Operator valued Fourier transforms on nilpotent Lie groups

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Reference

- I. Beltiţă, D. B., J. Ludwig, Fourier transforms of C^*-algebras of nilpotent Lie groups. Preprint arXiv:1411.3254 [math.OA].

Plan

1. Motivation: continuity properties of the Kirillov correspondence
2. Operator-valued Fourier transforms: continuity of operator fields
3. Tools from C^*-algebra extension theory: Busby invariant, completely positive lifting
4. C^*-algebras of nilpotent Lie groups: stratifications of the dual, C^*-solvability
5. Application to Heisenberg groups

Motivation (1): Lie group representations

• *Nilpotent Lie group* $G = (\mathfrak{g}, \cdot)$: finite-dim. $\mathbb{R}$-linear space $\mathfrak{g}$ with polynomial group law satisfying $(sx) \cdot (tx) = (s+t)x$ for $s, t \in \mathbb{R}$, $x \in \mathfrak{g}$

• $\widehat{G} :=$ unitary equivalence classes $[\pi]$ of unirreps $\pi\colon G \to U(\mathcal{H}_\pi)$

• Kirillov correspondence: $\boxed{\kappa\colon \widehat{G} \xrightarrow{\sim} \mathfrak{g}^*/\mathrm{Ad}_G^*}$ (=the coadjoint G-orbits)
where $\mathrm{Ad}_G^*\colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$

Recall: $[\pi] \overset{\kappa}{\longleftrightarrow} \mathcal{O} \iff (\forall \varphi \in \mathcal{C}_c^\infty(\mathfrak{g})) \quad \mathrm{Tr}\,\pi(\varphi) = \int\limits_{\mathcal{O}} \widehat{\varphi}$

Goal

continuity properties of the bijection κ

• *Regular representation* $\lambda\colon L^1(G) \to \mathcal{B}(L^2(G))$, $\lambda(f)\varphi = f * \varphi$

• $C^*(G) := \overline{\lambda(L^1(G))}^{\|\cdot\|} \subseteq \mathcal{B}(L^2(G)) \rightsquigarrow \boxed{\widehat{G} \simeq \widehat{C^*(G)}}$

Motivation (2): C^*-algebra representations

• C^*-alg. $\mathcal{A} \rightsquigarrow$ spectrum $\widehat{\mathcal{A}} := \{[\pi] \mid \pi\colon \mathcal{A} \to B(\mathcal{H}_\pi)$ irred. $*$-repres.$\}$
$\rightsquigarrow$ topology with open sets $\{[\pi] \in \widehat{\mathcal{A}} \mid \pi|_{\mathcal{J}} \not\equiv 0\}$ for closed 2-sided ideals $\mathcal{J} \subseteq \mathcal{A}$

• $\mathcal{A}_0 := \{a \in \mathcal{A} \mid \widehat{\mathcal{A}} \to [0,\infty),\ [\pi] \mapsto \mathrm{Tr}\,(\pi(a)\pi(a)^*)$ well-def. & cont$\}$

• $\mathcal{A}$ *has continuous trace* $\iff \overline{\mathcal{A}_0} = \mathcal{A}$
$\Rightarrow \widehat{\mathcal{A}}$ is loc. comp. Hausdorff and $\pi(\mathcal{A}) = \mathcal{K}(\mathcal{H}_\pi)$ for all $[\pi] \in \widehat{\mathcal{A}} \setminus \{[0]\}$

Example: $\mathcal{A} = \mathcal{C}_0(\Gamma, \mathcal{K}(\mathcal{H}))$ with Γ loc. comp. Hausdorff
$\Rightarrow \widehat{\mathcal{A}} \simeq \Gamma$ and $\mathcal{A}$ has continuous trace

Theorem (N.V. Pedersen, 1984)

If G is a nilpotent Lie group then there exist closed 2-sided ideals of $C^*(G)$

$$\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n = C^*(G)$$

with $\mathcal{J}_j/\mathcal{J}_{j-1}$ having continuous trace for $j = 1, \dots, n$

Question: Can we always arrange to have $\mathcal{J}_j/\mathcal{J}_{j-1} \simeq \mathcal{C}_0(\Gamma_j, \mathcal{K}(\mathcal{H}_j))$ $j = 1, \dots, n$?

Operator valued Fourier transforms of a C^*-algebra $\mathcal{A}$

- let $\Gamma \subseteq \widehat{\mathcal{A}}$
- select $\pi_\gamma \colon \mathcal{A} \to B(\mathcal{H}_\gamma)$ with $[\pi_\gamma] = \gamma$ for all $\gamma \in \Gamma$

$\rightsquigarrow$ *Fourier transform*

$$\mathcal{F}_\Gamma \colon \mathcal{A} \to \ell^\infty(\Gamma, \prod_{\gamma \in \Gamma} \mathcal{B}(\mathcal{H}_\gamma)), \quad a \mapsto \{\pi_\gamma(a)\}_{\gamma \in \Gamma}$$

Problem: What is the range of $\mathcal{F}_\Gamma$, particularly for $\mathcal{A} = C^*(G)$ & $\Gamma = \widehat{\mathcal{A}}$?

Continuity of operator fields

- Γ Hausdorff
- $(\forall \gamma \in \Gamma)\ \mathcal{H}_\gamma = \mathcal{H}$
- total subset $\mathcal{V} \subseteq \mathcal{H}$, dense $*$-subalg. $\mathcal{S} \subseteq \mathcal{A}$ satisfying

1. $(\forall a \in \mathcal{S})(\forall v_1, v_2 \in \mathcal{V}) \quad \Gamma \to \mathbb{C},\ \gamma \mapsto \langle \pi_\gamma(a) v_1, v_2 \rangle$ is continuous
2. $(\forall a \in \mathcal{S}) \quad \Gamma \to \mathbb{C},\ \gamma \mapsto \mathrm{Tr}\, \pi_\gamma(a)$ is well-defined & continuous

$\Longrightarrow (\forall a \in \mathcal{A}) \quad \mathcal{F}_\Gamma(a) \in \mathcal{C}_b(\Gamma, \mathcal{K}(\mathcal{H}))$

Tools from C^*-algebra extension theory

• Extension of C^*-algebras: $0 \to \mathcal{J} \hookrightarrow \mathcal{A} \xrightarrow{q} \mathcal{B} \to 0$
classified by *Busby's $*$-morphism* $\beta\colon \mathcal{B} \to M(\mathcal{J})/\mathcal{J}$

Example

$\mathcal{J} = \mathcal{C}_0(\Gamma, \mathcal{K}(\mathcal{H})) \Rightarrow M(\mathcal{J}) = \{\varphi\colon \Gamma \to \mathcal{B}(\mathcal{H}) \mid \varphi$ bounded strong*-cont.$\}$

• If the points of $\Gamma = \widehat{\mathcal{J}}$ are closed separated in $\widehat{\mathcal{A}}$, then
$\beta\colon \mathcal{B} \to \mathcal{C}_b(\Gamma, \mathcal{K}(\mathcal{H}))/\mathcal{C}_0(\Gamma, \mathcal{K}(\mathcal{H}))$ $(\subseteq M(\mathcal{J})/\mathcal{J})$

Choi-Effros completely positive lifting theorem

If $\mathcal{B}$ nuclear separable, then there exists $\nu\colon \mathcal{B} \to \mathcal{C}_b(\Gamma, \mathcal{K}(\mathcal{H}))$ linear, completely positive, $\|\nu\| \le 1$, satisfying

$$(\forall b \in \mathcal{B}) \quad \beta(b) - \nu(b) \in \mathcal{C}_0(\Gamma, \mathcal{K}(\mathcal{H})).$$

Also $\nu(b_1 b_2) - \nu(b_1)\nu(b_2) \in \mathcal{C}_0(\Gamma, \mathcal{K}(\mathcal{H}))$ for $b_1, b_2 \in \mathcal{B}$.

Examples: $C^*(G)$ is nuclear separable.

The class of nuclear separable C^*-algebras is closed under closed 2-sided ideals and quotients.

Boundary values of operator fields

- $\mathcal{A}$ separable C^*-algebra
- open sets $\emptyset = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \widehat{\mathcal{A}}$
- ideals $\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n = \mathcal{A}$ with $\widehat{\mathcal{J}_\ell} = V_\ell$, satisfying

1. $\Gamma_\ell := V_\ell \setminus V_{\ell-1}$ is dense in $\widehat{\mathcal{A}} \setminus V_{\ell-1}$;
2. there exist a complex Hibert space $\mathcal{H}_\ell$ and $\pi_\gamma \colon \mathcal{A} \to \mathcal{K}(\mathcal{H}_\ell)$ with $[\pi_\gamma] = \gamma$ for all $\gamma \in \Gamma_\ell$ such that for every $a \in \mathcal{A}$ the mapping $\Gamma_\ell \to \mathcal{K}(\mathcal{H}_\ell)$, $\gamma \mapsto \pi_\gamma(a)$ is norm continuous.

Define

$$\mathcal{L}_\ell := \{f \colon \widehat{\mathcal{A}} \setminus V_\ell \to \mathcal{K}(\mathcal{H}_{\ell+1}) \oplus \cdots \oplus \mathcal{K}(\mathcal{H}_n) \mid f(\gamma) \in \mathcal{K}(\mathcal{H}_j) \text{ if } \gamma \in \Gamma_j\}$$

and $\mathcal{F}_{\mathcal{A}/\mathcal{J}_\ell} \colon \mathcal{A}/\mathcal{J}_\ell \to \mathcal{L}_\ell$, $(\mathcal{F}_{\mathcal{A}/\mathcal{J}_\ell}(a + \mathcal{J}_\ell))(\gamma) := \pi_\gamma(a)$, $\ell = 0, \dots, n$.

There exist linear maps $\nu_\ell \colon \mathcal{F}_{\mathcal{A}/\mathcal{J}_\ell}(\mathcal{A}/\mathcal{J}_\ell) \to \mathcal{C}_b(\Gamma_\ell, \mathcal{K}(\mathcal{H}_\ell))$, which are completely positive, completely isometric, almost $*$-morphisms, with

$$\mathcal{F}_{\mathcal{A}}(\mathcal{A}) = \{f \in \mathcal{L}_0 \mid f|_{\Gamma_\ell} - \nu_\ell(f|_{\hat{\mathcal{A}} \setminus V_\ell}) \in \mathcal{C}_0(\Gamma_\ell, \mathcal{K}(\mathcal{H}_\ell)),\ \ell = 1, \dots, n-1\}$$

C^*-algebras of nilpotent Lie groups are solvable (1)

G nilpotent Lie group $\implies C^*(G)$ has a special solving series.

That is, a finite series of ideals $\{0\} = \mathcal{J}_0 \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n = \mathcal{A} := C^*(G)$ with $\mathcal{J}_j/\mathcal{J}_{j-1} \simeq \mathcal{C}_0(\Gamma_j, \mathcal{K}(\mathcal{H}_j))$ for $j = 1, \dots, n$, and moreover

1. $\widehat{\mathcal{A}}$ is a topological $\mathbb{R}$-space, Γ_j are $\mathbb{R}$-subspaces, $\widehat{\mathcal{A}} = \Gamma_1 \sqcup \cdots \sqcup \Gamma_n$
2. $\dim \mathcal{H}_n = 1$ and $\Gamma_n \simeq [\mathfrak{g}, \mathfrak{g}]^\perp$ as topological $\mathbb{R}$-spaces
3. $\dim \mathcal{H}_j = \infty$ if $j < n$, Γ_j is open dense, having closed and separated points in $\widehat{\mathcal{A}} \setminus \widehat{\mathcal{J}}_{j-1}$
4. $\Gamma_j \simeq$ semi-algebraic cone in a finite-dimensional vector space, which is a Zariski open set for $j = 1$, and its dimension is the *index of G*, denoted by $\operatorname{ind} G$.
5. there exists a homogeneous function $\varphi_j \colon \widehat{\mathcal{A}} \to \mathbb{R}$ such that $\varphi_j|_{\Gamma_1}$ is a polynomial function and

$$\Gamma_j = \{\gamma \in \widehat{\mathcal{A}} \mid \varphi_j(\gamma) \neq 0 \text{ and } \varphi_i(\gamma) = 0 \text{ if } i < j\}.$$

C^*-algebras of nilpotent Lie groups are solvable (2)

A *topological* $\mathbb{R}$-*space* is a topological space X with a continuous map $\mathbb{R} \times X \to X$, $(t, x) \mapsto t \cdot x$, and with a distinguished point $x_0 \in X$ satisfying

1. $(\forall x \in X) \quad 0 \cdot x = x_0$
2. $(\forall t, s \in \mathbb{R})(\forall x \in X) \quad t \cdot (s \cdot x) = ts \cdot x$
3. For every $x \in X \setminus \{x_0\}$ the map $\mathbb{R} \to X$, $t \mapsto t \cdot x$ is a homeomorphism onto its image.

An $\mathbb{R}$-*subspace* is any $\Gamma \subseteq X$ with $\mathbb{R} \cdot \Gamma \subseteq \Gamma \cup \{x_0\}$, so $\Gamma \cup \{x_0\}$ is a topological $\mathbb{R}$-space.

Examples: 1. Finite-dimensional $\mathbb{R}$-linear spaces are topological $\mathbb{R}$-spaces.
2. $G = (\mathfrak{g}, \cdot) \rightsquigarrow \widehat{G} \simeq \widehat{C^*(G)} \simeq \mathfrak{g}^*/\mathrm{Ad}_G^*$ topological $\mathbb{R}$-space via

$$t \cdot \mathcal{O}_\xi := \mathcal{O}_{t\xi}$$

where $\mathcal{O}_\xi = \mathrm{Ad}_G^*(G)\xi$. The linear space $[\mathfrak{g}, \mathfrak{g}]^\perp$ ($\simeq$ the singleton orbits) is an $\mathbb{R}$-subspace of $\mathfrak{g}^*/\mathrm{Ad}_G^*$.

C^*-algebras of nilpotent Lie groups are solvable (3)

- nilpotent Lie group $G = (\mathfrak{g}, \cdot)$ $G = (\mathfrak{g}, \cdot)$
- Jordan-Hölder sequence $\{0\} = \mathfrak{g}_0 \subseteq \cdots \subseteq \mathfrak{g}_m = \mathfrak{g}$
- duality pairing $\langle \cdot, \cdot \rangle \colon \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$
- coadjoint isotropy at $\xi \in \mathfrak{g}^*$: $\mathfrak{g}(\xi) := \{x \in \mathfrak{g} \mid \langle \xi, [x, \mathfrak{g}] \rangle = 0\}$
- jump set at $\xi \in \mathfrak{g}^*$: $J_\xi := \{j \in \{1, \dots, m\} \mid \mathfrak{g}_j \not\subset \mathfrak{g}(\xi) + \mathfrak{g}_{j-1}\}$
- $\mathcal{E}$ the set of all subsets of $\{1, \dots, m\}$

Piecewise continuity of trace wrt the coarse stratification

Define $\Omega_e := \{\xi \in \mathfrak{g}^* \mid J_\xi = e\}$ for $e \in \mathcal{E}$.

Coarse stratification: $\mathfrak{g}^* = \bigsqcup_{e \in \mathcal{E}} \Omega_e$, finite partition into G-invariant sets

$\rightsquigarrow$ $\boxed{\widehat{G} \simeq \mathfrak{g}^*/\mathrm{Ad}^*_G = \bigsqcup_{e \in \mathcal{E}} \Xi_e}$ where $\Xi_e := \Omega_e/\mathrm{Ad}^*_G$

For every $e \in \mathcal{E}$ one has:

1. The relative topology of $\Xi_e \subseteq \mathfrak{g}^*/\mathrm{Ad}^*_G$ is Hausdorff.
2. For every $\varphi \in \mathcal{C}_0^\infty(G)$ the function $\Xi_e \to \mathbb{C}$, $\mathcal{O} \mapsto \mathrm{Tr}\,(\pi_{\mathcal{O}}(\varphi))$ is well defined and continuous, where $[\pi_{\mathcal{O}}] \longleftrightarrow \mathcal{O}$.

C^*-algebras of nilpotent Lie groups are solvable (4)

Piecewise continuity wrt the refined stratification

Define $J^k_\xi := \{j \in \{1,\dots,k\} \mid \mathfrak{g}_j \not\subset \mathfrak{g}_k(\xi|_{\mathfrak{g}_k}) + \mathfrak{g}_{j-1}\}$ for $k = 1,\dots,m$, $\xi \in \mathfrak{g}^*$, and

$$(\forall \varepsilon \in \mathcal{E}^m) \quad \Omega_\varepsilon := \{\xi \in \mathfrak{g}^* \mid (J^1_\xi,\dots,J^m_\xi) = \varepsilon\}.$$

Fine stratification: $\mathfrak{g}^* = \bigsqcup_{\varepsilon\in\mathcal{E}^m} \Omega_\varepsilon$ finite partition into G-invariant sets

$\rightsquigarrow$ $\boxed{\widehat{G} \simeq \mathfrak{g}^*/\mathrm{Ad}^*_G = \bigsqcup_{\varepsilon\in\mathcal{E}^m} \Xi_\varepsilon}$ where $\Xi_\varepsilon := \Omega_\varepsilon/\mathrm{Ad}^*_G$

For $\varepsilon \in \mathcal{E}^m$ let $\Gamma_\varepsilon \subseteq \widehat{G}$ be the image of Ξ_ε through Kirillov's correspondence $\mathfrak{g}^*/\mathrm{Ad}^*_G \simeq \widehat{G}$.

For $\varepsilon \in \mathcal{E}^m$ there exist a Hilbert space $\mathcal{H}_\varepsilon$ & unirrep $\pi_\gamma\colon G \to \mathcal{B}(\mathcal{H}_\varepsilon)$ with $[\pi_\gamma] = \gamma$ for $\gamma \in \Gamma_\varepsilon$ such that the map $\Pi_a\colon \Gamma_\varepsilon \to \mathcal{B}(\mathcal{H}_\varepsilon)$, $\gamma \mapsto \pi_\gamma(a)$, is norm continuous for all $a \in C^*(G)$.

Prf. 1. Weak continuity for $a \in \mathcal{C}^\infty_0(G)$ suffices since trace continuity holds on Γ_ε.

C^*-algebras of nilpotent Lie groups are solvable (5)

2. *Models of representations via canonical coordinates on coadjoint orbits.* Let $2d = \dim \mathcal{O}$ for all $\mathcal{O} \in \Xi_\varepsilon$ and $\mathcal{H}_\varepsilon := L^2(\mathbb{R}^d)$.

(2a) Let $p_1, \dots, p_d, q_1, \dots, q_d$ be the coordinate functions on $\mathbb{R}^{2d}$. Then

$$\mathcal{E}^1(\mathbb{R}^{2d}) := \{\varphi \in \mathcal{C}^\infty(\mathbb{R}^{2d}) \mid \varphi = a_{\varphi,0}(q) + \sum_{j=1}^d a_{\varphi,j}(q)p_j\}$$

is a Lie algebra wrt the Poisson bracket, and $Q\colon \mathcal{E}^1(\mathbb{R}^{2d}) \to \mathrm{Diff}\,(\mathbb{R}^d)$,

$$Q(\varphi)f = \sum_{j=1}^d a_{\varphi,j}\partial_j f + \Big(\mathrm{i}a_{\varphi,0} + \frac{1}{2}\sum_{j=1}^d \partial_j a_{\varphi,j}\Big) f$$

is a Lie algebra morphism into the skew-symmetric differential operators.

(2b) There exist a semi-alg. set T and a homeo. $\Psi\colon T \times \mathbb{R}^{2d} \to \Xi_\varepsilon$ with

1. for every $t \in T$, $\Psi_t := \Psi(t,\cdot)$ is a symplectomorphism from $\mathbb{R}^{2d}$ onto a coadjoint orbit $\mathcal{O}_t$ of G;
2. $\psi^x := \langle \cdot, x\rangle$ on $\mathcal{O}_t$ satisfies $\psi^x \circ \Psi_t \in \mathcal{E}^1(\mathbb{R}^{2d})$ for $t \in T$, $x \in \mathfrak{g}$.
3. For all $t \in T$, $\boxed{\rho_t\colon \mathfrak{g} \to \mathrm{Diff}\,(\mathbb{R}^d)}$, $\rho_t(X) := Q(\psi^X \circ \Psi_t)$, is a Lie algebra morphism.
4. $f_1, f_2 \in \mathcal{C}_0^\infty(\mathbb{R}^d) \Rightarrow T \times \mathfrak{g} \to \mathbb{C},\ (t,x) \mapsto \langle \rho_t(x)f_1, f_2\rangle$ is continuous.

Uniqueness of Heisenberg groups via solvable C^*-algebras

$G = (\mathfrak{g}, \cdot)$ nilpotent Lie group $\Rightarrow$ Equivalent properties:
(1) $0 \to \mathcal{C}_0(\Gamma_1, \mathcal{K}(\mathcal{H})) \to C^*(G) \to \mathcal{C}_0([\mathfrak{g},\mathfrak{g}]^\perp) \to 0$ exact sequence with

- Γ_1 dense open $\mathbb{R}$-subspace of $\widehat{G}$ that is homeomorphic to $\mathbb{R} \setminus \{0\}$;
- $\mathcal{H}$ separable infinite-dimensional complex Hilbert space.

(2) There exists $d \geq 1$ with $\dim[\mathfrak{g},\mathfrak{g}]^\perp = 2d$ and $G \simeq \mathbb{H}_{2d+1}$.

Prf. (1) $\Rightarrow$ (2)
• $\widehat{G} \simeq \mathfrak{g}^*/\mathrm{Ad}_G^*$ via Kirillov's correspondence $\rightsquigarrow \mathfrak{g}^*/\mathrm{Ad}_G^* = \Gamma_1 \sqcup [\mathfrak{g},\mathfrak{g}]^\perp$
• $\mathcal{O}_\xi :=$ the coadjoint orbit of every $\xi \in \mathfrak{g}^*$
• G has infinite-dimensional unirreps $\Rightarrow$ G is non-commutative
$\Rightarrow (\exists \xi_1 \in \mathfrak{g}^*)\ \mathcal{O}_{\xi_1} \neq \{\xi_1\} \Rightarrow \mathfrak{g}^* = \bigsqcup\limits_{t \in \mathbb{R}\setminus\{0\}} \mathcal{O}_{t\xi_1} \sqcup [\mathfrak{g},\mathfrak{g}]^\perp$
$\Rightarrow (\exists x, y \in \mathfrak{g})\ z := [x,y] \in \mathcal{Z}(\mathfrak{g}) \setminus \{0\},\ \langle \xi_1, z\rangle \neq 0$
$\Rightarrow [\mathfrak{g},\mathfrak{g}] = \mathbb{R}z \Rightarrow (\exists d \geq 1, k \geq 0)\ \mathfrak{g} = \mathfrak{h}_{2d+1} \times \mathbb{R}^k \Rightarrow \mathrm{ind}\,\mathfrak{g} = k+1$
• Γ_1 is a dense open subset of $\widehat{G}$ that is homeomorphic to $\mathbb{R} \setminus \{0\}$
$\Rightarrow \mathrm{ind}\, G = 1 \Rightarrow k = 0 \Rightarrow \mathfrak{g} = \mathfrak{h}_{2d+1}$