Infinite-dimensional calculus with a view towards Lie theory

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Overview

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§1 Basics of ∞-dim calculus

Defn. E, F locally convex spaces, $U \subseteq E$ open. A map $f \colon U \to F$ is called C^1 if it is continuous, the directional derivatives

$$df(x,y) := (D_y f)(x) = \left.\frac{d}{dt}\right|_{t=0} f(x+ty)$$

exist for all $x \in U$, $y \in E$, and the map

$$df \colon U \times E \to F$$

is continuous. The map f is called C^k with $k \in \mathbb{N}_0 \cup \{\infty\}$ if the iterated directional derivatives

$$d^j f(x, y_1, \ldots, y_j) := (D_{y_j} \cdots D_{y_1} f)(x)$$

exist for all $j \in \mathbb{N}_0$ such that $j \leq k$ and define continuous functions

$$d^j f \colon U \times E^j \to F.$$

Rem f is C^{k+1} iff f is C^1 and $df \colon U \times E \to F$ is C^k.

C^∞-maps are also called *smooth*.

Basic facts

(a) $df(x,.)\colon E \to F$ is linear

(b) The Chain Rule holds: If $f\colon U \to V$ and $g\colon V \to F$ are C^k, then also $g \circ f\colon U \to F$ is C^k, with

$$d(g \circ f)(x,y) = dg(f(x), df(x,y)).$$

Defn. Smooth manifolds modelled on locally convex TVS E are defined as usual:

Hausdorff topological space M with an atlas of homeomorphisms $\phi\colon M \supseteq U \to V \subseteq E$ ("charts") between open sets such that the chart changes are smooth.

Defn. Lie group = group G, equipped with a smooth manifold structure modelled on a locally convex space such that the group operations are smooth maps.

$L(G) := T_eG$, with Lie bracket arising from the identification of $y \in L(G)$ with the corresponding left invariant vector field.

Comparison with other approaches to differential calculus

The approach to ∞-dimensional calculus presented here goes back to A. Bastiani and is also known under the name of *Keller's C^k_c-theory*.

Classical calculus in Banach spaces

A map $f\colon E \supseteq U \to F$ between Banach spaces is called *continuously Fréchet differentiable* (FC^1) if it is totally differentiable and

$$f'\colon U \to (\mathcal{L}(E,F), \|.\|_{op})$$

is continuous. If f is FC^1 and f' is FC^k, then f is called FC^{k+1}.

Fact: f is $C^{k+1} \Rightarrow f$ is $FC^k \Rightarrow f$ is C^k

Convenient differential calculus

If E is a Fréchet space, then a map $f\colon E \supseteq U \to F$ is C^∞ iff $f \circ \gamma\colon \mathbb{R} \to F$ is C^∞ for each C^∞-curve $\gamma\colon \mathbb{R} \to U$, i.e., iff f is smooth in the sense of the convenient differential calculus (developed by Frölicher, Kriegl and Michor).

Likewise if E is a *Silva space* (or (DFS)-space), i.e., a locally convex direct limit

$$E = \varinjlim E_n$$

of Banach spaces $E_1 \subseteq E_2 \subseteq \cdots$ such that all inclusion maps $E_n \to E_{n+1}$ are compact operators.

Beyond metrizable or Silva domains, the smooth maps of convenient differential calculus need not be C^∞ in the sense used here (they need not even be continuous).

Diffeological spaces

If E is a Fréchet space or a Silva space, then a map $f\colon E \supseteq U \to F$ is C^∞ if and only if $f \circ \gamma\colon \mathbb{R}^n \to F$ is C^∞ for each $n \in \mathbb{N}$ and C^∞-map $\gamma\colon \mathbb{R}^n \to U$ (and it suffices to take $n = 1$ as already mentioned).

Main classes of ∞-dim Lie groups

Linear Lie groups

$$G \leq A^\times$$

Mapping groups

e.g. $C^\infty(M, H)$

Diffeomorphism groups

$\mathrm{Diff}(M)$ M compact

Direct limit groups

$G = \bigcup_n G_n$ with

$G_1 \leq G_2 \leq \cdots$ fin-dim

Here A is a Banach algebra or a *continuous inverse algebra* (CIA)

$A^\times$ is open and $A^\times \to A$, $x \mapsto x^{-1}$ is continuous

Elementary facts for $f\colon E \supseteq U \to F$.

(a) If $f(U) \subseteq F_0$ for a closed vector subspace $F_0 \subseteq F$, then f is C^k iff $f|^{F_0}$ is C^k

(b) If $F = \prod_{j\in J} F_j$, then f is C^k iff each of its components f_j is C^k.

(c) If $F = \varprojlim F_n$ for a projective sequence

$$\cdots \to F_2 \to F_1,$$

then f is C^k iff $\pi_n \circ f$ is C^k for each $n \in \mathbb{N}$, where $\pi_n\colon F \to F_n$ is the limit map.

E.g. $C^\infty([0,1],\mathbb{R}) = \varprojlim C^n([0,1],\mathbb{R})$ for $n \in \mathbb{N}$;

$$C^{k+1}([0,1],\mathbb{R}) \to C([0,1],\mathbb{R}) \times C^k([0,1],E),$$
$$\gamma \mapsto (\gamma, \gamma')$$

linear topological embedding, closed image.

Hence a map f to $C^\infty([0,1],\mathbb{R})$ is smooth iff it is smooth as a map to $C^k([0,1],\mathbb{R})$ for each finite k.

A map to $C^{k+1}([0,1],\mathbb{R})$ is smooth iff it is smooth as a map to $C([0,1],\mathbb{R})$ and $x \mapsto f(x)'$ is smooth as a map to $C^k([0,1],\mathbb{R})$

$\rightsquigarrow$simple inductive proofs for smoothness of maps to function spaces

Mean Value Theorem. *If* $f\colon E \supseteq U \to F$ *is* C^1 *and* $x, y \in U$ *such that* $x + [0,1](y-x) \subseteq U$, *then*

$$f(y) - f(x) = \int_0^1 df(x + t(y-x), y-x)\, dt.$$

Defn. Let E be a locally convex space. A (nec. unique) element $z \in E$ is called the *weak integral* of a continuous path $\gamma\colon [a,b] \to E$ if

$$\lambda(z) = \int_a^b \lambda(\gamma(t))\, dt \quad \text{for all } \lambda \in E'.$$

Write $\int_a^b \gamma(t)\, dt := z$.

Mappings on non-open sets: Let $U \subseteq E$ be a subset with dense interior which is *locally convex*, i.e., each $x \in U$ has a relatively open, convex neighbourhood in U. Say that a continuous map $f\colon U \to F$ is C^k if $f|_{U^0}$ is C^k and

$$d^j(f|_{U^0})\colon U^0 \times E^j \to F$$

extends to a continuous map $d^j f\colon U \times E^j \to F$ for each $j \in \mathbb{N}$ such that $j \leq k$.

If $f\colon E \supseteq U \to F$, then the directional difference quotients
$$\frac{f(x+ty)-f(x)}{t}$$
make sense for all (x,y,t) in the set
$$U^{[1]} := \{(x,y,t) \in U \times E \times \mathbb{R} \colon x+ty \in U\}$$
such that $t \neq 0$.

Fact. *A continuous map f is C^1 if and only if there is a continuous map $f^{[1]}\colon U^{[1]} \to F$ with*
$$f^{[1]}(x,y,t) = \frac{f(x+ty)-f(x)}{t}$$
or all $(x,y,t) \in U^{[1]}$ such that $t \neq 0$.

Indeed, $df(x,y) = \lim_{t\to 0} f^{[1]}(x,y,t) = f^{[1]}(x,y,0)$ in this case and thus f is C^1. If f is C^1, define
$$f^{[1]}(x,y,t) := \begin{cases} \frac{f(x+ty)-f(x)}{t} & \text{if } t \neq 0; \\ df(x,y) & \text{if } t = 0. \end{cases}$$
By the Mean Value Theorem, for $|t|$ small have
$$f^{[1]}(x,y,t) = \int_0^1 df(x+sty,y)\,ds.$$
Since weak integrals depend continuously on parameters, $f^{[1]}$ is continuous.

First application of $f^{[1]}$: Very easy proof of the Chain Rule.

Another application, with a view towards the commutator formula:

If G is a Lie group and $\gamma_1, \gamma_2 \in C^1([0,r], G)$ with $\gamma_1(0) = \gamma_2(0) = e$, then $\eta\colon [0, r^2] \to G$,

$$\eta(t) := \gamma_1(\sqrt{t})\gamma_2(\sqrt{t})\gamma_1(\sqrt{t})^{-1}\gamma_2(\sqrt{t})^{-1}$$

is C^1.

Proof. η is C^1 on $]0, r^2]$. We show $(\eta|_{]0,r^2]})'$ has a continuous extension to $[0, r^2]$.

Let $U \subseteq G$, $V \subseteq U$ be open identity neighbourhoods with $VVV^{-1}V^{-1} \subseteq U$. Identify U with an open set in E using a chart, such that $e = 0$. The map

$$f\colon V \times V \to U, \quad f(x,y) := xyx^{-1}y^{-1}$$

is smooth with $df(0,0,v,w) = 0$ and

$$d^2f(0,0;x,y;x,y) = 2[x,y].$$

The assertion now follows with a lemma by K.-H. Neeb:

Lemma *If $U \subseteq E$ is open, $\gamma\colon [0,1] \to U$ is C^1 and $f\colon U \to F$ a C^2-map with $df(\gamma(0),.) = 0$, then*

$$\eta\colon [0,1] \to U, \quad t \mapsto f(\gamma(\sqrt{t}))$$

is C^1 with $\eta'(0) = \frac{1}{2}d^2f(\gamma(0), \gamma'(0), \gamma'(0))$.

Proof: We may assume that $\gamma(0) = 0$ and $f(0) = 0$. Noting that

$$\gamma(\sqrt{t}) = \sqrt{t}\,\frac{\gamma(\sqrt{t}) - \overbrace{\gamma(\sqrt{0})}^{=0}}{\sqrt{t}} = \sqrt{t}\,\gamma^{[1]}(0,1,\sqrt{t}),$$

we get for $t > 0$

$$\begin{aligned}\eta'(t) &= \frac{1}{2\sqrt{t}}df(\gamma(\sqrt{t}); \gamma'(\sqrt{t})) \underbrace{-\frac{1}{2\sqrt{t}}df(0, \gamma'(\sqrt{t}))}_{=0} \\ &= \frac{1}{2}(df)^{[1]}(0, \gamma'(\sqrt{t}); \gamma^{[1]}(0,1,\sqrt{t}), 0; \sqrt{t})\end{aligned}$$

The right-hand-side makes sense also for $t = 0$ and is continuous on $[0,1]$. Hence η is C^1, with

$$\begin{aligned}\eta'(0) &= \frac{1}{2}(df)^{[1]}(0, \gamma'(0); \gamma'(0), 0; 0) \\ &= \frac{1}{2}d^2f(0, \gamma'(0), \gamma'(0)).\end{aligned}$$

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- J. Milnor, *Remarks on infinite-dimensional Lie groups*, 1984.
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§2 Inverse functions and implicit functions

Implicit Function Theorem (HG'05) *Let E be a locally convex space, F be a Banach space, $G \subseteq E \times F$ be open, $(p_0, y_0) \in G$ and*

$$f\colon G \to F$$

be a C^k-map such that $f(p_0, y_0) = 0$ and

$$f_{p_0}\colon y \mapsto f(p_0, y)$$

has invertible differential at y_0. If F has finite dimension, assume $k \geq 1$; otherwise, assume that $k \geq 2$. Then there exist open neighbourhood $P \subseteq E$ of p_0 and $V \subseteq F$ of y_0 such that

$$\{(p, y) \in P \times V\colon f(p, y) = 0\} = \mathrm{graph}(\phi)$$

for a C^k-function $\phi\colon P \to V$.

(Compare Hiltunen 1999, Teichmann 2001 for related results in other settings of ∞-dim calculus)

Some ideas of the proof.

Let E be a locally convex space, $(F, \|.\|)$ be a Banach space, $P \subseteq E$ and $V \subseteq F$ be open sets. We say that a map

$$f\colon P \times V \to F$$

defines a *uniform family of contractions* if there is $\theta \in [0,1[$ such that

$$\|f(p,y_2) - f(p,y_1)\| \leq \theta \|y_2 - y_1\|$$

for all $p \in P$, $y_1, y_2 \in V$.

Fact (HG'05) *If $f\colon U \times V \to F$ is C^k and defines a uniform family of contractions, then the set Q of all $p \in P$ such that $f(p,.)\colon V \to F$ has a fixed point y_p is open in P, and the map*

$$Q \to V, \qquad p \mapsto y_p$$

is C^k.

This implies:

Inverse Functions with Parameters (HG'05)
Let E be a locally convex space, F be a Banach space, $P \subseteq E$ and $V \subseteq F$ be open sets, $p_0 \in P$ and $f\colon P \times V \to F$ be a C^k-map such that

$$f_{p_0} := f(p_0, .)\colon V \to F$$

has invertible differential at some $y_0 \in V$. If F has finite dimension, assume $k \geq 1$; otherwise, assume that $k \geq 2$. Then, after shrinking P and V if necessary, we may assume that, for each $p \in P$,

$$f_p\colon V \to f_p(V)$$

has open image and is a C^k-diffeomorphism. Moreover, the map

$$\theta\colon P \times V \to \bigcup_{p \in P} \{p\} \times f_p(V),\ (p, y) \mapsto (p, f_p(y))$$

is a C^k-diffeomorphism onto an open set Ω.

The inverse map is $\Omega \to P \times V$, $(p, z) \mapsto (p, f_p^{-1}(z))$. Thus $(p, z) \mapsto (f_p)^{-1}(z)$ is defined on an open set and is C^k.

Application: Submersions, regular value theorem, pre-images of submanifolds etc (Neeb and Wagemann 2008, HG 2015).

Another application:

Stimulated by related work by Hiltunen (2000) and Teichmann (2001), Eyni recently used the inverse function theorem with parameters to obtain **Frobenius theorems** on the integrabilty of vector distributions $(D_p)_{p\in M}$ on infinite dimensional manifolds M (see Eyni 2014 and the references therein). Three cases were discussed:

- Finite-dimensional vector spaces $D_p \subseteq T_pM$;
- Banach spaces $D_p \subseteq T_pM$;
- D_p is complemented in T_pM and T_pM/D_p is a Banach space.

As a consequence, a Lie subalgebra $\mathfrak{h} \subseteq L(G)$ integrates to an immersed Lie subgroup of a Lie grop G if $\mathfrak{h}$ is co-Banach or $\mathfrak{h}$ is Banach and G has (at least on $\mathfrak{h}$) a smooth exponential function. That is, there is a smooth function

$$\exp_G\colon \mathfrak{h} \to G$$

such that $t \mapsto \exp_G(ty)$ is a one-parameter group with derivative y at $t = 0$ in G (Eyni'14).

Eyni actually constructs foliated charts around each point, which shows that H locally has a smooth transversal. As a consequence,

$$G/H$$

is a manifold whenever the leaf H just described is a submanifold of G (see HG'15)

Literature on §2

- J. M. Eyni, *The Frobenius theorem for Banach distributions on infinite dimensional manifolds and applications in infinite dimensional Lie theory*, preprint, 2014; `arXiv:1407.3166`.
- HG, *Finite order differentiability properties, fixed points and implicit functions over valued fields*, preprint, 2005; `arXiv:math/0511218`. Improves:
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- K.-H. Neeb and F. Wagemann, *Lie group structures on groups of smooth and analytic maps on non-compact manifolds*, 2008
- J. Teichmann, *A Frobenius theorem on convenient manifolds*, 2001.

§3 Exponential laws for function spaces

Following Alzaareer 2013, we consider functions on products with different orders of differentiability in the two factors:

Defn. Let E_1, E_2, F be locally convex, $U \subseteq E_1$ and $V \subseteq E_2$ be open, and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. A map $f\colon U \times V \to F$ is called $C^{r,s}$ if the iterated directional derivatives

$$d^{i,j} f(x, y_1, \ldots, y_i, w_1, \ldots, w_j) :=$$

$$(D_{(y_i,0)} \cdots D_{(y_1,0)} D_{(0,w_j)} \cdots D_{(0,w_1)} f)(u, v)$$

exist for all $i, j \in \mathbb{N}_0$ such that $i \leq r$, $j \leq s$ and define continuous functions

$$d^{i,j} f\colon U \times E_1^i \times E_2^j \to F.$$

If U, V are locally convex with dense interior, again use continuous extensions of differentials.

Endow $C^{r,s}(U \times V, F)$ with the initial topology with respect to the maps

$$C^{r,s}(U \times V, F) \to C(U \times V \times E_1^i \times E_2^j)_{c.o.},\ f \mapsto d^{i,j} f.$$

Exponential law (Alzaareer 2013). *If $f \in C^{r,s}(U \times V, F)$, then the map*

$$f^\vee \colon U \to C^s(V, F), \quad f^\vee(x)(y) := f(x, y)$$

is C^r and the map

$$\Phi \colon C^{r,s}(U \times V, F) \to C^r(U, C^s(V, F)),\ f \mapsto f^\vee$$

is a linear topological embedding.

If $U \times V \times E_1 \times E_2$ is a k-space or V is locally compact, then Φ is an isomorphism of topological vector spaces.

Recall that a Hausdorff space X is called a $\boldsymbol{k}$**-space** if a subset $A \subseteq X$ is closed iff $A \cap K$ is closed for each compact subset $K \subseteq X$. For example, every metrizable topological space is a k-space, as well as every locally compact topological space.

For an application to ODE's with $C^{r,s}$ right hand sides, see Alzaareer und Schmeding 2013

Application: regularity of mapping groups

If G is a Lie group modelled on a locally convex space, then we obtain a smooth action

$$G \times TG \to TG, \quad (g, v) \mapsto g.v := T\lambda_g(v),$$

using the left translation $\lambda_g \colon G \to G$, $x \mapsto gx$ by g. Abbreviate $\mathfrak{g} := L(G)$.

Defn. Let $k \in \mathbb{N}_0 \cup \{\infty\}$. The Lie group G is called C^k*-semiregular* if, for each $\gamma \in C^k([0,1], \mathfrak{g})$, there exists a (necessarily unique) $\mathrm{Evol}(\gamma) := \eta \in C^{k+1}([0,1], G)$ such that

$$\eta'(t) = \eta(t).\gamma(t) \quad \text{and} \quad \eta(0) = e.$$

If, moreover, $\mathrm{Evol} \colon C^k([0,1], \mathfrak{g}) \to C^{k+1}([0,1], G)$ [or, equivalently, the map

$$\mathrm{evol} \colon C^k([0,1], \mathfrak{g}) \to G, \quad \gamma \mapsto \mathrm{Evol}(\gamma)(1) \;]$$

is smooth, then G is called C^k*-regular.* If G is C^∞-regular, then G is called *regular* (cf. Milnor 1984). This is the weakest regularity property: If G is C^k-regular and $\ell \geq k$, then G is also C^ℓ-regular.

Regularity is important to retain familiar facts in infinite dimensions. E.g.

Theorem. (Milnor 1984). *Let G be a 1-connected Lie group and H be a regular Lie group (modelled on locally convex spaces). If $\phi\colon L(G) \to L(H)$ is a continuous Lie algebra homomorphism, then there is a unique smooth group homomorphism $\psi\colon G \to H$ with $L(\psi) = \phi$.*

If both U and V are locally compact (e.g.), then the exponential law entails that

$$C^r(U, C^s(V, F)) \cong C^s(V, C^r(U, F)).$$

The isomorphism is the composition

$$C^r(U, C^s(V, F)) \to C^{r,s}(U \times V, F)$$

$$\to C^{s,r}(V \times U, F) \to C^s(V, C^r(U, F))$$

of isomorphisms.

Here is a typical application of the exponential law:

Prop. *Let $r, s \in \mathbb{N}_0 \cup \{\infty\}$. If H is a C^r-regular Lie group and M a compact smooth manifold, then also the mapping group $G := C^s(M, H)$ is C^r-regular.*

Sketch. Identify $\mathfrak{g} := L(G)$ with $C^s(M, \mathfrak{h})$, where $\mathfrak{h} := L(H)$. The main point is to get a candiate for $\mathrm{Evol}(\gamma)$ if $\gamma \in C^r([0,1], \mathfrak{g}) = C^r([0,1], C^s(M, \mathfrak{h}))$. We try to construct the evolution pointwise:

$$\mathrm{Evol}(\gamma)(t)(x) := \mathrm{Evol}_H(s \mapsto \gamma(s)(x))(t).$$

Let us write $\Psi(\gamma)$ for the right-hand-side. We can obtain Ψ as the composition of isomorphisms and the smooth map $f \mapsto \mathrm{Evol}_H \circ f$:

$$C^r([0,1], C^s(M, \mathfrak{h})) \to C^s(M, C^r([0,1], \mathfrak{h}))$$

$$\to C^s(M, C^{r+1}([0,1], H)) \to C^{r+1}([0,1], C^s(M, H)).$$

Thus Ψ takes its values in the desired Lie group and is smooth. Testing with point evaluations (which are smooth group homomorphisms and separate points), we see that $\Psi(\gamma)$ is the evolution $\mathrm{Evol}(\gamma)$.

Rem. In particular, exponential laws for spaces of smooth functions are available (as $C^{\infty,\infty}$ maps on products coincide with C^{∞}-mps). This special case was known longer. Moreover, exponential laws in the sense of **bornological** isomorphisms play a key role in the Convenient Differential Calculus of Frölicher, Kriegl and Michor.

References for §3:

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§4 Non-linear mappings on locally convex direct limits

For example, consider the space $E := C^\infty_c(\mathbb{R})$ of real-valued test functions. Then

$$E = \bigcup_{n\in\mathbb{N}} E_n$$

with the Fréchet spaces $E_n := C^\infty_{[-n,n]}(\mathbb{R})$ of smooth functions supported in $[-n,n]$. Thus

$$E_1 \subseteq E_2 \subseteq \cdots$$

Moreover, $E = \varinjlim E_n$ as a locally convex space. Hence a linear map

$$f\colon E \to F$$

is continuous if and only if each restriction $f|_{E_n}$ is continuous. What about non-linear maps:

If $f\colon E \to F$ is a map such that $f|_{E_n}$ is C^k for each $n \in \mathbb{N}$, will f be C^k?

The answer is **no** in general. For example,

$$f\colon C^\infty_c(\mathbb{R}) \to C^\infty_c(\mathbb{R}\times\mathbb{R}), \qquad f(\gamma)(x,y) := \gamma(x)\gamma(y)$$

is discontinuous although $f|_{E_n}$ is a continuous quadratic polynomial for all n (cf. Hirai et al'01)

Well-behaved situations:

(a) (HG'02+04) If $f\colon C^\infty_c(\mathbb{R}) \to C^\infty_c(\mathbb{R})$ is C^k on each of the spaces $C^\infty_{[-m,m]}(\mathbb{R})$ and f is *local* in the sense that $f(\gamma)(x)$ only depends on the germ of γ at x, then f is C^k.

Likewise if f is **almost local**, and for maps between spaces of sections in vector bundles

$\rightsquigarrow$group operations on $\mathrm{Diff}_c(M)$ are C^∞ for σ-compact M.

Follows from:

(b) (HG'03) If $(f_n)_{n\in\mathbb{N}}$ is a sequence of C^k-maps $f_n\colon E_n \supseteq U_n \to F_n$ on open 0-neighbourhoods with $f_n(0) = 0$, then also the map

$$\oplus_{n\in\mathbb{N}} f_n\colon \bigoplus_{n\in\mathbb{N}} U_n \to \bigoplus_{n\in\mathbb{N}} F_n$$

$(x_n)_{n\in\mathbb{N}} \mapsto (f_n(x_n))_{n\in\mathbb{N}}$ is C^k.

(c) If each E_n is a complex Banach space, the inclusion maps do not increase norms and $f|_{B^{E_n}_r(0)}\colon B^{E_n}_r(0) \to F$ is complex analytic **and bounded** for all $n \in \mathbb{N}$, then

$$f\colon \bigcup_{n\in\mathbb{N}} B^{E_n}_r(0) \to F$$

is complex analytic (Dahmen 2011).

$\rightsquigarrow$Lie group structures on unions of Banach-Lie groups

(d) Let $E = \bigcup_{n\in\mathbb{N}} E_n$ be a Silva space (i.e., each E_n is a Banach space and the inclusion map $E_n \to E_{n+1}$ is a compact operator for each $n \in \mathbb{N}$). Then $f\colon E \to F$ is C^k iff $f|_{E_n}$ is C^k for each $n \in \mathbb{N}$ (see. e.g., HG'07).

(e) If E is a Silva space and $k \in \mathbb{N}_0$, then

$$C^k([0,1],E) = \bigcup_{n\in\mathbb{N}} C^k([0,1],E_n)$$

with the direct limit topology by Mujica's Theorem.

However, the path space is **not** a Silva space. One can show:

If $f\colon C^k([0,1],E) \to F$ restricts to a C^ℓ-map on each $C^k([0,1],E_n)$, then

$$f|_{C^{k+1}([0,1],E)}$$

is C^ℓ (HG'15).

$\leadsto$ $\mathrm{Diff}_{C^\omega}(M)$ is C^1-regular for each compact real analytic manifold M.

A typical application of (b) (see, e.g., HG'15)

Prop. *If M is a σ-compact smooth manifold and H a C^r-regular Lie group for some $r \in \mathbb{N}_0$, then also $C^s_c(M,H)$ is C^r-regular for each $s \in \mathbb{N}_0 \cup \{\infty\}$.*

Sketch. Let $(M_n)_{n\in\mathbb{N}}$ be a locally finite sequence of compact submanifolds of M whose interiors cover M. We know that $G_n := C^s(M_n, H)$ is C^r-regular for each n. Now the map

$$C^s_c(M,H) \to \bigoplus_{n\in\mathbb{N}} C^s(M_n,H),\ \gamma \mapsto (\gamma|_{M_n})_{n\in\mathbb{N}}$$

co-restricts to an isomorphism onto the Lie subgroup

$$\{(\gamma_n)_{n\in\mathbb{N}} \colon (\forall x \in M_n \cap M_m)\ \gamma_n(x) = \gamma_m(x)\}$$

of the weak direct product G on the right. As this subgroup is an equalizer of smooth group homomorphisms, we need only show that the weak direct product is C^r-regular. This is true since evol_G can be identified with $\oplus_{n\in\mathbb{N}}\mathrm{evol}_{G_n}$:

$$\bigoplus_{n\in\mathbb{N}} C^r([0,1], L(G_n)) = C^r([0,1], \bigoplus_{n\in\mathbb{N}} L(G_n)) \to \bigoplus_{n\in\mathbb{N}} G_n = G.$$

References on §4

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§5 Measurable regularity properties of infinite-dimensional Lie groups

Defn. If F is a Fréchet space, let $L^1([a,b],F)$ be the space of equivalence classes of absolutely integrable measurable mappings $\gamma\colon [a,b] \to F$ **with separable image**.

Continuous paths $\eta\colon [a,b] \to F$ of the form

$$\eta(t) := \int_a^t \gamma(s)\, ds$$

with $\gamma \in L^1([a,b],F)$ are called **absolutely continuous**.

Defn. Let F be a Fréchet space, $G \subseteq \mathbb{R}\times F$ and $(t_0, y_0) \in G$. A map $\eta\colon I \to F$ on an interval containing t_0 is called a **Caratheodory solution** to

$$y' = f(t,y), \;\; y(t_0) = y_0$$

if graph$(\eta) \subseteq G$, the map $t \mapsto f(t,\eta(t))$ is in L^1 and

$$\eta(t) = y_0 + \int_{t_0}^t f(s,\eta(s))\, ds \text{ for all } t \in I.$$

Rem. If η is absolutely continuous and ϕ is smooth, then $\phi \circ \eta$ is absolutely continuous. Hence absolutely continuous mappings to manifolds can be defined. Moreover,

$$AC([0,1], G)$$

is a Lie group for each Fréchet-Lie group G.

Defn. G is called L^1-regular if a Caratheodory solution $\mathrm{Evol}(\gamma) \in AC([0,1], G)$ exists to

$$y'(t) = y(t).\gamma(t), \quad y(0) = e$$

and $\mathrm{Evol}\colon L^1([0,1], \mathfrak{g}) \to AC([0,1], G)$ is smooth.

Rem. (a) Replacing L^1 with L^p yields L^p-regular Fréchet-Lie groups.

(b) $L^\infty_{rc}([a,b], E)$ (γ has metrizable compact closure) and $AC_{L^\infty_{rc}}([a,b], E)$ even works for arbitrary locally convex spaces E which are **integral complete** in the sense that each continuous curve has a weak integral. In there have space $R([a,b], E)$ of classes of regulated functions.

Then L^1-regularity is the strongest notion of measurable regularity, regulated regularity the weakest:

L^p-regularity implies L^q- regularity for all $q \geq p$

L^∞-regularity implies L^∞_{rc}-regularity, which implies regulated regularity.

Theorem. (HG) *Every Banach-Lie group is L^1-regular.*

Theorem. (HG) $\mathrm{Diff}_c(M)$ *is L^∞_{rc}-regular for each paracompact finite-dimensional smooth manifold M.*

Following a suggestion by K.-H. Neeb:

Theorem *If G is regulated regular, then G has the strong Trotter property, i.e.*

$$(\gamma(t/n))^n \to \exp_G(t\gamma'(0)) \text{ as } n \to \infty$$

for each C^1-map $\gamma\colon [0,1] \to G$, uniformly for t in compact sets.

Prop. *If G has the strong Trotter property, then G also has the strong commutator property, i.e.,*

$$\Big(\gamma_1(\sqrt{t}/n)\gamma_2(\sqrt{t}/n)\gamma_1(\sqrt{t}/n)^{-1}\gamma_2(\sqrt{t}/n)^{-1}\Big)^{n^2}$$

$$\to \exp_G(t[\gamma_1'(0),\gamma_2'(0)])$$

uniformly for t in compact sets.

Proof. Apply the n^2-subsequence of the Trotter formula to the C^1-map

$$\gamma(t) := \gamma_1(\sqrt{t})\gamma_2(\sqrt{t})\gamma_1(\sqrt{t})^{-1}\gamma_2(\sqrt{t})^{-1}.$$

Rem. (a) $L^p([a,b],E)$ can be defined not only for Fréchet spaces, but at least for some more general locally convex spaces, including spaces of compactly supported smooth vector fields. $\mathrm{Diff}_c(M)$ actually is L^1-regular.

(b) This section compiles material from HG 2015b.

(c) L^∞_{rc}-regularity of Banach-Lie groups was first announced in HG 2013; the L^∞_{rc}-regularity of diffeomorphism groups was conjectured there.

(d) That evol: $C^0([0,1],\mathfrak{g}) \to G$ is smooth with respect to the L^1 topology on $C^0([0,1],\mathfrak{g})$ for each Banach-Lie group G was already shown in HG 2015a.

Literature on §5.

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