3) \\ &= [g_1\sigma(g_2, g_3)g_1^{-1}]\sigma(g_1, g_2g_3)F(g_1g_2g_3) \end{aligned}$$

which proves the 2-cocycle relation for σ.

3-cocycle α for G

Lifting the loop group elements σ to inner automorphisms $\hat{\sigma}$ through a projective representation of ΩG we can write

$$\hat{\sigma}(g_1, g_2)\hat{\sigma}(g_1 g_2, g_3) = \mathrm{Aut}(g_1)[\hat{\sigma}(g_2, g_3)]\hat{\sigma}(g_1, g_2 g_3)\alpha(g_1, g_2, g_3),$$

where $\alpha : G \times G \times G \to S^1$ is some phase function arising from the fact that the projective lift is not necessarily a group homomorphism.

Since (in the case of a Lie group) the function $F(\cdot)$ is smooth only in a neighborhood of the neutral element, the same is true also for σ and finally for the 3-cocycle α.

The Lie algebra 3-cocycle

An equivalent point of view to the construction of the 3-cocycle α is this: We are trying to construct a central extension $\hat{P}$ of the group P of paths in G (with initial point $e \in G$) as an extension of the central extension over the subgroup ΩG. The failure of this central extension is measured by the cocycle $\alpha,$ as an obstruction to associativity of $\hat{P}$.

On the Lie algebra level, we have a corresponding cocycle $c_3 = d\alpha$ which is easily computed. The cocycle c of $\Omega\mathfrak{g}$ extends to the path Lie algebra $P\mathfrak{g}$ as

$$c(X, Y) = \frac{1}{4\pi i} \int_{[0,2\pi]} \mathrm{tr}\,(XdY - YdX).$$

This is an antisymmetric bilinear form on $P\mathfrak{g}$ but it fails to be a Lie algebra 2-cocycle. The coboundary is given by

The Lie algebra 3-cocycle

$$\begin{aligned}(\delta c)(X,Y,Z) &= c(X,[Y,Z]) + c(Y,[Z,X]) + c(Z,[X,Y]) \\ &= -\frac{1}{4\pi i}\operatorname{tr} X[Y,Z]|_{2\pi} = d\alpha(X,Y,Z).\end{aligned}$$

Thus δc reduces to a 3-cocycle of the Lie algebra $\mathfrak{g}$ of G on the boundary $x = 2\pi$. This cocycle defines by left translations on G the left-invariant de Rham form $-\frac{1}{12\pi i}\operatorname{tr}(g^{-1}dg)^3$; this is normalized as $2\pi i$ times an integral 3-form on G.

Transgression

Let ω_3 represent a class in the singular cohomology $\mathrm{H}^3(H, \mathbf{Z})$. We shall now make the following assumptions: 1) The pull-back $\pi^*(\omega_3) = d\theta_2$ is trivial on G. 2) H and G are simply connected and $\mathrm{H}_2(G, \mathbf{Z}) = \mathrm{H}_2(H, \mathbf{Z}) = 0$. Using the exact homotopy sequence from the fibration $N \to G \to H$ we conclude that N is connected and $\pi_1(N) = 0$ and thus also $\mathrm{H}_1(N, \mathbf{Z}) = 0$. For each $g \in G$ we select a path $g(t)$ with end points $g(0) = 1 \in G$ and $g(1) = g$. We can make the choice $g \to g(t)$ in a locally smooth manner close to the neutral element $1 \in G$. In addition, since also N is connected, we may assume that $g(t) \in N$ if $g \in N$. For a triple $g, g_1, g_2 \in G$ we make a choice of a singular 2-simplex $\Delta(g; g_1, g_2)$ such that its boundary is given by the union of the 1-simplices $gg_1(t), gg_1(1)g_2(t)$ and $g(g_1g_2)(1-t)$. All this can be made in a locally smooth manner since locally the Lie groups are open contractible sets in a vector space.

$$c_2(g; g_1, g_2) = \exp 2\pi i < \Delta(g; g_1, g_2), \theta_2 >$$

using the duality pairing of singular 2-simplices and 2-cochains. This formula does not in general define a group cocycle for G but it gives a 2-cocycle for the group N with the right action of N on G and the corresponding action of N on $A = Map(G, S^1)$. To prove that indeed

$$\begin{aligned} (\delta c)_2(g; n_1, n_2, n_3) &= \\ c_2(g; n_1, n_2)c_2(g; & n_1 n_2, \quad n_3)c_2(g; n_1, n_2 n_3)^{-1} c_2(g n_1; n_2, n_3)^{-1} \\ &= \quad 1 \end{aligned}$$

we just need to observe that the product is given through pairing the cochain θ_2 with the singular cycle defined as the union of the singular 2-simplices involved in the above formula. All these 2-simplices are in the same N orbit gN and since $d\theta_2 = \pi^*\omega_3$ the cochain θ_2 is actually an integral cocycle on the N orbits and the pairing gives an integer k and $\exp 2\pi ik = 1$. For arbitrary $g_i \in G$ the coboundary δc_2 does not vanish but its value

$$(\delta c_2)(g; g_1, g_2, g_3) = \exp 2\pi i < \Delta(g; g_1, g_2, g_3), d\theta_2 >$$

is given by pairing $d\theta_2 = \pi^*\omega_3$ with the singular 3-simplex V with the boundary consisting of the sum of the faces $\Delta(g; g_1, g_2), \Delta(g; g_1g_2, g_3), \Delta(g; g_1, g_2g_3)$, $\Delta(gg_1; g_2, g_3)$. But this is the same as $\exp 2\pi i < \pi(V), \omega_3 >$ and therefore it depends only on the projections $\pi(g), \pi(g_i) \in H$. Denote by $c_3 = c_3(h; h_1, h_2, h_3)$ this locally smooth 3-cocycle on H. (This construction can be extended to higher cocycles under appropriate conditions on the homology groups of H.)

We may think of the cohomology class $[c_3]$ as an obstruction to prolonging the principal N bundle G over H to a bundle $\hat{G}$ with the structure group $\hat{N}$. Namely, if such a prolongation exists then there is a 2-cocycle c_2 on G which when restricted to N orbits in G is equal to $c_2(g; n_1, n_2)$. If c_2' is another such a 2-cocycle then $(\delta c_2')(\delta c_2)^{-1}$ projects to a a trivial 3-cocycle on H. Conversely, if c_3 on H is a coboundary of some ξ_2 then $c_2' = c_2(\pi^* \xi)^{-1}$ agrees with c_2 on the N orbits and so the obstruction depends only on the cohomology class $[c_3]$.

Wagemann and Wockel defined a map from the locally smooth cohomology of a Lie group H to its Čech cohomology. There is also a map from the locally smooth group cohomology $\mathrm{H}^2_s(N, A)$ to the Čech cohomology $\check{\mathrm{H}}^2(H, A)$ by the formula

$$c_{ijk}(x) = \hat{\eta}_{ij}(x)\hat{\eta}_{jk}(x)\hat{\eta}_{ki}(x)$$

where $\psi_i(x)\eta_{ij}(x) = \psi_j(x)$, $\psi_i : U_i \to G$ are local smooth sections for an open good cover $\{U_i\}$ of H and the $\hat{\eta}_{ij}$'s are lifts of the transition functions $\eta_{ij} : U_i \cap U_j \to N$ to the extension $\hat{N}$; the product on the right is determined by an element in $\mathrm{H}^2_s(N, A)$. Although these Čech cocycles have values in A they correspond to a cocycle in $\mathrm{H}^3(H, \mathbf{Z})$ by the usual way, taking differences of logarithms $\log c_{ijk}/2\pi i$ on intersections U_{ijkl} which must be integer constants for a good cover.