Genericity in Equivariant Dynamical Systems and Equivariant Fuller Index Theory

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Introduction

Index theory is a branch of mathematics which aims to assign algebraic invariants to certain topological objects, encoding more or less substantial information on the object. The most common example is the Brouwer degree, which was established at the beginning of the 20th century. It assigns an integer to continuous maps of euclidean spaces and, roughly speaking, it is an algebraic measure of the cardinality of the preimage of a point. There are several features which qualify such an assignment to be called an index. Foremost, the potential index should be invariant under homotopies. This implies that it will be locally constant, but also contains information on large perturbations in a certain way. Secondly, non-vanishing of the index should imply existence of some critical element connected to the objects under consideration. In the case of the Brouwer degree, its non-vanishing implies existence of zeros (or more generally, existence of points with a certain value).

There are several generalizations in different directions. Clearly it is no problem to define an index for maps of manifolds and counting fixed points instead of zeros, by locally passing to the map $\mathbb{1} - f$. In his fundamental work [Ful67], Fuller defines an index for vector fields which counts periodic orbits of the flow instead of fixed points of a map. There are several obstacles one has to overcome. A major problem is the period of a given orbit, which need not be minimal. This results in the fact that the Fuller index is a rational number rather than an integer.

Fullers original work uses differential forms and so actually de Rham cohomology. There were several reformulations of his theory, namely an approach using bifurcation theory, initally developed by Chow and Mallet-Paret [CMP78], and an approach using singular homology, developed by Franzosa [Fra90]. The bifurcation theoretic approach is of particular interest for us. It only uses the local structure of a flow around a periodic orbit and assigns local indices to isolated periodic orbits. These indices are summed up in a way such that the sum remains unchanged during homotopies of the vector field. This requires a thorough understanding of bifurcations of periodic orbits. The general idea is to use Poincaré systems and assign to a periodic orbit the fixed point index of an associated Poincaré map.

A generalization in another direction is to take symmetries into account. That is, one has manifolds with an action of a compact group G and looks at equivariant maps. An equivariant Brouwer degree was defined by Ize and others, compare [IV03], [BKS06]. It counts group orbits of zeros of an equivariant map, that is, non-vanishing of certain components of the equivariant degree implies the existence of a group orbit of zeros of a certain symmetry. In this case, however, there is no obvious generalization to an index counting fixed points. The reason is that counting fixed points in fact is not enough. In the equivariant setting, one should look for fixed group orbits of a map, that is, one seeks to solve the equation $f(x) = gx$ for some $g \in G$. But the map $g^{-1} \circ f$ will in general not be equivariant, so even if one can count group orbits of fixed points, it is still not clear how to count a fixed orbit which has a non-trivial action of f on itself. In the direction of orbits of fixed points, Lück and Rosenberg developed an invariant counting these including symmetries, compare [LR03]. In the direction of fixed orbits, Zdzislaw

Dzedzej in 2001 defined a local index for a fixed orbit of a G-map. Summing up the local indices, if possible, one obtains a G-homotopy invariant counting fixed orbits. In 2003, Chorny in [Cho03] defined an equivariant Lefschetz number that can count fixed orbits of self-maps according to their symmetry. He also did so in a setting far more general than group actions on topological spaces. It is to be expected that the fixed orbit index is a local version of the equivariant Lefschetz number, though this remains to be proven.

The existence of an equivariant fixed orbit index suggests that it should be possible to define an equivariant Fuller index, assigning to relative periodic orbits of equivariant flows the orbit index of an associated equivariant Poincaré map. To the authors knowledge, this idea has not been carried out yet, nor has any sort of equivariant Fuller index been defined. The main goal of this work is the proof that the definition of an equivariant Fuller index as indicated above satisfies the properties an equivariant index should have.

The work is build up as follows. In the first chapter, we develop the theory of critical elements of maps and vector fields without symmetries, namely, we identify certain open and dense subsets in spaces of maps (vector fields) for which a direct assignment of an index is possible. The theorems for maps and vector fields are well-known, the main new feature of the presentation here is a unified treatment. This results in some modifications of the techniques of proof. The theorems for homotopies of maps and vector fields seem to be known as well, however, the author could not find any reference in which they are proven. This essentially can be done by giving parametrized versions of the various lemmas leading to the non-parametrized results.

We proceed to define the classical Fuller index in a bifurcation theoretic way and prove its most important properties. Some of the proofs, especially the one for homotopy invariance, are new and were constructed to be easily generalizable to systems with symmetry. In particular, we emphasize the use of the fixed point index of Poincaré maps.

In the second chapter, we proceed in the same way for systems with symmetry. However, the main notion of non-degeneracy in the equivariant setting is G-hyperbolicity, and this property is not flexible enough to establish the genericity results we need. So we begin instead with the development of a notion of equivariant non-degeneracy, which is rooted in the theory of jet transversality, developed by Bierstone in [Bie77b]. We will not use the full complexity of his results, since we only have to deal with zero jets. On the other hand, we have to make a slight generalization, since we need transversality to locally semialgebraic sets rather than submanifolds. However, this generalization comes at almost no cost because Bierstone in fact never makes uses of the manifold structure. The equivariant Thom-Mather Theorem 2.2.3.6 related to this theory is the main achievement here and allows to proceed further to genericity in equivariant systems.

Using the notion of equivariant non-degeneracy, we prove the genericity theorems for equivariant systems analogous to the theorems of chapter one. Again, the theorems for G-maps and G-vector fields are well-known, at least the versions involving G-hyperbolicity. Our use of equivariant non-degeneracy simplifies some of the proofs, but it comes to essential use only when it comes to proving the parametrized versions, i.e. genericity theorems for equivariant homotopies.

Finally, the third chapter is devoted to the construction of the equivariant Fuller index, based on the fixed orbit index of Dzedzej, and the subsequent proofs of its main properties, which are based on the genericity theorems developed in chapter two.

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1 Genericity in Non-Equivariant Dynamical Systems

The first chapter is an overview over the theory of genericity in dynamical systems. The term genericity will be used ambiguously. Most of the time, "generic" will simply mean that a certain property holds in an open and dense subset of an appropriate space. One has to be careful with the topologies, for some openness results hold in the $\mathcal{C}^r$-topology for some finite r, whereas density will usually hold in the $\mathcal{C}^\infty$-topology. So the various aspects of this terminology must be distinguished. However, we mostly use the term to help the intuition and will be more precise when it comes to concrete statements. Furthermore, the exact degree of differentiability which can be obtained is not of interest for us and we will be satisfied if the results hold for some $r > 0$, possibly $r = \infty$.

Most of the results of this chapter are well-known in one way or the other. The main purposes to gather all of them are threefold. First of all, the results of the second chapter heavily depend on the results of the first. Either we have to use them directly, for example as a base for an induction, or the techniques can be generalized to systems with symmetry. Since symmetry sometimes clouds the view on the essentials, one can often grasp the basic ideas of proofs and constructions by going into the non-symmetric case. Secondly, there seems to be no exhaustive treatment of genericity in our sense in the literature, especially the genericity theorems for homotopies remain rather obscure. So it seemed worthwhile to bring all the various results together in one place, proving them properly and emphasizing the interplay between them. Finally, in the first chapter we will lay the ground for all our notation and conventions, so the reader will know what we are talking about in later chapters after having read through chapter one.

The chapter is build up as follows. We begin with a short introduction on topologies of mapping spaces, followed by an overview of transversality theory, which will be basic for a large part of the work. Then we give an overview of the foundations of the theory of dynamical systems and bifurcation theory, mainly to establish a common notation. The main new aspect of these sections is the notion of a Poincaré system, which differs a bit from the usual one. Namely, we do not require the system to be centered at a fixed point. This is due to the fact that we want to capture jug handle bifurcations of periodic orbits by pushing a Poincaré system across the bifurcation parameter. But then, on one side of the parameter we will have two fixed points, whereas on the other side there will be none. In the classical sense, this is not possible using a Poincaré system. We will go into detail later.

What follows are the main genericity theorems in the non-equivariant setting. We will prove density of non-degenerate maps, vector fields and homotopies of these objects. In most cases, openness of these sets will be proven as well. We will also sharpen some of the theorems to the case of hyperbolic objects. This is not necessary to do generic bifurcation theory, but when it comes to calculations, it is much easier to work with hyperbolic objects rather than with non-degenerate ones. Also, it is not at all difficult to extend the results to the hyperbolic case, in some cases, the proofs are even identical besides changing some minor wording.

In the final part of the first chapter we will apply our theorems to develop the well-known Fuller index and give a new proof of its homotopy invariance (though it is closely related to the proof in [CMP78]).

We will give references to literature in the respective sections. Here it shall be said that most of the material on genericity is taken from the very readable book of Palis and de Melo [PdM82]. That book does not deal with bifurcation theory, but one can easily modify many of its results to parametrized versions, i.e. versions for homotopies of the given objects. The main other source were the works of Mike Field [Fie77], [Fie80], [Fie91], [Fie07] on equivariant bifurcation theory. For more details on this topic, see also [CL00].

1.1 Preliminaries

As already mentioned in the introduction, in this opening section we are mainly concerned with establishing a common background and notation. Some of the notions established here will be generalized several times, e.g. the notion of transversality to a submanifold, and it is important to keep the origin in mind. The non-standard part will be section 1.1.4 with its definition of Poincaré systems, branches of critical elements and some non-standard propositions.

1.1.1 Topologies on Mapping Spaces

Since we are concerned with genericity of maps between smooth manifolds, we have to specifiy topologies on spaces of such maps which will allow us to prove openness and density results. There are mainly two topologies on the mapping space $\mathcal{C}^r(M,N)$, the space of $\mathcal{C}^r$-maps between two smooth manifolds M and N. The first only uses the structure of a manifold and therefore is the more elementary one. The second uses the Whitney embedding theorem which allows us to assume $M \subseteq \mathbb{R}^p, N \subseteq \mathbb{R}^q$ for some large p,q where, in fact, $p = 2\dim M, q = 2\dim N$ is possible. Luckily, we will see that these definitions coincide in the cases of interest, so we can use both topologies, whenever necessary.

Intuitively, $\mathcal{C}^r$-maps should be close to one another if the induced maps in some chart are close to one another for every chart of given atlasses of the manifolds. This is the idea of the *Whitney topology* on $\mathcal{C}^r(M,N)$. Let $f : M \to N$ be a $\mathcal{C}^r$-map and let $(U_i,\varphi_i)_{i\in I}$ be a locally finite subatlas of a given maximal atlas for M, $K_i \subseteq U_i$ compact sets such that $f(K_i) \subseteq V_i$, where (V_i,ψ_i) is a chart of N for $i \in I$. Let $\varepsilon_i > 0$ be real numbers for $i \in I$. We define a basic neighbourhood of f by

$$\begin{aligned} &\mathcal{U}(f,(K_i,U_i,\phi_i)_{i\in I},(V_i,\psi_i)_{i\in I},\{\varepsilon_i\}_{i\in I}) \\ &\quad = \left\{ g : M \to N \mid g(K_i) \subseteq V_i,\ \left\| \psi_i \circ g\big|_{K_i} \circ \varphi_i^{-1} - \psi_i \circ f\big|_{K_i} \circ \varphi_i^{-1} \right\|_{\mathcal{C}^r(\varphi(K_i),\mathbb{R}^n)} < \varepsilon_i \right\}. \end{aligned}$$

Then these sets, for the various choices of $(U_i,\varphi_i),(V_i,\psi_i),K_i,\varepsilon_i$ form a neighbourhood basis for f and thus constitute a topology on $\mathcal{C}^r(M,N)$, called the *Whitney-$\mathcal{C}^r$-topology* (see [PdM82], chapter 0). From the definition it is clear that $\mathcal{C}^r(M,N) \subseteq \mathcal{C}^{r-1}(M,N)$ as topological spaces and that the topology on $\mathcal{C}^0(M,N)$ is the compact-open topology. Furthermore, by taking the limit topology under the filtration $\mathcal{C}^\infty(M,N) \subseteq \cdots \subseteq$

$\mathcal{C}^r(M,N) \subseteq \ldots$, we also obtain a topology on the space of $\mathcal{C}^\infty$-maps which we will call the Whitney-$\mathcal{C}^\infty$-topology.

We now turn to the second method of constructing topologies on $\mathcal{C}^r(M,N)$. We begin by assuming $M = \mathbb{R}^p$, $N = \mathbb{R}^q$. Then for every compact subset K of M, we have the norm topology on $\mathcal{C}^r(K,N)$. For an $f \in \mathcal{C}^r(M,N)$, $K \subseteq M$ compact and $\varepsilon > 0$, we define a basic neighbourhood

$$\mathcal{N}(f,K,\varepsilon) = \{g \in \mathcal{C}^r(M,N) \mid \left\| f\big|_K - g\big|_K \right\|_{\mathcal{C}^r(K,N)} < \varepsilon\}.$$

This constitutes a topology on $\mathcal{C}^r(M,N)$ which we call the $\mathcal{C}^r$-topology.

In the general case, let $i : M \to \mathbb{R}^p$, $j : N \to \mathbb{R}^q$ be embeddings of M, N into some euclidean spaces. Let U be a tubular neighbourhood of $i(M)$ in $\mathbb{R}^p$, $r : U \to i(M)$ the tubular retraction. Let $\rho : \mathbb{R}^p \to [0,1]$ be a smooth Urysohn function such that $\rho^{-1}(0) = \overline{\mathbb{R}^p - U}$ and $\rho^{-1}(1) = \overline{i(M)}$. We define an extension of a map $f : M \to N$ to be the map

$$\tilde{f} : \mathbb{R}^p \to \mathbb{R}^q, \quad \tilde{f}(x) = \rho(x) \cdot j \circ f \circ i^{-1}(rx),$$

which we interprete as 0 if rx is not defined. Using this construction, we can view $\mathcal{C}^r(M,N)$ as a subspace of $\mathcal{C}^r(\mathbb{R}^p, \mathbb{R}^q)$ and provide it with the subspace topology.

One can check that this defines a topology on $\mathcal{C}^r(M,N)$ which is independent of the choices made during the construction. It is called the *weak topology* on $\mathcal{C}^r(M,N)$. It is again obvious that $\mathcal{C}^r(M,N) \subseteq \mathcal{C}^{r-1}(M,N)$ in these topologies, so we obtain a limit topology on $\mathcal{C}^\infty(M,N)$.

The relationship between these two topologies is clarified by the next proposition.

Proposition 1.1.1.1 *Let M, N be compact manifolds.*

1. *The weak topology on $\mathcal{C}^r(M,N)$ coincides with the $\mathcal{C}^r$-Whitney topology on $\mathcal{C}^r(M,N)$ for $0 \leq r \leq \infty$.*
2. *$\mathcal{C}^r(M,N)$ becomes a separable space of second category.*
3. *$\mathcal{C}^k(M,N)$ is dense in $\mathcal{C}^r(M,N)$ for $0 \leq r \leq k \leq \infty$.*
4. *$\mathcal{C}^r(M,N)$ is a Banach manifold.*

Proof. 1.-3. is established in chapter one, §2 of [PdM82]. 4. is proven in [AR67]. □

By using 4. of the last proposition, the following result is clear. We will give a different proof, however, since this is not difficult and the Banach manifold structure is not easy to grasp.

Lemma 1.1.1.2 *Let $f : M \to M$ be a $\mathcal{C}^r$-map and $\mathcal{V}$ a neighbourhood of f. Then there is a neighbourhood $\mathcal{U} \subseteq \mathcal{V}$ of f such that all elements of $\mathcal{U}$ are homotopic to f via a homotopy not leaving $\mathcal{U}$.*

Proof. Let $i : M \to \mathbb{R}^N$ be an embedding into some euclidean space $\mathbb{R}^N$ and let $U \subseteq \mathbb{R}^N$ be a tubular neighbourhood of $i(M)$, $r : U \to i(M)$ the tubular retraction. The set

$$\mathcal{U} = \{g \in \mathcal{V} \mid t \cdot i \circ f(x) + (1-t) \cdot i \circ g(x) \in U \; \forall t \in [0,1], x \in M\}$$

is an open subset of $\mathcal{V}$. For $g \in \mathcal{U}$, define

$$H_g : M \times [0,1] \to M, \quad H_g(x,t) = i^{-1} \circ r(t \cdot i \circ f(x) + (1-t) \cdot i \circ g(x)).$$

Then H_g is $\mathcal{C}^r$ and a homotopy between f and g. □

1.1.2 Transversality Theory

The notion of transversality of a map to a submanifold is fundamental to the theory of genericity. It can be reduced to the question of general position of submanifolds, for $f : M \to N$ is transverse to a submanifold P of N if and only if the graph of f is in general position to the submanifold $M \times P$ of $M \times N$. General position of submanifolds is a geometric concept accessible to intuition. Take two submanifolds P, Q of some euclidean space $\mathbb{R}^m$. The manifolds are in general position, if the intersection is stable under small perturbations. That is, if Q_v is the manifold Q translated by the vector v, the intersections $P \cap Q_v$ are diffeomorphic for all sufficiently small $v \in \mathbb{R}^m$. Figure 1 shows some intersections of two spheres in $\mathbb{R}^3$. In (a) and (c), the spheres are in general position, in (b) they are not.

As mentioned above, transversality of a map to a submanifold could be defined using this concept of general position. But there is also another definition, the standard definition, doing the same, and handling more directly the concept of transversality of a mapping.

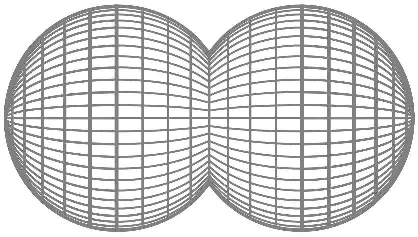
Definition 1.1.2.1 *Let M, N be arbitrary smooth manifolds, $P \subseteq N$ a submanifold. A smooth map $f : M \to N$ is said to be* transverse *to P at $x \in M$, if either $f(x) \notin P$, or else*

$$T_x f(T_x M) + T_{f(x)} P = T_{f(x)} N.$$

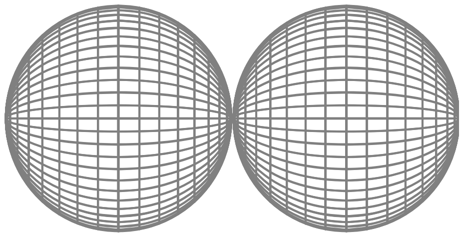
It is said to be transverse *to P* at a subset *$A \subseteq M$, if it is transverse to P at every $x \in A$. If a map is transverse to P at all of M, we simply say it is* transverse to *P.*

In view of this definition, two submanifolds $P, Q \subseteq M$ are in general position, if the inclusion $i : P \to M$ is transverse to Q (or vice versa). We give a useful alternative characterization of transversality.

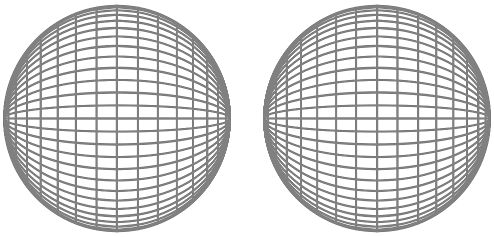
Lemma 1.1.2.2 *Let $f : M \to N$ be a smooth map of manifolds, $P \subseteq N$ a submanifold, $x \in M$ and $f(x) \in P$. Let U be a neighbourhood of x, $\varphi : U \to \mathbb{R}^m$ be a chart mapping x to zero, $V \subseteq N$ a neighbourhood of $f(x)$ in N, $\psi : V \to \mathbb{R}^n$ a chart mapping $f(x)$ to 0 and $P \cap V$ onto the subspace $\mathbb{R}^p \times \{0\} \subseteq \mathbb{R}^n$, $p = \dim P$. We can achieve $f(\overline{U}) \subseteq V$. Let $\pi_2 : \mathbb{R}^n \to \{0\} \times \mathbb{R}^{n-p}$ be the projection. Then f is transverse to P at x if and only if $\pi_2 \circ \psi \circ f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^{n-p}$ has a surjective differential at 0.*



(a) transverse



(b) not transverse



(c) transverse

Figure 1: Intersections of 2-spheres

Proof. Proposition 1.3.1 of [PdM82]. $\square$

A basic feature of submanifolds in general position is that their intersection is again a submanifold. In terms of transversality of a map, this reads as follows.

Proposition 1.1.2.3 *Let $f : M \to N$ be a smooth map of smooth manifolds,* $\dim M = m, \dim N = n$*, $P \subseteq N$ a p-dimensional submanifold. If f is transverse to P, $f^{-1}(P)$ is either empty, or an $m - n + p$-dimensional submanifold of M.*

Proof. This is immediate from Lemma 1.1.2.2 and the regular value theorem. Locally, $f^{-1}(P)$ is the preimage of 0 under a map $\mathbb{R}^m \to \mathbb{R}^{n-p}$ which has 0 as a regular value. Hence, this is a manifold of dimension $m - n + p$. $\square$

Proposition 1.1.2.3 provides us with an easy criterion for a map to have no values in a given submanifold: If $m - n + p < 0$ and f is transverse to P, then $f^{-1}(P)$ must be empty.

We turn to the main result in basic transversality theory, namely the transversality theorem of Thom. It states that the set of smooth maps transverse to a given submanifold is generic. This allows the identification of many other generic subsets by describing them through transversality properties.

Theorem 1.1.2.4 (Thom's Transversality Theorem) *Let M, N be smooth manifolds, $P \subseteq N$ a closed submanifold. Then the set of maps $f : M \to N$ that are transverse to P is residual in $\mathcal{C}^\infty(M, N)$, i.e. it is the countable intersection of open and dense sets. If M is compact, this set is open.*

Proof. Theorem 1.3.4 in [PdM82]. □

Note that, since $\mathcal{C}^\infty(M, N)$ is a Baire space, residual subsets are dense.

Corollary 1.1.2.5 *Let $K \subseteq M$ be a compact subset of any manifold M, $P \subseteq N$ a closed submanifold. Then the set of maps in $\mathcal{C}^\infty(M, N)$ that are transverse to P at K is open and dense.*

Proof. Density is clear, since already the maps transverse to P in all of M are dense. But openness follows immediately from the characterization of transversality via surjectivity of a differential in Lemma 1.1.2.2, since the set of surjective linear maps is open and K is compact. □

The next result is important when dealing with genericity of non-bifurcation parameters. It relates transversality of a parametrized map to transversality of the maps induced at a fixed parameter. We will come across similar theorems in later chapters.

If $f : X \times Y \to Z$ is any map, $x \in X, y \in Y$, we denote by $f_x : Y \to Z$ the map $y \mapsto f(x, y)$, similarly $f_y : X \to Z$, $x \mapsto f(x, y)$. We will never use the symbols f_x, f_y for partial derivatives, so no confusion should be possible.

Proposition 1.1.2.6 *Let M, N, Λ be manifolds. Let $P \subseteq N$ be a compact submanifold and $F : M \times \Lambda \to N$ be transverse to P. Then the set*

$$\Lambda_P = \{\lambda \in \Lambda \mid F_\lambda \textit{ is transverse to } P\}$$

is open and dense in Λ.

Proof. Openness: By Thom's Transversality Theorem, if F_λ is transverse to P for some $\lambda \in \Lambda$, then so is every map in a neighbourhood $\mathcal{U}$ of F_λ. In particular, we find a neighbourhood $U \subseteq \Lambda$ of λ such that $F_\mu \in \mathcal{U}$ for $\mu \in U$.

Density: Since F is transverse to P, $F^{-1}(P)$ is either empty, in which case we are done, or a submanifold of $M \times \Lambda$ and we have

$$T_{(x,\lambda)}F(T_{(x,\lambda)}(M \times \Lambda)) + T_{F(x,\lambda)}P = T_{F(x,\lambda)}N.$$

Now,
$$T_{(x,\lambda)}F(T_{(x,\lambda)}(M\times\Lambda)) = T_\lambda F_x(T_\lambda\Lambda) + T_x F_\lambda(T_x M).$$
By Sards theorem, the set of regular values of the map $\pi_2 : F^{-1}(P) \to \Lambda$ is dense. Hence, in a dense subset of Λ, either $F_\lambda(M) \cap P = \emptyset$, in which case F_λ is trivially transverse to P, or else for $(x,\lambda) \in F^{-1}(P)$,
$$T_{(x,\lambda)}\pi_2(T_{(x,\lambda)}F^{-1}(P)) = T_\lambda\Lambda.$$
We have a splitting
$$T_{(x,\lambda)}F^{-1}(P) = T_{(x,\lambda)}\pi_1(T_{(x,\lambda)}F^{-1}(P)) \times T_{(x,\lambda)}\pi_2(T_{(x,\lambda)}F^{-1}(P)).$$
This yields
$$\begin{array}{rcl} T_{F(x,\lambda)}P & \supseteq & T_{(x,\lambda)}F(T_{(x,\lambda)}F^{-1}(P)) \\ & = & T_{(x,\lambda)}F\left(T_{(x,\lambda)}\pi_1(T_{(x,\lambda)}F^{-1}(P)) \times T_{(x,\lambda)}\pi_2(T_{(x,\lambda)}F^{-1}(P))\right) \\ & = & T_{(x,\lambda)}F(T_{(x,\lambda)}\pi_1(T_{(x,\lambda)}F^{-1}(P)) \times T_\lambda\Lambda) \\ & \supseteq & T_{(x,\lambda)}F(\{0\} \times T_\lambda\Lambda) \\ & = & T_\lambda F_x(T_\lambda\Lambda). \end{array}$$
So we see that
$$\begin{array}{rcl} T_xF_\lambda(T_xM) + T_{F_\lambda(x)}P & = & T_xF_\lambda(T_xM) + T_\lambda F_x(T_\lambda\Lambda) + T_{F_\lambda(x)}P \\ & = & T_{(x,\lambda)}F(T_{(x,\lambda)}(M\times\Lambda)) + T_{F(x,\lambda)}P \\ & = & T_{F(x,\lambda)}N, \end{array}$$
showing transversality of F_λ. □

Note that in fact, with the above proof, we have shown that the parameters λ such that F_λ is transverse to P are just the regular values of the projection map $\pi : F^{-1}(P) \to \Lambda$. This observation will be useful later on.

1.1.3 Discrete Semi-Dynamical Systems

The next two sections shall establish the basic notions of dynamical systems and bifurcation theory. There are many different ways how a dynamical system can be defined mathematically exact, the common basis being the evolution of a state in time. Since a large part of this work is concerned with the theory of group actions, we define a dynamical system to be given by an action of $\mathbb{Z}$, in the disrete case, or $\mathbb{R}$, in the continuous case, on some manifold M. That is, if G is $\mathbb{R}$ or $\mathbb{Z}$, a *dynamical system* on a smooth manifold M is given by a smooth map
$$\varphi : M \times G \to M$$

satisfying $\varphi(x,0) = x$, $\varphi(x,a+b) = \varphi(\varphi(x,a),b)$. Note that a dynamical system is, in this definition, always a right action, whereas symmetry groups usually will act on the left. We will return to the general notion of a group action in chapter two.

The investigations of such group actions naturally focus on the points with non-trivial stabilizers first. That is, we are looking for points being fixed by a non-trivial subgroup of $\mathbb{R}$ (in the continuous case), called the *stabilizer* of the point. Since such a subgroup necessarily is closed, it must be either all of $\mathbb{R}$, or else it must be isomorphic to $\mathbb{Z}$. In the first case, we are dealing with fixed points of the dynamical system, while in the second case, we have periodic orbits. The same is true in the case of discrete systems, where the stabilizers are either all of $\mathbb{Z}$, or else are given by $n\mathbb{Z}$ for some $n \in \mathbb{N}$. Elements with non-trivial stabilizers are called *critical elements* of the system.

One is interested in systems with two properties. Firstly, the system should have a simple structure of critical elements, for example, just a finite number. Secondly, the behaviour of the system locally around a critical element should be easy to understand. In addition, one would like to know that almost all systems exhibit this simple behaviour, e.g. systems taken from an open and dense set. To find systems having the first two properties, the notions of non-degeneracy and hyperbolicity are introduced. As we will see, non-degenerate systems have a finite number of critical elements, whereas hyperbolic systems have a finite number of critical elements and a local dynamical behaviour around these elements which is easy to understand. In the following sections, genericity of non-degenerate and hyperbolic systems will be established, which will complete the outlined program.

Having understood the generic structure of dynamical systems, it is natural to ask how the system changes under large perturbations, that is, under homotopies. A major aim of bifurcation theory is to find a way to relate the structure of an initial system with the structure of the final system of a homotopy, possibly under some genericity assumptions. We will see that generically, a homotopy will not change the topological structure of the set of critical elements, except for finitely many so called *bifurcations.* Together with our genericity results for fixed systems, we will see that the large picture should be as indicated in Figure 2. At the initial parameter, we begin with finitely many critical elements. These elements start to run through the homotopy without interfering with each other. Then, at a first bifurcation parameter, some of these "branches" can merge. Whatever happens, after passing the bifurcation parameter, we end up again with finitely many critical elements. Such bifurcations occur finitely many times until we end up with the final system. Our aim is to show that this picture is the generic picture for all the critical elements we will deal with, namely fixed points of maps, periodic orbits of flows and the equivariant analoga.

Note that in many cases, we can do even better than indicated in Figure 2. The set of critical elements will generically be a manifold of dimension one or dimension two, according to whether we are dealing with fixed points or periodic orbits, respectively. The situation is not as simple for equivariant critical elements and the bifurcation pattern indicated in the figure is what we will use in proofs. So we are satisfied with this weaker result.

We now turn our attention to the discrete time case. The objects under consideration

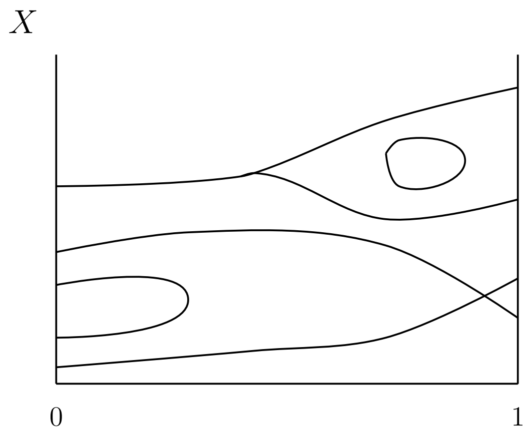


Figure 2: Generic bifurcation of critical elements

are iterates of a self-map $f : M \to M$. To be a dynamical system in the above sense, f must be a diffeomorphism. We will be a bit more general and take f arbitrary. This gives a semi-dynamical system, because we only have the semigroup $\mathbb{N}$ acting on M, but this generalization does not make anything harder. Critical elements of f are *fixed points*, i.e. points $x \in M$ such that $f(x) = x$, or *periodic points*, i.e. points $x \in M$ such that $f^n(x) = x$ for some $n \in \mathbb{N}$, $n > 1$. We will only be interested in fixed points.

Fix a smooth manifold M and a smooth self map $f : M \to M$. A fixed point $x \in M$ of f is called *non-degenerate*, if the differential $T_x f$ of f at x has not 1 as an eigenvalue. It is called *hyperbolic*, if $T_x f$ has no eigenvalues of absolute value 1. The meaning of this definition is clear. If a fixed point is non-degenerate, then the map $\mathbb{1} - f$, defined locally around x, has invertible derivative in x. Hence by the inverse function theorem, x is isolated. An important observation is that non-degeneracy of a fixed point is in fact a transversality property.

Proposition 1.1.3.1 *Let $f : M \to M$ be smooth. A fixed point $x \in M$ of f is non-degenerate, if and only if the map $F : M \to M \times M$, $x \mapsto (x, f(x))$ is transverse to the diagonal $\Delta = \Delta(M) = \{(x, x) \mid x \in M\}$ at x.*

Proof. The map F is transverse to Δ in x if and only if

$$T_x F(T_x M) + T_{(x,x)}\Delta = T_{(x,x)} M \times M.$$

But $T_x F(T_x M) = \{(v, T_x f v) \mid v \in T_x M\}$ and $T_{(x,x)}\Delta = \{(w, w) \mid w \in T_x M\}$. So F is transverse to Δ in x if and only if the map $\mathbb{1} - T_x f : T_x M \to T_x M$ is surjective, i.e. 1 is not an eigenvalue of $T_x f$. □

Of course, a map $f : M \to M$ is called non-degenerate if every fixed point is non-degenerate or, equivalently, the map F from above is transverse to Δ (in all of M). Non-degenerate maps have a simple fixed point structure and one would like to know what happens if one disturbs the map slightly. In general, there can be quite unpleasant behaviour, like submanifolds of fixed points of positive dimension emerging from isolated fixed points. This is a quite radical change in the structure of the fixed point set. A more convenient phenomenon might be the change of the number of connected components of the set of fixed points.

At any rate, the interesting parameters are those for which it is possible for the topological structure of the set of fixed points to change. These parameters of a homotopy are called bifurcation parameters.

Definition 1.1.3.2 *Let $H : M \times I \to M$ be a homotopy. A parameter $\lambda \in I$ is called a* regular parameter*, if H_λ is non-degenerate. Otherwise, λ is called a* bifurcation parameter*.*

The question arises if it is possible to find a generic set of homotopies of maps such that the fixed point set at every stage of the homotopy has a simple structure, even after passing a bifurcation parameter. We will deal with this question in section 1.2. For now, we just want to describe what happens between two bifurcation parameters. This should be the unique continuation of the fixed point. Making this precise leads to the notion of a branch of fixed points. Since we do not want to care about parametrizations, we first have to define prebranches and then pass to an equivalence class, combining all prebranches that are equal up to reparametrization. These notions are adaptions from [MPY82].

Definition 1.1.3.3 *Let $H : M \times I \to M$ be a homotopy. A* prebranch of fixed points of H emanating from *$(x, \lambda) \in M \times I$ is a continuous map $x \times \mu : (0, 1) \to M \times I$ such that $x(t)$ is an isolated fixed point of $H_{\mu(t)}$, $x(t) \to x$ for $t \to 0$, $\mu(t) \to \lambda$ for $t \to 0$ and for $t \neq s$, if $\mu(t) = \mu(s)$ then $x(t) \neq x(s)$.*

There is an equivalence relation on the set of prebranches given by reparametrization, i.e. the prebranches $x \times \mu : (0, 1) \to M \times I, y \times \nu : (0, 1) \to M \times I$ are equivalent, if there is an increasing homeomorphism

$$\zeta : (0, 1) \to (0, 1)$$

and $\nu \circ \zeta = \mu$, $y \circ \zeta = x$. An equivalence class of prebranches is called a branch of fixed points*.*

We will not explicitly distinguish between a branch and a representing prebranch. The next proposition gives the desired result that, as long as a homotopy is regular between two bifurcation parameters, isolated fixed points lie on a unique branch connecting these two parameters.

Proposition 1.1.3.4 *Let $H : M \times I \to M$ be a smooth homotopy. Let $\lambda_0 \in I$ be a regular parameter and $[\lambda_1, \lambda_2] \subseteq I$ be the maximal interval such that (λ_1, λ_2) consists of regular parameters and $\lambda_0 \in [\lambda_1, \lambda_2]$. Then for every fixed point x_0 of H_{λ_0}, all prebranches $x \times \mu : (0, 1) \to M \times I$ such that $\mu(t) \to \lambda_1$ for $t \to 0$, $\mu(t) \to \lambda_2$ for $t \to 1$ and $x \times \mu(t) = (x_0, \lambda_0)$ for some $t \in I$ are equivalent, i.e. there is a unique branch of fixed points passing through x_0.*

Proof. Choose a chart neighbourhood U of x_0 and a smaller neighbourhood U' such that $H_\lambda(U') \subseteq U$ for λ in a neighbourhood $J = (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ of λ_0. Let $\psi : U \to \mathbb{R}^n$ be a chart and $\psi(x_0) = 0$, $\psi(U') = \mathbb{B}_1(0)$. Denote

$$\tilde{H}_\lambda = \psi \circ H_\lambda \circ \psi^{-1}\big|_{\mathbb{B}_1(0)} : \mathbb{B}_1(0) \to \mathbb{R}^n.$$

We obtain a map

$$\tilde{H} : \mathbb{B}_1(0) \times J \to \mathbb{R}^n, \quad \tilde{H}(0, \lambda_0) = 0$$

and the differential of $\mathbb{1} - \tilde{H}$ with respect to the first variable is invertible. By the implicit function theorem we find a neighbourhood $W \subseteq \mathbb{B}_1(0)$ of 0, $K \subseteq J$ of λ_0 and a smooth map $\vartheta : K \to W$ such that

$$\tilde{H}(v, \lambda) = v,\ v \in W,\ \lambda \in K \iff v = \vartheta(\lambda).$$

We can assume that $J = K$. With $V = \psi^{-1}(W)$, we have

$$H(x, \lambda) = x,\ x \in V,\ \lambda \in J \iff x = \psi^{-1} \circ \vartheta(\lambda).$$

This constitutes a prebranch

$$J \to M \times I, \quad \lambda \mapsto (\psi^{-1} \circ \vartheta(\lambda), \lambda)$$

which is unique up to reparametrization. Clearly, $\psi^{-1} \circ \vartheta(\lambda)$ converges to fixed points of $H_{\lambda_0 \pm \varepsilon}$ for $\lambda \to \lambda_0 \pm \varepsilon$ and we can repeat the procedure for these fixed points and the maps $H_{\lambda_0 \pm \varepsilon}$. We obtain prebranches on intervals containing $\lambda_0 \pm \varepsilon$ and by uniqueness, they have to coincide with the old prebranch, where this is defined. Continuing in this fashion, we obtain a prebranch

$$[\lambda_1, \lambda_2] \to M \times I,\ \lambda \mapsto (x(\lambda), \lambda),$$

$x(\lambda_0) = x_0$ and after reparametrizing to a prebranch defined on I, every prebranch running through x_0 is equivalent to this one. □

1.1.4 Dynamical Systems and Bifurcation Theory

We are now turning our attention to continuous dynamical systems. Beside the obvious analoga of the definitions and results of the last chapter for periodic orbits of continuous systems, we also want to have possibilities at hand to reduce the investigation of periodic orbits of continuous systems to fixed points of discrete systems. This happens by means of Poincaré systems.

Since the tangent bundle TM is a smooth manifold, we have a topology on the space of vector fields $\mathfrak{X}(M)$, namely the subspace topology of $\mathcal{C}^\infty(M, TM)$. The whole theory also works in the $\mathcal{C}^r$ case when keeping in mind that TM is in general only $\mathcal{C}^{r-1}$, if M is $\mathcal{C}^r$. Note that the flow of a smooth vector field $\xi : M \to TM$, i.e. the entirety of solution curves of the ordinary differential equation $\dot{x} = \xi(x)$ on M, constitutes a continuous dynamical system if M is compact. On the other hand, every continuous dynamical system on a compact manifold comes from an ordinary differential equation and its flow. Hence, we will not distinguish between a dynamical system and the flow of a vector field from now on. The situation is a bit more complicated if we allow $\mathcal{C}^r$-fields. One should keep in mind that Fuller index theory deals with flows rather than vector fields, and this distinction vanishes if everything is smooth.

Definition 1.1.4.1 *Let M be a smooth compact manifold, $\xi : M \to TM$ a vector field on M, $\varphi : M \times \mathbb{R} \to M$ its flow. A point $(x,T) \in M \times \mathbb{R}^+$ is called a* periodic point *of ξ or of φ, if $\varphi(x,T) = x$. In this case, T is called a* period *of x. If there is a minimal $p > 0$ such that $\varphi(x,p) = x$, p is called the* minimal period *of x and (x,T) a proper periodic point. Every period T of x is an integer multiple of p, $T = k \cdot p$. k is called the* multiplicity *of T.*

If (x,T) is a periodic point, then the set

$$\gamma = \varphi(x,\mathbb{R}) \times \{T\}$$

is called a periodic orbit *of ξ or φ. The set $\varphi(x,\mathbb{R})$ is called a* geometric orbit *of ξ (or φ).*

It is important for us to distinguish between geometric periodic orbits, which are subsets of M, and periodic orbits, which are geometric orbits with a fixed period in the parameter space $\mathbb{R}^+$. This distinction will sometimes make phrasing a bit harder but it unfolds the dynamic behaviour at period multiplying bifurcations.

The problem setting in Fuller index theory is a bit different from that of fixed point theory. In general, we are only interested in periodic orbits in a given open subset Ω of M (we could have done this for fixed points as well). As is standard in index theory, we have to make sure that no periodic orbits lie on the boundary of Ω. But from a more general point of view, we should take $\Omega \times (a,b)$ as a subset of $M \times \mathbb{R}^+$ and require no periodic points to lie on the boundary of $\Omega \times (a,b)$. This means that we have no periodic orbits of period $T \in [a,b]$ on $\partial\Omega$, and we have no periodic orbits of period a or b at all. So our period is a priori bounded in a compact interval. We denote by $\mathfrak{X}(M,\Omega,a,b)$ the subset of $\mathfrak{X}(M)$ of vector fields having no periodic points on $\partial(\Omega \times (a,b))$.

In this setting it makes sense to ask when two vector fields are homotopic, because an existing homotopy might leave the set $\mathfrak{X}(M,\Omega,a,b)$. So the following statement is non-trivial.

Lemma 1.1.4.2 *Let $\Omega \subseteq M$ be open, $\xi : M \to TM$ a vector field without critical elements on $\partial(\Omega \times (a,b))$. Then for every neighbourhood $\mathcal{V}$ of ξ there is a neighbourhood $\mathcal{U} \subseteq \mathcal{V}$ of ξ such that no field in $\mathcal{U}$ has critical elements on $\partial(\Omega \times (a,b))$ and any two elements of $\mathcal{U}$ are homotopic via a homotopy not leaving $\mathcal{U}$.*

PROOF. Since ξ has no critical elements on $\partial(\Omega\times(a,b))$ and since this last set is compact, the distance function $d(\varphi(x,t),x)$ attains a positive minimum on $\partial(\Omega \times (a,b))$. Hence, the same is true in a neighbourhood $\mathcal{U}_1$ of ξ and no element of $\mathcal{U}_1$ has a critical element on $\partial(\Omega \times (a,b))$. For $\eta \in \mathcal{U}_1$, let $H_\eta : M \times [0,1] \to M$ be the convex homotopy between ξ and η. Define

$$\mathcal{U} = \{\eta \in \mathcal{U}_1 \mid (H_\eta)_t \in \mathcal{U}_1 \ \forall\, t \in [0,1]\}.$$

Then $\mathcal{U}$ is open and all elements in $\mathcal{U}$ are homotopic to ξ. This proves the lemma. □

Our next aim is to define Poincaré systems for a given flow. As mentioned in the introduction, we have to generalize the usual definition, because we want Poincaré systems to be not necessarily centered around a periodic point. The notion of a disc transverse to a flow is essential.

Let M be a smooth n-dimensional manifold and $\psi : \mathbb{B}_1(0) \to M$ a diffeomorphism of the unit ball in $\mathbb{R}^{n-1}$ onto its image. Then $D = \psi(\mathbb{B}_1(0))$ is called a *disc in M centered at $\psi(0)$*. A *subdisc D' of D* is the image of some disc $\mathbb{B}_r(0)$, $0 < r < 1$, under ψ in D. We will not explicitly mention the diffeomorphism ψ and just speak of discs in M, centered at an element $x \in M$. We say that a disc is *transverse to a flow* $\varphi : M \times \mathbb{R} \to M$, if the map $\varphi_x : \mathbb{R} \to M$ is transverse to D for every $x \in D$.

The next result will help to prove the continuation lemma for Poincaré systems, compare Lemma 1.1.4.5. Note that it does not follow trivially from Thom's transversality theorem, since discs are submanifolds with boundary. One could repair this, but we can also give a direct proof.

Lemma 1.1.4.3 *If D is a disc transverse to the flow φ of a vector field $\xi : M \to TM$, then for every proper subdisc $D' \subseteq D$ there is a neighbourhood $\mathcal{U}$ of ξ such that D' is transverse to every element of $\mathcal{U}$.*

Proof. For $x \in \overline{D'}$, $\dot\varphi_x(0)$ is linearly independent to T_xD. Hence, by choosing a local trivialization of the tangent bundle of M and a norm on the tangent spaces, the function $d : U_x \to \mathbb{R},\ y \mapsto \operatorname{dist}(\dot\varphi_y(0), T_yD)$ is positive in a neighbourhood U_x of x and the same holds for η in a neighbourhood $\mathcal{U}_x$ of ξ and the flow of η instead of φ. The sets U_x cover $\overline{D'}$, so we find a finite subcover, corresponding to points $x_1, \ldots, x_m \in \overline{D'}$. Then $\overline{D'}$ is transverse to every element of $\mathcal{U} = \mathcal{U}_{x_1} \cap \cdots \cap \mathcal{U}_{x_m}$. □

Now we can give the definition of Poincaré systems, which will be crucial in many aspects of the theory.

Definition 1.1.4.4 *Let $\xi : M \to TM$ be a vector field, $\varphi : M \times \mathbb{R} \to M$ its flow. Let $x \in M$ be a point and $D \subseteq M$ a disc centered at x transverse to the flow. Assume that $\varphi(x,T) \in D$ for some $T > 0$. Then there is a subdisc $D' \subseteq D$ centered at x and a continuous map $t : D' \to \mathbb{R}^+$ such that*

$$P(y) = \varphi(y, t(y)) \in D.$$

(D, D', P, t) is called a Poincaré system centered at *x. P is called the* Poincaré map *and t the* period map *of the system.*

The existence of t follows from the implicit function theorem because locally, being in D means lying in a 1-codimensional subspace or alternatively, lying in the kernel of a projection.

Note that, although we called t the "period" map, we do not require x to be a periodic point, nor do we require P to have any fixed points at all.

Obviously, Poincaré systems capture all the dynamical behaviour of a periodic orbit of a flow and translate it into the dynamical behaviour of a fixed point. But we can also use Poincaré systems to investigate local bifurcations of periodic orbits by investigating the bifurcation of a fixed point. This is the continuation lemma for Poincaré systems which is the key of our proof of homotopy invariance of the Fuller index.

Lemma 1.1.4.5 (Continuation Lemma) *Let $\xi_0 : M \to TM$ be a smooth vector field and (D, D', P_0, t_0) a Poincaré system for ξ_0, centered at $x_0 \in M$. Then, after possibly shrinking D and D', there is a neighbourhood $\mathcal{U} \subseteq \mathfrak{X}(M)$ of ξ and continuous maps $P : \mathcal{U} \to \mathcal{C}^\infty(D', D)$, $t : \mathcal{U} \to \mathcal{C}(D', \mathbb{R}^+)$, $P(\xi_0) = P_0$, $t(\xi_0) = t_0$, such that for $\xi \in \mathcal{U}$, $(D, D', P(\xi), t(\xi))$ is a Poincaré system for ξ centered at x_0.*

Proof. After possibly shrinking D and D', by Lemma 1.1.4.3 we find a neighbourhood $\mathcal{U}$ of ξ_0 such that D is transverse to all elements of $\mathcal{U}$. Take $\xi \in \mathcal{U}$ and $x \in D'$, ψ the flow of ξ. Then we find a continuous map $t : D' \to \mathbb{R}^+$ such that $\psi(y, t(y)) \in D$ for all $y \in D'$. If we require $t(x_0)$ to be close to $t_0(x_0)$, by continuous dependence of the flow on the vector field, t is close to t_0, i.e. the map $\xi \mapsto t$ is continuous. Then clearly, the map $\xi \mapsto P_\xi$ is continuous as well, where $P_\xi(y) = \psi(y, t(y))$. $\square$

Of course, the structure of the set of periodic orbits of a field can be arbitrarily complicated, as was the case with fixed points of maps. So we need additional requirements which will control the periodic orbits.

Definition 1.1.4.6 *If $\varphi : M \times \mathbb{R} \to M$ is a flow of a vector field $\xi : M \to TM$, a periodic orbit γ is called* non-degenerate*, if for some choice of Poincaré system (D, D', P, t) around some $(x, T) \in \gamma$, x is a non-degenerate fixed point of the Poincaré map. Similarly, γ is called a* hyperbolic *periodic orbit, if x is a hyperbolic fixed point of P.*

An interesting phenomenon occurs if the Poincaré map P for a periodic orbit with its minimal period has roots of unity as eigenvalues. If 1 is not an eigenvalue, this orbit will be non-degenerate by definition. But if ζ is a primitive k-th root of unity and an eigenvalue of P, then 1 is an eigenvalue of P^k, and so the same orbit, now considered with multiplicity k, will be degenerate. Hyperbolic orbits cannot exhibit a similar behaviour.

One could define non-degeneracy and hyperbolicity of periodic orbits directly, without using Poincaré systems, but this is a nice example demonstrating the principle of investigating periodic orbits through associated Poincaré maps. The price we have to pay consists in proving the following lemma and thus is bearable to pay.

Lemma 1.1.4.7 *The definition of non-degenerate and hyperbolic periodic orbits does not depend on the choice of the Poincaré system.*

PROOF. Let (D, D', P, t) be a Poincaré system centered at x, $t(x) = T$, $\varphi(x, T) = x$. Let (E, E', Q, s) be another Poincaré system centered at y, $\varphi(y, S) = y$. Since both x and y lie on the same geometric periodic orbit, $s(y) = T$. We have $\varphi(x, r_0) = y$ for some minimal $0 \leq r_0 < T$. The condition

$$\varphi(z, r(z)) \in E',\ r(x) = r_0,\ r \text{ continuous},$$

defines a map F from a subdisc $D'' \subseteq D'$ to E' which is clearly a diffeomorphism onto its image. Take a subdisc $D''' \subseteq D''$ such that $Q \circ F(D''') \subseteq F(D'')$. Then obviously, $F \circ P = Q \circ F$. Hence, if $T_x P$ has no eigenvalue 1, then so does

$$T_x F \circ T_x P \circ (T_x F)^{-1} = T_y Q$$

and vice versa. The hyperbolic case follows identically. □

A vector field is called non-degenerate or hyperbolic, if all its periodic orbits are non-degenerate or hyperbolic, respectively. More generally, an element of $\mathfrak{X}(M, \Omega, a, b)$ is called non-degenerate or hyperbolic, if all periodic orbits through periodic points in $\Omega \times (a, b)$ are non-degenerate or hyperbolic, respectively. The periodic orbits in $\Omega \times (a, b)$ will be called the essential periodic orbits of the field. Of course, when $H : M \times I \to TM$ is a homotopy of vector fields, a bifurcation parameter of H is a parameter such that H_λ is not non-degenerate.

We will now give the notion of branches and prebranches in the context of periodic orbits.

Definition 1.1.4.8 *Let $H : M \times I \to TM$ be a homotopy of vector fields, $\lambda \in I$ any parameter. A* prebranch of periodic orbits emanating from (x, T, λ) *is a continuous map*

$$x \times \mu \times T : (0, 1) \to M \times I \times \mathbb{R}^+$$

such that $(x(t), T(t))$ is an isolated periodic point of $H_{\mu(t)}$, $\mu(t) \to \lambda$ for $t \to 0$, $(x(t), T(t)) \to (x, T)$, and for $t \neq s$, $x(t) \neq x(s)$. We also say that the prebranch emanates from (x, T) at λ.

There is an equivalence relation on the set of prebranches given by reparametrization, i.e. the prebranches $x \times \mu \times T : I \to M \times I \times \mathbb{R}^+, y \times \nu \times S : I \to M \times I \times \mathbb{R}^+$ are equivalent, if there is an increasing homeomorphism $\zeta : (0, 1) \to (0, 1)$ and $y \circ \zeta \cong x$, $\nu \circ \zeta = \mu$, $S \circ \zeta = T$. Here, $y \circ \zeta \cong x$ means that $y \circ \zeta(t)$ and $x(t)$ are on the same flow-orbit for every $t \in (0, 1)$. An equivalence class of prebranches is called a branch of periodic orbits.

The uniqueness result 1.1.3.4 of branches in a region of regular parameters also holds for branches of periodic orbits.

Proposition 1.1.4.9 *Let $H : M \times I \to TM$ be a smooth homotopy of vector fields. Let $\lambda_0 \in I$ be a regular parameter and $[\lambda_1, \lambda_2] \subseteq I$ be the maximal interval such that (λ_1, λ_2) consists of regular parameters and $\lambda_0 \in [\lambda_1, \lambda_2]$. Then for a periodic orbit γ_0 through (x_0, T_0) of H_{λ_0}, all prebranches $x \times \mu \times T : (0, 1) \to M \times I \times \mathbb{R}^+$ such that $\mu(0) = \lambda_1$, $\mu(1) = \lambda_2$ and $x \times \mu \times T(t) = (x_0, \lambda_0, T_0)$ for some $t \in I$ are equivalent, i.e. there is a unique branch of periodic orbits passing through γ_0.*

Proof. This is a trivial consequence of the result for fixed points, using Poincaré systems for γ_0. $\square$

Bifurcation theory deals with the process when a branch of critical elements runs into a bifurcation parameter. Examples of bifurcations of fixed points where shown in Figure 2 and by use of Poincaré systems, similar phenomena will occur when periodic orbits bifurcate (note that, since we always assume the period to be a priori bounded in a positive interval, we neither have Hopf bifurcations nor bifurcations into homoclinic orbits). But in the case of periodic orbits, there is another interesting phenomenon, namely the period multiplying bifurcation. This is a bifurcation where, in the geometric sense, a periodic orbit whose period is an integer multiple of the period of the initial orbit bifurcates. If we use periodic points, where we had assigned a fixed period to a point on a geometric periodic orbit, this is nothing special, for if (x, T) is periodic, then so is (x, kT) for $k \in \mathbb{Z}$ and period multiplying by k just means that the bifurcation takes place at the periodic point (x, kT), not at (x, T). It is still worthwhile to say some words about this phenomenon.

Lemma 1.1.4.10 *If $(x_n, T_n) \in M \times [a, b]$ are periodic points, $0 < a < b < \infty$ and the x_n converge to a point $x \in M$ such that (x, p) is a periodic point, $p > 0$ the minimal period. Then for any subsequence (x_{n_k}, T_{n_k}) such that the minimal periods of the x_{n_k} converge to some $q \in \mathbb{R}$, we have $q = k \cdot p$ for some integer $k \in \mathbb{N}$.*

Proof. Let p_n be the minimal period of the point x_n. We cannot have $p_n \to 0$ for a subsequence, since then x would be a fixed point. Thus, the minimal periods are bounded in a compact interval $[c, b]$, $c > 0$, and we can assume that p_n converges to some $q \in [c, b]$. We obtain

$$x_n = \varphi(x_n, p_n) \to \varphi(x, q),$$

hence q is a period of x, i.e. $q = k \cdot p$ for some $k \in \mathbb{N}$. $\square$

The geometric picture one should have in mind is the following.

Proposition 1.1.4.11 *Let $H : M \times I \to TM$ be a homotopy of vector fields, 0 an isolated bifurcation parameter of H and γ_0 a critical geometric periodic orbit with minimal period p. Let $x \times T : (0, 1) \to M \times \mathbb{R}$ be a branch of periodic orbits (we assume the parameter to be t for simplicity), bifurcating from γ_0, such that the minimal period of the orbits on $x \times T$ is approximately $k \cdot p$ for some $k \in \mathbb{N}$. For a fixed $\lambda \in (0, 1)$, let $x(\lambda) = x_1$, $\lim_{\mu \to 0} x(\mu) = x_0$ and let γ_λ be the geometric orbit of H_λ through x_1. Define a map*

$$q : \gamma_\lambda \to \gamma_0,\ q(\varphi_\lambda(x_1, t)) = \varphi_{\lambda_0}(x_0, t).$$

Then q is a k-fold covering map.

Proof. We have to check that around any point $x \in \gamma_0$ there is a neighbourhood $U \subseteq \gamma_0$ of x and a homeomorphism $\psi : q^{-1}(U) \to \coprod_{j=1}^{k} U$. Take $x \in \gamma_0$, then $x = \varphi_{\lambda_0}(x_0, t)$ for some $t \in (0, p]$. Let $U = \varphi_{\lambda_0}(x_0, (t-\varepsilon, t+\varepsilon))$, where $\varepsilon > 0$ is chosen such that $t - \varepsilon > 0$ and $t + \varepsilon - p < t - \varepsilon$. U is an open neighbourhood of x. We have

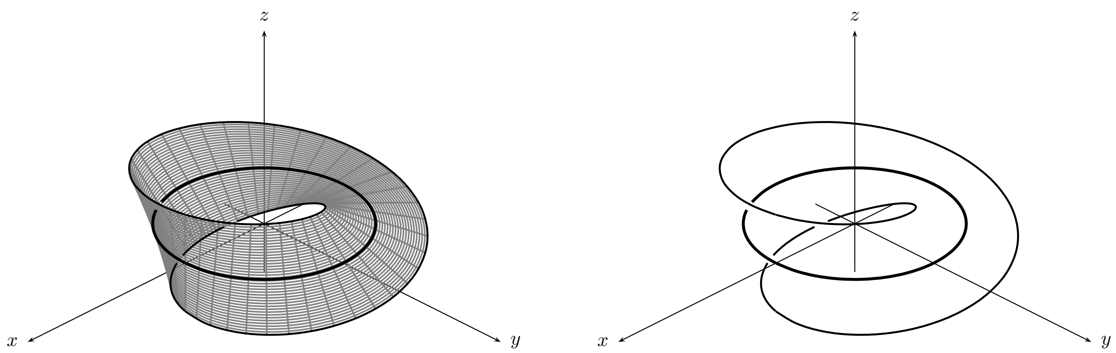
$$q^{-1}(U) = \{\varphi_\lambda(x_1, s) \mid s \in (t - \varepsilon + j \cdot p, t + \varepsilon + j \cdot p) \text{ for some } j \in \{0, \dots, k-1\}\}.$$

Let j_s be the unique integer in $\{0, \dots, k-1\}$ such that $s \in (t - \varepsilon + j \cdot p, t + \varepsilon + j \cdot p)$ and define

$$\psi : q^{-1}(U) \to \coprod_{j=1}^{k} U, \; \psi(\varphi_\lambda(x_1, s)) = (\varphi_{\lambda_0}(x_0, s), j_s).$$

ψ is obviously surjective and if $\psi(\varphi_\lambda(x_1, s)) = \psi(\varphi_\lambda(x_1, s'))$, then $j_s = j_{s'}$, i.e. $s - s' \in [-\varepsilon, \varepsilon]$. But $\varphi_{\lambda_0}(x_0, s) = \varphi_{\lambda_0}(x_0, s')$ implies $s = s' \mod p$, hence $s = s'$ and ψ is shown to be injective. It is apparent that the definition of ψ defines a map in the other direction as well which is smooth and the inverse map to ψ. Hence, ψ is a diffeomorphism. $\square$

Figure 3 illustrates a period doubling bifurcation. The equator of a Moebius strip is a periodic orbit from which a periodic orbit bifurcates which is given by the boundary of the strip. This orbit has approximately twice the period of the initial one.



(a) moebius strip (b) boundary and equator

Figure 3: Period doubling bifurcation

1.2 Genericity Theorems

The second main part and first non-introductory part of this work will give proofs of genericity of the spaces of maps and vector fields defined in part 1.1. Namely, this are the sets of non-degenerate maps, hyperbolic maps, non-degenerate vector fields and hyperbolic vector fields. The basic tool is transversality. Thom's theorem will play the major rôle in the proofs of genericity for non-degenerate maps. The method to prove genericity of hyperbolic maps is to show that hyperbolic maps are generic in the set of non-degenerate maps. Once we know that generically there are only finitely many fixed points, we can make these hyperbolic with an arbitrary small perturbation and take care that we do not generate additional fixed points elsewhere. A similar method works for homotopies. These theorems are rather simple and the material or at least the ideas are due to [PdM82].

For vector fields, it is more difficult, because non-degeneracy is now connected to transversality of the flow rather than the field. This makes it difficult to use Thom's transversality theorem. We use direct proofs instead, inspired by [PdM82] and [Fie80], to show that hyperbolic fields are generic. Transversality comes in when it comes to prove openness. The homotopy case is again similar, depending on parametrized versions of the methods used for fields.

An additional remark on smoothness conditions seems in order. Most of the results we prove will hold for $\mathcal{C}^r$-maps in the $\mathcal{C}^r$-topologies, for $r \geq 1$. Sometimes $r \geq 2$ is necessary. Since we are ultimately interested in doing index theory, the details of the varying degrees of smoothness are of no interest for us. So if we do not specify the degree of smoothness, we will implicitly assume that we are dealing with $\mathcal{C}^\infty$-maps. In this sense, the results are far from being sharp and one might want to read "sufficiently smooth" instead of "smooth", meaning that the result is true for $\mathcal{C}^r$-maps and some r sufficiently large, possibly $r = \infty$.

1.2.1 Genericity in the Space of Maps

Genericity of smooth non-degenerate self maps of a smooth manifold follows almost immediately from Thom's Transversality Theorem, as we shall see shortly. So the main focus of this section is on the technique of hyperbolization of non-degenerate fixed points, which will allow to deduce genericity of hyperbolic maps from genericity of non-degenerate ones. Our first result shows that non-degenerate fixed points are not only isolated in the manifold M, but also remain so under perturbations of the map. This should be no surprise, since we already know the set of regular parameters of a homotopy to be open (Proposition 1.1.2.6), and small perturbations of a map are all homotopic (Lemma 1.1.1.2).

Proposition 1.2.1.1 *Let $f_0 : M \to M$ be a $\mathcal{C}^r$-map, $x_0 \in M$ a non-degenerate fixed point of f_0. Then there is a neighbourhood U of x_0, a neighbourhood $\mathcal{U}$ of f_0 and a continuous map $x : \mathcal{U} \to U$ such that $x(f)$ is the unique fixed point of f in U and $x(f)$ is non-degenerate. If x_0 happens to be hyperbolic, by possibly shrinking U and $\mathcal{U}$ one can achieve that $x(f)$ is hyperbolic.*

Proof. Take a chart neighbourhood V of x_0 such that the chart ψ maps V onto $\mathbb{R}^n$, $\psi(x_0) = 0$. Since x_0 is fixed, we find a neighbourhood $V' \subseteq V$ of x_0 such that $f(\overline{V'}) \subseteq V$ for all f in a neighbourhood $\mathcal{U}_0$ of f_0 and we can arrange that $\psi(V') = \mathbb{B}_1(0)$. 0 is a non-degenerate fixed point of the map $\tilde{f}_0 = \psi \circ f_0 \circ \psi^{-1} : \mathbb{B}_1(0) \to \mathbb{R}^n$.

Let X be the Banach space of bounded $\mathcal{C}^r$-maps from $\mathbb{B}_1(0)$ to $\mathbb{R}^n$ and define a map

$$F : X \times \mathbb{B}_1(0) \to \mathbb{R}^n, \; F(h, v) = v - h(v).$$

If $r \geq 1$, F is differentiable and the partial derivative in $(\tilde{f}_0, 0)$ with respect to v is given by $D_v F(\tilde{f}_0, 0) = \mathbb{1} - D\tilde{f}(0)$. Since 0 is non-degenerate, this operator is invertible and by the implicit function theorem we find neighbourhoods $W \subseteq \mathbb{B}_1(0)$ of 0 and $\mathcal{V} \subseteq X$ of $\tilde{f}_0$ such that the equation $F(h,v) = 0$ is uniquely solved in $\mathcal{V} \times W$ by a function $\vartheta : \mathcal{V} \to W$, i.e.

$$F(h,v) = 0,\ h \in \mathcal{V}, v \in W \Leftrightarrow v = \vartheta(h).$$

Let $U = \psi^{-1}(W)$ and let $\mathcal{U}$ be the subset of maps $f \in \mathcal{U}_0$ such that $\psi \circ f \circ \psi^{-1}\big|_{\mathbb{B}_1(0)} \in \mathcal{V}$. Then $\mathcal{U}$ is a neighbourhood of f_0 and if v_f is the unique fixed point of $\psi \circ f \circ \psi^{-1}$ in W, then $\psi^{-1}(v_f) \in U$ is a unique fixed point of f in U. Since 0 was non-degenerate for $\tilde{f}_0$, we can choose $\mathcal{V}$ and W such that v_f is non-degenerate for f, since the set of linear operators without eigenvalue 1 is open and dense in the set of linear operators. Similarly, if 0 is hyperbolic for $\tilde{f}_0$, we can arrange that v_f is hyperbolic for f, since the set of linear hyperbolic operators is open and dense in the set of linear operators. So the definition $x(f) = \psi^{-1}(v_f)$ concludes the proof. $\square$

Now we deal with the hyperbolization of a non-degenerate fixed point. By this, we mean a local perturbation of a map around a non-degenerate fixed point, such that this fixed point becomes hyperbolic and we do not obtain new fixed points by the perturbation. Since non-degeneracy and hyperbolicity are characterized by the spectrum of the differential of a map, it should be possible to prove such a result using density of linear hyperbolic operators. Namely, if A is any operator, then $A + c\mathbb{1}$ is hyperbolic for $c > 0$ sufficiently small. Furthermore, Proposition 1.2.1.1 guarantees that in a certain neighbourhood, there will be no fixed points except the one we just made hyperbolic. Then the following result is easy to deduce.

Lemma 1.2.1.2 *Let $f : M \to M$ be $\mathcal{C}^r$ and $x \in M$ a non-degenerate fixed point of f. Then for any neighbourhood $\mathcal{U}$ of f and any sufficiently small neighbourhood U of x there is a map $f' \in \mathcal{U}$ such that x is a hyperbolic fixed point of f', is the only fixed point of f' in U and f' equals f outside of U.*

Proof. Take a chart (ψ, U) around x such that $\psi(U) = \mathbb{R}^n$ and a subset $U' \subseteq U$ such that f maps $\overline{U'}$ into U. We can arrange that $\psi(U') = \mathbb{B}_1(0)$. Furthermore, we can choose U such that the conclusion of Proposition 1.2.1.1 is satisfied, i.e. for a neighbourhood $\mathcal{U}_1 \subseteq \mathcal{U}$, every element of $\mathcal{U}_1$ has a unique non-degenerate fixed point in U. Let $\tilde{f} = \psi \circ f \circ \psi^{-1}\big|_{\mathbb{B}_1(0)} : \mathbb{B}_1(0) \to \mathbb{R}^n$. Then for small $c > 0$, the map $\tilde{f} + c \cdot \mathbb{1}$ is smooth, has 0 as fixed point and its differential at 0 is hyperbolic. Take a smooth Urysohn function $\rho : \mathbb{B}_1(0) \to [0,1]$ that is equal to 1 on $\mathbb{B}_1(0) - \mathbb{B}_{\frac{3}{4}}(0)$ and equal to 0 on $\mathbb{B}_{\frac{1}{4}}(0)$. Define a map

$$\tilde{h}_c : \mathbb{B}_1(0) \to \mathbb{R}^n, \quad v \mapsto \tilde{f}(v) + c \cdot (1 - \rho(v)) \cdot v.$$

Then $\tilde{h}_c$ is equal to $\tilde{f}$ near the boundary of $\mathbb{B}_1(0)$ and equal to $\tilde{f} + c \cdot \mathbb{1}$ near 0. Furthermore, by choosing c sufficiently small, $\tilde{h}_c$ is arbitrarily close to $\tilde{f}$. Define

$$h_c : M \to M, \quad x \mapsto \begin{cases} f(x) & x \notin U' \\ \psi^{-1} \circ \tilde{h}_c \circ \psi(x) & x \in U' \end{cases}$$

which by definition of $\tilde{h}_c$ is a smooth function. Since $\tilde{h}_c$ was arbitrarily close to $\tilde{f}$, we can choose c such that h_c is an element of $\mathcal{U}_1$. Thus, h has a unique fixed point in U, which has to be x, and x is hyperbolic. □

The last two results are enough to prove genericity of hyperbolic maps from genericity of non-degenerate ones. As already mentioned, this last result is rather trivial.

Proposition 1.2.1.3 *The set of maps $f : M \to M$ such that all fixed points of f are non-degenerate is open and dense in the set of all maps.*

PROOF. By Proposition 1.1.3.1, the fixed points of f are non-degenerate if and only if the map $\mathbb{1} \times f : M \to M \times M$ is transverse to the diagonal $\Delta \subseteq M \times M$. By Thom's Transversality Theorem, the set of maps $M \to M \times M$ that are transverse to the diagonal is open and dense. This immediately gives openness of our set. For density, note that if $F = f_1 \times f_2$ is sufficiently close to $\mathbb{1} \times f$, then f_1 is a diffeomorphism. If $f_1 \times f_2$ is transverse to the diagonal, then so is $\mathbb{1} \times (f_2 \circ f_1^{-1}) = (f_1 \times f_2) \circ f_1^{-1}$ and by choosing f_1 sufficiently close to $\mathbb{1}$, this last map is arbitrarily close to $\mathbb{1} \times f$, proving the proposition. □

This result immediately allows the proof of the next theorem.

Theorem 1.2.1.4 *The set of hyperbolic maps is open and dense in the set of all maps.*

PROOF. For openness, take any hyperbolic map $f : M \to M$. Using Proposition 1.2.1.1 we find a neighbourhood $\mathcal{U}_0$ of f and pairwise disjoint neighbourhoods $U_1, \ldots, U_m$ of the finitely many fixed points $x_1, \ldots, x_m$ of f such that all elements of $\mathcal{U}_0$ have a unique hyperbolic fixed point in U_j for $1 \leq j \leq m$. Furthermore, f has no fixed points outside of $U = U_1 \cup \cdots \cup U_m$, so the same is true for all maps in a neighbourhood $\mathcal{U}_1$ of f. Then $\mathcal{U}_0 \cap \mathcal{U}_1$ is a neighbourhood of f consisting of hyperbolic maps.

For density, by Proposition 1.2.1.3 it suffices to show that hyperbolic maps are dense in the set of non-degenerate maps. Thus, take $f : M \to M$ non-degenerate and let $\mathcal{U}$ be any neighbourhood of f in the set of non-degenerate maps. We have finitely many fixed points $x_1, \ldots, x_m$ of f and, using Proposition 1.2.1.1 again, pairwise disjoint neighbourhoods $U_1, \ldots U_m$ such that any element in a neighbourhood $\mathcal{U}_0$ of f has a unique non-degenerate fixed point in U_j, $1 \leq j \leq m$. Using the argument of the proof of openness, we can assume that the elements of $\mathcal{U}_0$ have no fixed points outside $U = U_1 \cup \cdots \cup U_m$. Using Lemma 1.2.1.2 repeatedly, we find a map f' arbitrary close to f that is equal to f outside of U and that has the x_j's, $1 \leq j \leq m$, as hyperbolic fixed points. In particular we can achieve that $f' \in \mathcal{U} \cap \mathcal{U}_0$, i.e. f' is hyperbolic and in the given neighbourhood $\mathcal{U}$ of f. This proves density of hyperbolic maps. □

1.2.2 Genericity in the Space of Homotopies

In what follows, we want to prove results similar to those of the preceeding section, but now for homotopies. As before, the essence of these results is well-known. Most of the genericity theory for homotopies was established in [Bru70]. However, our emphasis is on the bifurcation theoretic aspect and our main aim is to establish the generic bifurcation scenario we described in the introduction. So we can skip some details of Brunovský's paper. For example, in the treatment of hyperbolic homotopies, we do not introduce any eigenvalue crossing condition. Such a condition would control the eigenvalues of a hyperbolic homotopy when they cross the unit circle during the homotopy. The standard condition would be that the radial speed of the crossing eigenvalue is non-zero. For our purposes, it suffices for the homotopy to be hyperbolic at all non-bifurcation parameters and we need almost no control of the bifurcations other than non-degeneracy of the homotopy (but compare Corollary 1.2.2.9).

We should start with a definition of non-degenerate and hyperbolic homotopies. Of course we cannot expect the set of homotopies all of whose fibre maps H_λ are non-degenerate to have a satisfyingly generic structure. Non-degeneracy of all these maps would imply, by our results on the continuation of branches, that the number of fixed points remains unchanged during the homotopy, which is clearly no generic condition. As we will see shortly, the assumption that a homotopy has finitely many bifurcation parameters is rooted in transversality theory and thus, a natural and generic assumption. Since the unit interval I is a manifold with boundary, we have to be a bit technical when defining non-degeneracy. The purpose, however, should be clear.

Definition 1.2.2.1 *A homotopy $H : M \times I \to M$ is called* non-degenerate*, if there is an extension $\tilde{H} : M \times \mathbb{R} \to M$ of H such that the map*

$$F_H : M \times \mathbb{R} \to M \times M,\ (x, t) \mapsto (x, \tilde{H}(x, t))$$

is transverse to the diagonal.

From the parametrized transversality lemma 1.1.2.6, we obtain

Proposition 1.2.2.2 *A non-degenerate homotopy has finitely many bifurcation parameters.*

Proof. Let $H : M \times I \to M$ be non-degenerate and $\tilde{H} : M \times \mathbb{R} \to M$ an extension of H such that F_H defined as above is transverse to the diagonal. By Proposition 1.1.2.6, the set of $\lambda \in \mathbb{R}$ such that $(F_H)_\lambda$ is transverse to the diagonal is open and dense, hence, its complement is discrete and locally finite and so its intersection with I is finite. But $(F_H)_\lambda : M \to M \times M$ is just the map $x \mapsto (x, H_\lambda(x))$ for $\lambda \in I$, so H has only finitely many bifurcation parameters. □

By a *fixed point of a homotopy $H : M \times I \to M$*, we mean a point $(x, \lambda) \in M \times I$ such that $H_\lambda(x) = x$. Using Proposition 1.1.2.3, if $H : M \times I \to M$ is non-degenerate, then either H has no fixed points at all, or the set of fixed points is a manifold of dimension 1. This could simplify some of our proofs, where we will just use the fact that we have finitely many bifurcation points and finitely many branches connecting them. The reason is that in the equivariant case, there is no similarly simple structure, in particular we do not know if generically there are only equivariant jug-handle bifurcations. Since we are developing the non-equivariant theory in a way to have easy generalizations to the equivariant setting, this approach seems to be reasonable.

We can also define *hyperbolic homotopies*. This will be homotopies that are non-degenerate and all non-degenerate fixed points are hyperbolic. As is shown in [Bru70], this set of homotopies will contain a generic subset. But the set itself is not open in general. The reason is of course that we have no control of the bifurcations, so we can approximate a hyperbolic homotopy by a non-degenerate (but not hyperbolic) homotopy, shrinking the part where the approximating homotopy is not hyperbolic to a point (see also the following example). Since the more special results of Brunovský are of no use for us, we will be satisfied with our definition.

Example 1.2.2.3 For each $n \in \mathbb{N}$ let $\rho_n : \mathbb{R} \to \left[-1, -\frac{1}{2}\right]$ be a smooth function such that $\rho_n^{-1}(-1) = \left[1 - \frac{1}{n}, 1\right]$ and such that the sequence of ρ_n converges to a function ρ with $\rho^{-1}(-1) = \{1\}$. Consider the homotopy

$$H^n(x, y, \lambda) = (x^2 + 3 \cdot x + \frac{\lambda + 1}{2}, \rho_n(\lambda) \cdot y).$$

Since $\rho_n(\lambda) \neq 1$, a fixed point (x, y) must satisfy $y = 0$ and in addition,

$$x^2 + 2 \cdot x + \frac{\lambda + 1}{2} = 0,$$

giving

$$x = -1 \pm \sqrt{\frac{1 - \lambda}{2}}.$$

This yields two fixed points for every $\lambda < 1$, the two branches merging at $\lambda = 1$. Denote these fixed points by $x_-(\lambda)$ and $x_+(\lambda)$, corresponding to the sign.

The derivative of H^n at these fixed points is given by

$$\begin{pmatrix} 2x + 3 & 0 \\ 0 & \rho_n(\lambda) \end{pmatrix}.$$

$2x + 3$ is one of the eigenvalues of this matrix, and substituting $x = x_-(\lambda)$, we obtain a curve strictly larger than -1 for all $\lambda \in I$ and lesser than 1 for all $\lambda < 1$, approaching 1 for $\lambda \to 1$. Substituting $x = x_+(\lambda)$ gives a curve larger than 1, approaching 1 for $\lambda \to 1$. So all fixed points are non-degenerate for $\lambda \in [0, 1)$. They are even hyperbolic for $\lambda \in \left[0, 1 - \frac{1}{n}\right)$ and not hyperbolic for $1 - \frac{1}{n} \leq \lambda \leq 1$ (by definition of ρ_n). The only bifurcation parameter is $\lambda = 1$, meaning that for non-degeneracy of H^n, we only have to check whether the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ H^n_{1\,x} & H^n_{1\,y} & H^n_{1\,\lambda} & 1 & 0 \\ H^n_{2\,x} & H^n_{2\,y} & H^n_{2\,\lambda} & 0 & 1 \end{pmatrix}$$

has maximal rank in the fixed point $(x_-(1), 0) = (x_+(1), 0) = (-1, 0)$. The derivative of H_1^n with respect to λ is just $\frac{1}{2}$. So we have to check the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix},$$

which indeed has rank 4. We see that H^n is a non-degenerate homotopy. Clearly, the sequence of the H^n converges to a homotopy H which is defined just as H^n but exchanging ρ_n for ρ. By the same calculations as above, H is a non-degenerate homotopy and at every regular parameter, H_λ is hyperbolic. We conclude that the set of hyperbolic homotopies is not open.

To deduce genericity theorems for homotopies, we just need to prove parametrized versions of the two crucial results concerning hyperbolization of fixed points. The first result, the continuation of fixed points in a homotopy, is in principle already proven by Proposition 1.2.1.1.

Proposition 1.2.2.4 *Let $H : M \times I \to M$ be a $\mathcal{C}^r$ map, $x_0 \in M$ a non-degenerate fixed point of H_{λ_0}. Then there is a neighbourhood U of x_0 in M, a neighbourhood $\mathcal{U} \times J$ of (H, λ_0) and a continuous map $x : \mathcal{U} \times J \to U$, $x(H, \lambda_0) = x_0$, such that $x(K, \mu)$ is a unique fixed point of K_μ in U and $x(K, \mu)$ is non-degenerate. If x_0 happens to be hyperbolic, by possibly shrinking U and $\mathcal{U} \times J$ one can achieve that $x(K, \mu)$ is hyperbolic.*

Proof. By the non-parametrized version of this claim, Proposition 1.2.1.1, we find a neighbourhood $\mathcal{U}_1$ of H_λ, a neighbourhood U of x_0 and a continuous map $x_1 : \mathcal{U}_1 \to U$ such that for $f \in \mathcal{U}_1$, $x_1(f)$ is a unique non-degenerate fixed point of f in U. Choose a neighbourhood $\mathcal{U}$ of H such that for $K \in \mathcal{U}$, $K_\lambda \in \mathcal{U}_1$ for all $\lambda \in [\lambda_0 - \varepsilon, \lambda_0 + \varepsilon] = J$. Then $x_1(K_\lambda)$ is a unique fixed point of K_λ in U. Define $x : \mathcal{U} \times J \to U$, $x(K, \lambda) = x_1(K_\lambda)$. x is obviously continuous and has all the required properties. The statement for hyperbolicity follows in exactly the same way. $\square$

Contrary to the continuation of fixed points of maps, the replacement of a non-degenerate fixed point of a homotopy by a hyperbolic one does not follow trivially from the original lemma. This is essentially because we want the homotopy to become $\mathcal{C}^r$. That is, if we excise some part of $[0, 1]$ and replace the homotopy there with a restricted homotopy with only hyperbolic fixed points near the non-degenerate fixed points of the original homotopy, we have to make sure that this replacement takes place in a smooth or, at least, $\mathcal{C}^r$ way. We even show a bit more, namely that we can make a given branch of non-degenerate fixed points hyperbolic and glue the new branch into the old homotopy without changing it outside of the interval we were replacing.

Lemma 1.2.2.5 *Let $H : M \times I \to M$ be $\mathcal{C}^r$ and $x_\lambda \in M$ a non-degenerate fixed point of H_λ. Then for any neighbourhood $\mathcal{U}$ of H, any sufficiently small neighbourhood U of x_λ and any sufficiently small neighbourhood J of λ there is a homotopy $K \in \mathcal{U}$ such that the following holds:*

1. *If x_μ is the unique and non-degenerate fixed point of H_μ in U (and U is chosen such that these fixed points exist for $\mu \in J$, according to Proposition 1.2.2.4), x_μ is a hyperbolic fixed point of K_μ for $\mu \in J$.*

2. *x_μ is the only fixed point of K_μ in U for all $\mu \in J$.*

3. *K_μ equals H_μ outside of U for $\mu \in [0,1]$.*

4. *$K_\mu = H_\mu$ for $\mu \notin J$.*

Proof. Take a chart (ψ, V), $V \subseteq U$, around x_λ such that H_μ maps a subset $V' \subseteq V$ into V for any μ in a neighbourhood J of λ. Furthermore, $\psi(V) = \mathbb{R}^n, \psi(V') = \mathbb{B}_1(0)$, $\psi(x_\lambda) = 0$ and $\psi(x_\mu) \in V'$ for $\mu \in J$. Let

$$\tilde{H}_\mu = \psi \circ H_\mu \circ \psi^{-1}\big|_{\mathbb{B}_1(0)} : \mathbb{B}_1(0) \to \mathbb{R}^n.$$

We can assume $J = (\lambda - \varepsilon, \lambda + \varepsilon)$ for some $\varepsilon > 0$. Let ρ be a smooth function

$$\rho : \mathbb{R} \to [0,1]\,,\ \rho(\lambda) = 1,\ \rho(\mu) = 0 \text{ for } \mu \notin J$$

and $\eta : \mathbb{B}_1(0) \to [0,1]$ a smooth Urysohn function that is equal to 1 on $\mathbb{B}_1(0) - \mathbb{B}_{\frac{3}{4}}(0)$ and equal to 0 on $\mathbb{B}_{\frac{1}{4}}(0)$. Define a map

$$\tilde{K}_\mu : \mathbb{B}_1(0) \to \mathbb{R}^n, \quad v \mapsto \eta(v) \cdot \tilde{H}_\mu + \rho(\mu) \cdot c \cdot (1 - \eta(v)) \cdot v$$

for $\mu \in J$, $c > 0$. By the same reasoning as in Lemma 1.2.1.2, when choosing c sufficiently small, we obtain maps $K_\mu : M \to M$ arbitrarily and uniformly close to H_μ, K_μ has x_μ as hyperbolic fixed point, no other fixed points in U and is equal to H_μ outside of U. Since $K_{\lambda \pm \varepsilon} = H_{\lambda \pm \varepsilon}$ and ρ runs smoothly into the constant zero function, we can extend the K_μ to a $\mathcal{C}^r$-homotopy $K : M \times I \to M$ satisfying 1. to 4. $\square$

The proof of genericity of non-degenerate homotopies is quite similar to the one for maps and uses only Thom's Transversality Theorem.

Proposition 1.2.2.6 *The set of non-degenerate homotopies is open and dense in the set of all homotopies.*

Proof. Let H be a given non-degenerate homotopy. Then there is an extension $\tilde{H} : M \times \mathbb{R} \to M$ of H such that

$$F_H : M \times \mathbb{R} \to M \times M,\ (x, \lambda) \mapsto (x, \tilde{H}(x, \lambda))$$

is transverse to the diagonal. By the corollary of Thom's theorem, Corollary 1.1.2.5, the set of maps $F : M \times \mathbb{R} \to M \times M$ transverse to the diagonal in $M \times [-1, 2]$ is open and dense. Hence, if H' is sufficiently close to H, we find an extension to $M \times \mathbb{R}$ that is transverse to the diagonal in $M \times [-1, 2]$. Certainly we can modify this homotopy to be a homotopy extending H' and being transverse to the diagonal on all of $M \times \mathbb{R}$. So the set of non-degenerate homotopies is open.

For density, let H be an arbitrary homotopy, $\mathcal{U}$ a given neighbourhood of H. Let $\tilde{H}$ be any extension of H to $M \times \mathbb{R}$. Then there is a map $F : M \times \mathbb{R} \to M \times M$ arbitrarily close to F_H and transverse to the diagonal in $M \times \mathbb{R}$, again by Corollary 1.1.2.5 and the argument from above. Denote with π_1, π_2 the projections from $M \times \mathbb{R}$ to the first and second component, respectively. If $F = (f_1, f_2)$, then f_1 is close to π_1, so $f_1 \times \pi_2$ is close to the identity. We can choose F so close to F_H that $f_1 \times \pi_2$ is a diffeomorphism locally around $M \times I$ and $K = f_2 \circ (f_1 \times \pi_2)^{-1}\big|_{M \times I} \in \mathcal{U}$. $F \circ (f_1 \times \pi_2)^{-1}$, defined locally around $M \times I$, is transverse to the diagonal and of the form $(x, \lambda) \mapsto (x, K(x, \lambda))$, so K is a non-degenerate homotopy in $\mathcal{U}$. □

Again, density of hyperbolic homotopies is merely a corollary from this genericity theorem and Lemma 1.2.2.5. As already explained, openness does not hold, at least with our definition.

Proposition 1.2.2.7 *The set of hyperbolic homotopies is dense in the set of all homotopies.*

Proof. It suffices to show density in the set of non-degenerate homotopies. Let $H : M \times I \to M$ be any non-degenerate homotopy, $\lambda_1, \dots, \lambda_m$ the bifurcation parameters of H. Choose any parameter $\mu_1 \in [0, \lambda_1]$. Let I_1 be the maximal connected subset of $[0, \lambda_1]$ containing μ_1 such that there is a homotopy K in a given neighbourhood $\mathcal{U}$ of H such that K_μ is hyperbolic for μ in the interior of I_1 and $K_\mu = H_\mu$ for $\mu \notin I_1$. Assume $\partial I_1 \neq \{0, \lambda_1\}$. Then $\partial I_1 = \{\mu_1^-, \mu_1^+\}$ and at least one of these parameters is non-degenerate, say, μ_1^+. Applying Lemma 1.2.2.5 to K and μ_1^+ we find a homotopy $K' \in \mathcal{U}$ which is hyperbolic in some interval J strictly containing I_1 and equal to K, hence to H, outside of J. This contradicts the definition of I_1. We obtain a homotopy K^1 which is hyperbolic in $[0, \lambda_1)$ and equal to H outside of this interval. Applying the same construction to K^1 and the interval $[\lambda_1, \lambda_2]$, inductively we obtain a homotopy $K^{m+1} \in \mathcal{U}$ which is hyperbolic at all points different from the critical parameters of H. This proves the claim. □

We will close this chapter with deriving one more density theorem. Although having stated that we do not need control over the bifurcations along a hyperbolic homotopy as long as the homotopy is non-degenerate, this is not the whole truth. We can circumvent the usage of eigenvalue crossing conditions as in [Bru70] to find generic subsets of hyperbolic homotopies. But when we pass from a homotopy of vector fields to a homotopy of Poincaré maps, to make use of index theory, we have to make sure that no fixed points of the Poincaré maps occur on the boundary of the Poincaré discs. The solution lies in the dimension formula for the preimage of submanifolds under a transverse map, Proposition 1.1.2.3, which allows us to deduce that homotopies avoiding submanifolds of codimension two are generic. Boundaries of discs in a manifold are submanifolds of codimension two, so the index condition may be fulfilled generically.

Proposition 1.2.2.8 *Let $S \subseteq M$ be a compact submanifold of codimension $c \geq 2$. Then the set of homotopies $H : M \times I \to M$ that have no fixed points in S at every stage is open and dense.*

PROOF. By Thom's Transversality Theorem, the set of maps $S \times I \to S \times M$ that are transverse to the S-diagonal $\{(s, s) \in S \times M \mid s \in S\} = \Delta_S \subseteq S \times M$ is open and dense. For $H : M \times I \to M$, denote the map

$$S \times I \to S \times M,\ (s, \lambda) \mapsto (s, H(s, \lambda))$$

by $\tilde{H}$. We show that the set of homotopies $H : M \times I \to M$ such that $\tilde{H}$ is transverse to the S-diagonal is open and dense in the set of homotopies. In this case, $\tilde{H}^{-1}(\Delta_S)$ is a submanifold of $S \times I$ of dimension $1 - c$. Since $c \geq 2$, $\tilde{H}^{-1}(\Delta_S)$ is empty, i.e. H has no fixed points in S at every stage.

Openness: Let $H : M \times I \to M$ be a homotopy such that $\tilde{H}$ is transverse to Δ_S. Then if $K : M \times I \to M$ is close to H, $\tilde{K}$ is close to $\tilde{H}$, hence is transverse to Δ_S.

Density: Let $H : M \times I \to M$ be arbitrary and $\mathcal{U}$ a given neighbourhood of H. Choose a map $f = f_1 \times f_2 : S \times I \to S \times M$ that is transverse to the diagonal. f can be chosen arbitrarily close to $\tilde{H}$, so the map $f_1 \times \pi_2$ is close to the identity and we can assume it is a diffeomorphism. The map

$$f \circ (f_1 \times \pi_2)^{-1}$$

will still be transverse to the S-diagonal and it is of the form

$$f \circ (f_1 \times \pi_2)^{-1}(s, \lambda) = (s, \tilde{f}(s, \lambda))$$

for some map $\tilde{f} : S \times I \to M$. $\tilde{f}$ is a homotopy arbitrarily close to $\tilde{H}$. So we are done if we can show that we can extend $\tilde{f}$ to a map $M \times I \to M$ that is an element of $\mathcal{U}$.

Embed M into some euclidean space $\mathbb{R}^N$ and let U be a tubular neighbourhood of M in $\mathbb{R}^N$. Furthermore, let V be a tubular neighbourhood of S in M. Choose a smooth Urysohn function $\eta : M \to [0, 1]$ with $\eta^{-1}(0) = S$, $\eta^{-1}(1) = M - V$. Let $r : V \to S$ be the normal retraction, similarly $R : U \to M$. Define a homotopy

$$F : M \times I \to M,\ (x, \lambda) \mapsto R(\eta(x) \cdot H(x, \lambda) + (1 - \eta(x)) \cdot \tilde{f}(r(x), \lambda)).$$

Since $\eta(x) = 1$ if $x \notin V$, this is well defined. F equals H outside of $M - V$ and $\tilde{f}$ on S. Furthermore, it is arbitrarily close to H when choosing the tubular neighbourhoods appropriately, in particular we can achieve that $F \in \mathcal{U}$. Hence, the set of homotopies without fixed points in S is dense and the proposition is proven. □

Corollary 1.2.2.9 *The set of non-degenerate homotopies that have no fixed points in a given compact submanifold $S \subseteq M$ of codimension 2 is open and dense in the set of all homotopies.*

PROOF. This set is the intersection of two open and dense subsets, hence it is open and dense. □

1.2.3 Genericity in the Space of Vector Fields

The genericity theorems for vector fields we will present in this chapter are of a somewhat different nature as the ones for maps. Recall the spaces $\mathfrak{X}(M, \Omega, a, b)$, introduced in the introductory section, which consist of the smooth vector fields on M without periodic orbits meeting $\partial(\Omega \times (a, b))$. We will prove all theorems in the spaces $\mathfrak{X}(M, \Omega, a, b)$. So we always have a priori bounds on the periods of periodic orbits. This is in fact essential to obtain openness. When speaking of hyperbolic or non-degenerate vector fields, we will always mean that the essential periodic orbits are hyperbolic or non-degenerate, respectively. We will also employ geometric techniques rather than techniques from transversality theory. This will have the effect that we can prove the hyperbolicity statements along the lines of the non-degeneracy statements by just substituting some words. The reason of course is the technique of deducing the results by substitution of fields by their Poincaré maps, and on the level of maps, we already have all genericity results at hand. For the remainder of this section, if ξ is a vector field, φ will always denote the flow of ξ without further labelling, if no confusion is possible.

The definition of non-degeneracy still depends on transversality, the reason being the following lemma.

Lemma 1.2.3.1 *Let $\xi : M \to TM$ be a vector field. A periodic orbit γ of ξ is non-degenerate if and only if the map*

$$F_\xi : M \times \mathbb{R} \to M \times M,\ (x, t) \mapsto (x, \varphi(x, t))$$

is transverse to the diagonal at (x, T), where (x, T) is a periodic point in γ.

PROOF. γ being non-degenerate means that the differential of any Poincaré map centered at x with return time T does not have 1 as an eigenvalue. Transversality of F_ξ to the diagonal means that

$$\{(v, T_x\varphi_T(v) + \lambda \cdot \xi(x)) + (w, w) \mid v, w \in T_xM, \lambda \in \mathbb{R}\} = T_{(x,x)}M \times M.$$

Clearly, this is equivalent to simplicity of the eigenvalue 1 of $T_x\varphi_T$ (since $\xi(x)$ is an eigenvector to the eigenvalue 1). But the non-trivial eigenvalues of $T_x\varphi_T$ are just the eigenvalues of the Poincaré map. This proves the claim. $\square$

So when speaking of a non-degenerate vector field in $\mathfrak{X}(M, \Omega, a, b)$, we mean a field such that the map $F_\xi : M \times \mathbb{R} \to M \times M,\ (x, t) \mapsto (x, \varphi(x, t))$ is transverse to the diagonal in $\Omega \times (a, b)$. The requirement that there are no periodic points on the boundary of this set will make genericity proofs possible.

Our general strategy is to reduce questions of the dynamical behaviour of vector fields locally around their periodic orbits to the dynamical behaviour of associated Poincaré maps. For this cause it is important that, locally, all maps can occur as Poincaré maps of a vector field (which is not true globally, compare [KH95]). One main idea in the proof of genericity of vector fields will be to replace the Poincaré map of a given field by one that is hyperbolic and arbitrarily close to the initial one. Then we want to deduce that there is a field having this map as Poincaré map. The following lemma shows that this approach may work. It is essentially taken from [Fie80].

Lemma 1.2.3.2 *Let γ be a periodic orbit of a vector field $\xi : M \to TM$. Let (D, D', P, t) be a Poincaré system around γ centered at x_0 and let V, U be open neighbourhoods of the underlying geometric orbit of γ in M such that*

$$\overline{U} \subseteq \bigcup_{x \in D'} \varphi(x, [0, t(x)])$$

and

$$\varphi(x, [0, t(x)]) \subseteq U$$

for $x \in \overline{V} \cap D'$. Then there is a neighbourhood $\mathcal{U}$ of P in the set of maps $D' \to D$ equal to P outside of $V \cap D'$ and a continuous map $\chi : \mathcal{U} \to \mathfrak{X}(M)$, satisfying

1. *For $Q \in \mathcal{U}$, $\chi(Q)$ has Poincaré map Q.*
2. *For $Q \in \mathcal{U}$, $\chi(Q)$ equals ξ outside of U.*
3. *$\chi(P) = \xi$.*

Proof. Figure 4 shows the setup with the various neighbourhoods of γ. Let φ be the

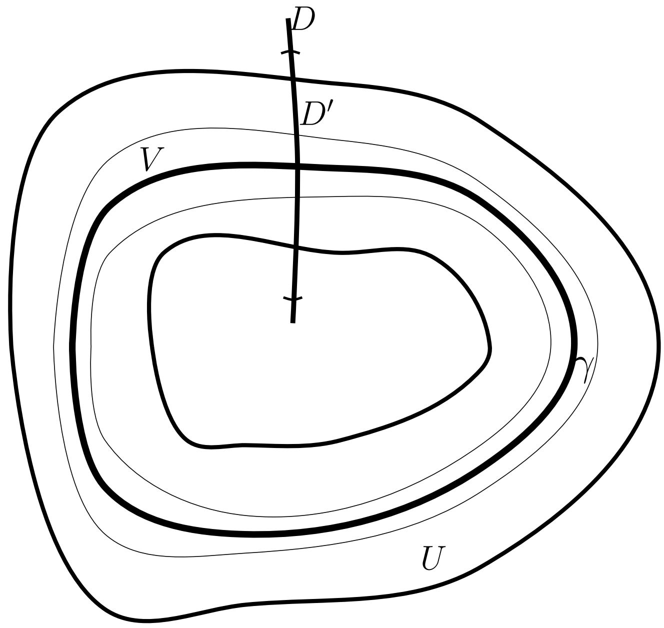


Figure 4: Suitable tubular neighbourhoods of a periodic orbit

flow of ξ. The period map t is bounded from below on D' by, say, $\tau > 0$. Choose real numbers $0 < a < b < \tau$. Now let $Q \in \mathcal{C}^\infty(D', D)$ be any map equal to P outside of $V \cap D'$. For Q close to P, $P^{-1} \circ Q : D' \to D$ is close to the inclusion $D' \to D$. We can extend $P^{-1} \circ Q$ to $\overline{D'}$ by taking it to be the identity on the boundary, since Q equals P there. But in a neighbourhood of the inclusion $i : \overline{D'} \to D$, all maps are isotopic to i and are embeddings, hence we find an isotopy $H : \overline{D'} \times [a, b] \to D$ joining i and $P^{-1}Q$. We extend H to a map $\overline{D'} \times [0, \rho] \to D$, where ρ is the maximum of $t(x)$ in D', by taking $H(x, s)$ to be $i(x)$ for $s < a$ and $H(x, s) = P^{-1}Q(x)$ for $s > b$. Certainly we can also achieve that H is smooth by a small perturbation. Define

$$\psi(y, s) = \varphi(H(y, s), s), \; s \in [0, t(y)] .$$

Since H is close to i at any stage s, if $y \in V \cap D'$, $\psi(y, s)$ does not meet $M - U$. Furthermore we can achieve that $\psi\big|_{D' \times [a,b]}$ is an embedding, because the set of embeddings is open and $\varphi\big|_{D' \times [a,b]}$ is an embedding. Thus, the curves $\psi(y, \cdot)$ are pairwise disjoint. Define a vector field ξ'' on the image of ψ by setting

$$\xi''(\psi(y, s)) = \dot{\psi}(y, s).$$

Since the image of ψ and $M - U$ are disjoint closed sets, we find a field ξ' extending ξ'' on the image of ψ and ξ on $M - U$. The integral curves of ξ' are, up to reparametrization, just the curves $\psi(y, \cdot)$. The Poincaré map of ξ' thus is calculated by the equation

$$\psi(y, t'(y)) \in D$$

for some map $t' : D' \to \mathbb{R}^+$ close to t. Let $t'(y) = t(P^{-1} \circ Q(y))$. Then

$$\psi(y, t'(y)) = \varphi(H(y, t'(y)), t'(y)) = \varphi(P^{-1} \circ Q(y), t'(y)) = Q(y)$$

by definition of P and t, i.e. Q is the Poincaré map of ξ'. Define $\chi(Q) = \xi'$. Then all the properties stated are obvious. □

The following result is in analogy to Proposition 1.2.1.1 and is needed to show openness of non-degenerate or hyperbolic vector fields.

Lemma 1.2.3.3 *Let $\xi_0 : M \to TM$ be a vector field and γ_0 a non-degenerate periodic orbit of ξ_0, $(x_0, T_0) \in \gamma_0$. Then there is a neighbourhood $U \times J \subseteq M \times \mathbb{R}$ of γ_0, a neighbourhood $\mathcal{U}$ of ξ_0 and a continuous map $x \times T : \mathcal{U} \to U \times J$ such that $(x(\xi), T(\xi))$ is a periodic point of $\xi \in \mathcal{U}$, $x(\xi_0) = x_0$, $T(\xi_0) = T_0$, and the orbit $\gamma(\xi)$ through $(x(\xi), T(\xi))$ is the unique periodic orbit of ξ in $U \times J$ and is non-degenerate. If γ_0 happens to be hyperbolic, then, after possibly shrinking $U \times J$ and $\mathcal{U}$, $\gamma(\xi)$ is hyperbolic as well. Moreover, a similar result holds for geometric periodic orbits, forgetting the period component.*

Proof. Take a Poincaré system (D, D', P, t) for γ_0, centered at $x_0 \in \gamma_0$. Then x_0 is a non-degenerate fixed point of P. By Proposition 1.2.1.1, every map $Q \in \mathcal{C}^r(D', D)$ in a neighbourhood $\mathcal{V}$ of P has a unique non-degenerate fixed point in some open neighbourhood $W \subseteq D'$ of x_0. Let

$$U = \bigcup_{x \in W} \varphi(x, [0, t(x)])$$

and $\mathcal{U} \subseteq \mathfrak{X}(M)$ be the subset of those vector fields whose Poincaré map is defined on D' and is an element of $\mathcal{V}$. By Lemma 1.1.4.5, $\mathcal{U}$ is open. For $\xi \in \mathcal{U}$, let $P(\xi) \in \mathcal{V}$ be the Poincaré map of ξ on D', obtained from P by continuation. Then $P(\xi)(y)$ is characterized by the property $\varphi_\xi(y, t_\xi(y)) \in D$, where $t_\xi(y)$ is close to $t(y)$. Let $x(\xi)$ be the unique fixed point of $P(\xi)$ in W and $T(\xi) = t_\xi(x(\xi))$. Clearly, $(x(\xi), T(\xi))$ is a periodic point of ξ and its orbit is the only periodic orbit of ξ in $U \times J$, where J is a sufficiently small neighbourhood of T. Furthermore, since $x(\xi)$ is a non-degenerate fixed point of $P(\xi)$, the corresponding periodic orbit of ξ is non-degenerate. Continuity of T is clear and continuity of x follows from the continuous dependence of the fixed point on P. The hyperbolicity statement follows in the same way from Proposition 1.2.1.1. The statement for the geometric orbits follows by applying what we have shown to the minimal period and using that the period of periodic orbits near a periodic orbit with minimal period must be minimal as well, compare Proposition 1.1.4.11. □

Before we can proceed to the main theorem, we have to deal with the question of the behaviour of the period of a periodic orbit under small perturbations. As already mentioned, we want to make periodic orbits with period in a compact interval hyperbolic, and push all other orbits into a region where their period must become large. So the unwanted orbits should become inessential. The following lemma shows that this method works.

Lemma 1.2.3.4 *Let $K \subseteq M$ be compact and assume that the vector field ξ has no periodic orbits meeting K of a period in a compact interval $[a, b]$. Then there is a neighbourhood $\mathcal{U}$ of ξ such that if a periodic orbit of $\eta \in \mathcal{U}$ meets K, its period does not lie in $[a, b]$.*

Proof. Take $x \in K$. We have three possibilities.

1. The orbit through x is not periodic.

   In this case there is an $\varepsilon_x > 0$ and a neighbourhood U_x of x such that $\varphi(y, t) \notin U_x$ for $y \in U_x$ and $t \in [\varepsilon_x, b + \varepsilon_x]$.

2. The geometric orbit through x has minimal period larger than b.

   We can find neighbourhoods U_x and ε_x as above.

3. The geometric orbit through x has minimal period p and $k \cdot p < a < b < (k+1) \cdot p$ for some $k \in \mathbb{N}$.

   In this case we find a neighbourhood U_x of x and $\varepsilon_x > 0$ such that $\varphi(y, t) \notin U_x$ for $y \in U_x$ and $t \in [a - \varepsilon_x, b + \varepsilon_x]$.

In any case we find neighbourhoods $\mathcal{U}_x$ of ξ such that the respective property holds for all fields in $\mathcal{U}_x$. The sets U_x cover K and we find a finite subcover, corresponding to elements $x_1, \dots, x_m \in K$. Let

$$\mathcal{U} = \mathcal{U}_{x_1} \cap \cdots \cap \mathcal{U}_{x_m}.$$

Then if x is a point in K, the orbit of $\eta \in \mathcal{U}$ through x has no period inside of $[a, b]$. □

This lemma allows us to prove an openness statement for hyperbolic vector fields that is stronger than just openness of the set of hyperbolic vector fields. We need this result later in the proof of density.

Proposition 1.2.3.5 *Let $K \subseteq \Omega$ be a compact subset and $\xi \in \mathfrak{X}(M, \Omega, a, b)$ such that all essential periodic orbits of ξ in K are hyperbolic. Then the same is true for all fields in a neighbourhood $\mathcal{U}$ of ξ.*

Proof. Since ξ is hyperbolic in K, it is in particular non-degenerate in K, i.e. the map

$$\Omega \times (a, b) \to M \times M, \ (x, \lambda) \mapsto (x, \varphi(x, \lambda))$$

is transverse to the diagonal in $K \times [a, b]$. Since the set of such maps is open in the set of all maps by Corollary 1.1.2.5, every field in a neighbourhood $\mathcal{U}_1$ of ξ induces a map that is transverse to the diagonal in $K \times [a, b]$, i.e. all fields in $\mathcal{U}_1$ are non-degenerate in K. Now by Lemma 1.2.3.3, since ξ was hyperbolic, all non-degenerate fields in a neighbourhood $\mathcal{U} \subseteq \mathcal{U}_1$ of ξ are hyperbolic as well. This proves the lemma. □

We will briefly sketch the idea of the proof of the genericity theorem. Since we have no periodic orbits on the boundary of $\Omega \times [a, b]$, the union of all geometric essential periodic orbits is compact. We cover this set by tubular neighbourhoods of the essential orbits such that we can replace our initial field by a field that is hyperbolic in the tubular neighbourhoods. We do this by using Lemma 1.2.3.2 and Poincaré systems. By a compactness argument, this process will result in a field that is close to the initial one and is hyperbolic locally in a neighbourhood of the set of essential periodic orbits of the initial field. But we will also take care that all periodic orbits that might occur outside of this neighbourhood are inessential, using Lemma 1.2.3.4. Then the new field is hyperbolic. The proof presented here is a version based on methods from [Fie80] and [PdM82]. The result itself is well-known as the Kupka-Smale theorem. This theorem makes the even stronger statement that the stable and unstable manifolds of different critical elements generically meet transversally. We do not need this property so we will not consider it any further.

Theorem 1.2.3.6 *The set of hyperbolic vector fields is open and dense in $\mathfrak{X}(M, \Omega, a, b)$.*

Proof. Openness follows trivially from Lemma 1.2.3.5 by taking $K = \overline{\Omega}$. For the sake of comprehensibility, we divide the proof of density into several steps. Since the proof of the same theorem for homotopies of vector fields is very similar, this also facilitates the comparison of the proofs.

1. Take any vector field $\xi \in \mathfrak{X}(M, \Omega, a, b)$ and let $\mathcal{U}$ be a given neighbourhood of ξ. Let φ be the flow of ξ. Define

   $$\Gamma = \{x \in \Omega \mid \varphi(x, t) = x \text{ for some } t \in [a, b]\}.$$

   The set Γ is compact. For every essential periodic orbit $\gamma \subseteq \Gamma$, choose a Poincaré system $(D_\gamma, D'_\gamma, P_\gamma, t_\gamma)$ such that in a neighbourhood $\mathcal{U}_\gamma \subseteq \mathcal{U}$ of ξ, all the Poincaré maps of elements of $\mathcal{U}_\gamma$ are defined as maps $D'_\gamma \to D_\gamma$.

2. Choose open neighbourhoods $W_\gamma \subseteq V_\gamma \subseteq U_\gamma$ of the underlying geometric orbit of γ such that $\overline{W}_\gamma \subseteq V_\gamma$ and for all flows $\tilde{\varphi}$ of elements in $\mathcal{U}_\gamma$, we have

$$\overline{U}_\gamma \subseteq \bigcup_{x \in D'_\gamma} \tilde{\varphi}(x, [0, \tilde{t}(x)])$$

and

$$\tilde{\varphi}(x, [0, \tilde{t}(x)]) \subseteq U_\gamma \quad \text{for } x \in \overline{V}_\gamma \cap D'_\gamma.$$

3. The sets W_γ cover Γ, hence we can find a finite subcover, corresponding to orbits $\gamma_1, \dots, \gamma_m$. Let

$$W = W_{\gamma_1} \cup \dots \cup W_{\gamma_m},\ \mathcal{U}_1 = \mathcal{U}_{\gamma_1} \cap \dots \cap \mathcal{U}_{\gamma_m} \subseteq \mathcal{U}.$$

The set $K = \overline{\Omega} - W$ is compact and all periodic orbits of ξ meeting K have period not in $[a, b]$. Hence, by Lemma 1.2.3.4, the same is true for all elements of a neighbourhood $\mathcal{U}_2$ of ξ and we can assume $\mathcal{U}_1 \subseteq \mathcal{U}_2$. Every element η of $\mathcal{U}_1$ has the following properties:

   (i) $\eta \in \mathcal{U}$.

   (ii) Periodic orbits of η meeting K are inessential.

   (iii) Lemma 1.2.3.2 is applicable to η and all the sets $V_{\gamma_j}, U_{\gamma_j}$ with the corresponding Poincaré systems for $j = 1, \dots, m$.

4. Assume that we have constructed a field $\xi_k \in \mathcal{U}_1$ that is hyperbolic in $\overline{W_1 \cup \dots \cup W_k}$ for some $0 \leq k \leq m-1$, where $k = 0$ simply means $\xi_0 \in \mathcal{U}_1$. We find a neighbourhood $\mathcal{W}_k$ of ξ_k such that every element of $\mathcal{W}_k$ is hyperbolic in $\overline{W_1 \cup \dots \cup W_k}$. Apply Lemma 1.2.3.2 to the sets $V_{\gamma_{k+1}}, U_{\gamma_{k+1}}$ and the corresponding Poincaré system to obtain a neighbourhood $\mathcal{V}_{k+1}$ of the Poincaré map $P_{\gamma_{k+1}}$ in the set of maps $D'_{\gamma_{k+1}} \to D_{\gamma_{k+1}}$ equal to $P_{\gamma_{k+1}}$ outside of $V_{\gamma_{k+1}} \cap D'$ and the map $\chi : \mathcal{V}_{k+1} \to \mathfrak{X}(M, \Omega, a, b)$. Now take a hyperbolic map $\overline{W_{\gamma_{k+1}}} \cap D'_{\gamma_{k+1}} \to D_{\gamma_{k+1}}$ so close to $P|_{W_{\gamma_{k+1}} \cap D'_{\gamma_{k+1}}}$ that there is an extension to a map $Q : D'_{\gamma_{k+1}} \to D_{\gamma_{k+1}}$ equal to P outside of $V_{\gamma_{k+1}} \cap D'$ and $Q \in \mathcal{V}_{k+1}$. By continuity of χ, we can achieve that $\xi_{k+1} = \chi(Q) \in \mathcal{U}_1 \cap \mathcal{W}_k$. Thus, ξ_{k+1} has the following properties:

   - All periodic orbits of ξ_{k+1} in $\overline{W_{\gamma_1} \cup \dots \cup W_{\gamma_{k+1}}}$ are hyperbolic.
   - All periodic orbits of ξ_{k+1} meeting K are inessential.
   - $\xi_{k+1} \in \mathcal{U}_1$.

5. Inductively, we find that ξ_m is a hyperbolic vector field in our sense, proving the theorem.

□

We state, for completeness, the canonical corollary that non-degenerate vector fields are open and dense in the set of vector fields. The proof is almost trivial, since hyperbolic fields are non-degenerate, which is the density part, and Proposition 1.2.3.5 in fact proves openness of non-degenerate fields along its lines. We could also take the proof of Theorem 1.2.3.6 and, in the induction, take a non-degenerate map instead of a hyperbolic one. This would yield genericity of non-degenerate vector fields instead of hyperbolic ones.

Corollary 1.2.3.7 *The set of non-degenerate vector fields in $\mathfrak{X}(M, \Omega, a, b)$ is open and dense in that set.*

1.2.4 Genericity in the Space of Homotopies of Vector Fields

The final section on non-equivariant genericity theorems is at the same time the most important one. We will prove parametrized versions of the genericity results of the previous section, i.e. we will do generic bifurcation theory of periodic orbits with a priori bounds. We work with non-degenerate homotopies instead of hyperbolic ones, because, as we saw in the case of maps, we cannot expect the set of hyperbolic homotopies to be open. Our strategy will be the same as for fields. We use geometric techniques to deduce directly the density of non-degenerate homotopies. Transversality theory will help us with the openness part. We begin with the definition of non-degenerate homotopies of vector fields and the observation that, since we are dealing with vector fields, our standing assumptions hold that all fields involved are elements of $\mathfrak{X}(M, \Omega, a, b)$ for fixed $\Omega \subseteq M$ open, $0 < a < b < \infty$.

Definition 1.2.4.1 *Let M be a compact manifold, $\Omega \subseteq M$ an open subset, $0 < a < b$ real numbers. Define the set $h\mathfrak{X}(M, \Omega, a, b) \subseteq h\mathfrak{X}(M)$ to be the set of homotopies of vector fields on M such that any H_λ, $\lambda \in I$, is an element of $\mathfrak{X}(M, \Omega, a, b)$. We call the periodic orbits in Ω of a period in $[a, b]$ the* essential *periodic orbits of the homotopy.*

We call H non-degenerate*, if there is an extension to a homotopy $\tilde{H} : M \times \mathbb{R} \to TM$ with all maps $\tilde{H}_\lambda$, $\lambda \in \mathbb{R}$ in $\mathfrak{X}(M, \Omega, a, b)$, such that the map*

$$F_H : \Omega \times (a, b) \times \mathbb{R} \to M \times M, \ (x, t, \lambda) \mapsto (x, \tilde{\varphi}_\lambda(x, t))$$

is transverse to the diagonal. Here, $\tilde{\varphi}$ is the flow of $\tilde{H}$.

In this section, we will denote the flow of a homotopy H by φ, i.e. $\dot{\varphi}(x, \lambda) = H(\varphi(x, \lambda), \lambda)$.

Non-degeneracy again implies finiteness of bifurcations, which brings us closer to our generic bifurcation scenario for periodic orbits.

Proposition 1.2.4.2 *If a homotopy H of vector fields is non-degenerate, then H_λ is non-degenerate for all but a finite number of parameters.*

Proof. Take any extension $\tilde{H} : M \times \mathbb{R} \to M \times M$ of H such that

$$F_H : M \times \mathbb{R} \times \mathbb{R} \to M \times M, \ (x, t, \lambda) \mapsto (x, \tilde{\varphi}_\lambda(x, t))$$

is transverse to the diagonal. By Proposition 1.1.2.6, the set of parameters $t \in \mathbb{R}$ such that

$$(x, t) \mapsto (x, \varphi_\lambda(x, t))$$

is transverse to the diagonal is open and dense in $\mathbb{R}$, so its complement is discrete and locally finite. Hence, the set of bifurcation parameters in I is finite. $\square$

If γ is a periodic orbit of a homotopy at some stage λ, then, by choosing a Poincaré system accordingly, we obtain a homotopy of Poincaré maps, locally around λ. Using this fact, we are going to generalize Lemma 1.2.3.2 by proving that, if a homotopy is close to the "Poincaré homotopy" of H locally, then it is itself a Poincaré homotopy of another homotopy of vector fields. Furthermore, if the Poincaré homotopies are close, then so are the homotopies. It should be clear that such a result should enable us to easily generalize the proof of density of non-degenerate vector fields to the case of homotopies.

Lemma 1.2.4.3 *Let H be a homotopy of vector fields, γ_λ a periodic orbit of H_λ. Choose a Poincaré system $(D, D', P_\lambda, t_\lambda)$ for γ_λ, centered at x_0, such that the Poincaré maps of all fields in a neighbourhood $\mathcal{U}_1$ of H_λ are defined as maps $D' \to D$. Let V, U be open neighbourhoods of γ_λ such that*

$$\overline{U} \subseteq \bigcup_{x \in D'} \varphi_\mu(x, [0, t_\mu(x)])$$

and

$$\varphi_\mu(x, [0, t_\mu(x)]) \subseteq U$$

for $x \in \overline{V} \cap D'$ and μ in a neighbourhood of λ, say, $|\lambda - \mu| \leq 3\varepsilon$, $\varepsilon > 0$. The Poincaré maps of the fields H_μ constitute a homotopy of maps

$$P : D' \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon] \to D.$$

Then there is a neighbourhood $\mathcal{U}$ in the set of homotopies $D' \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon] \to D$ that are equal to P outside of $V \cap D' \times [\lambda - 2\varepsilon, \lambda + 2\varepsilon]$ and a continuous map $\chi : \mathcal{U} \to h\mathfrak{X}(M, \Omega, a, b)$ such that

1. *for $Q \in \mathcal{U}$, $\chi(Q)_\mu$ has Poincaré map Q_μ for $|\lambda - \mu| \leq \varepsilon$.*
2. *for $Q \in \mathcal{U}$, $\chi(Q)$ equals H outside of $U \times [\lambda - 2\varepsilon, \lambda + 2\varepsilon]$.*
3. *$\chi(P) = H$.*

Proof. Let t_ξ be the period map of $\xi \in \mathcal{U}_1$. Then

$$m = \inf_{x \in D'} \inf_{\xi \in \mathcal{U}_1} t_\xi(x) > 0.$$

Choose real numbers a, b with $0 < a < b < m$. Let $Q : D' \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon] \to D$ be any homotopy that is equal to P outside of $V \cap D' \times [\lambda - 2\varepsilon, \lambda + 2\varepsilon]$. There is an isotopy

$K : \overline{D'} \times [a,b] \times [\lambda - 2\varepsilon, \lambda + 2\varepsilon]$ connecting the inclusion $\overline{D'} \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon]$ and the map $P^{-1} \circ Q$. Here, P^{-1} is defined fibrewise and on $\partial(D' \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon])$, $P^{-1} \circ Q$ is extended to be the identity. Define

$$\psi(y, s, \mu) = \varphi(K(y, s, \mu), s)$$

for $y \in D'$ und $s \in [0, t_\mu(y)]$, where K is taken to be constant outside of $[a, b]$. Then $\psi(y, a, \mu) = \varphi(y, s, \mu)$ for (y, μ) in a neighbourhood of the boundary of $D' \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon]$. Furthermore, by choosing Q sufficiently close to P, none of the curves $s \mapsto \psi(y, s, \mu)$ meets $M - U$. Since the set of embeddings is open, we can achieve in addition that the curves $s \mapsto \psi(y, s, \mu)$ for fixed μ and $s \in [a, b]$ are pairwise disjoint, since φ_μ is an embedding when restricted to $[a, b]$. With $\mu \in [\lambda - \varepsilon, \lambda + \varepsilon]$, define a field Ψ_μ on the image of ψ_μ by

$$\Psi_\mu(\psi(y, s, \mu)) = \frac{d}{dr}\psi(y, r, \mu)\Big|_{r=s}.$$

This yields a smooth homotopy Ψ. Take $\Psi_\mu = H_\mu$ outside of U. We can extend Ψ to a homotopy $M \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon]$ of vector fields equal to H outside of $U \times [\lambda - 2\varepsilon, \lambda + 2\varepsilon]$. The integral curves of Ψ_μ, with $|\lambda - \mu| < \varepsilon$, coincide, as sets, with the image of the curves $\psi(y, \cdot, \mu)$. We calculate

$$\psi(y, t_\mu(y), \mu) = \varphi(K(y, t_\mu(y), \mu), t_\mu(y), \mu) = \varphi(P_\mu^{-1} \circ Q_\mu(y), t_\mu(y), \mu) = Q_\mu(y).$$

Hence, the Poincaré homotopy of Ψ in $[\lambda - \varepsilon, \lambda + \varepsilon]$ is given by Q. The definition $\chi(Q) = \Psi$ gives the required result. $\square$

Now we are going to deal with openness of non-degenerate homotopies, where we will use the transversality characterization of non-degeneracy.

Lemma 1.2.4.4 *Let $H \in h\mathfrak{X}(M, \Omega, a, b)$ and assume that H is non-degenerate in a compact subset $K \times J \subseteq \Omega \times [0, 1]$. Then there is a neighbourhood $\mathcal{U}$ of H such that every element of $\mathcal{U}$ is non-degenerate in $K \times J$.*

Proof. The fact that H is non-degenerate translates into transversality to the diagonal of the map

$$M \times \mathbb{R} \times \mathbb{R} \to M \times M, \ (x, \lambda, t) \mapsto (x, \tilde{\varphi}_\lambda(x, t))$$

in $K \times J \times [a, b]$, where $\tilde{\varphi}$ is the flow of an extension of H. By Corollary 1.1.2.5, the set of maps $M \times \mathbb{R} \times \mathbb{R} \to M \times M$ transverse to the diagonal in $K \times J \times [a, b]$ is open. Clearly, given a neighbourhood $\mathcal{U}_1$ of $\tilde{H}$ of maps such that this transversality condition is fulfilled, we find a neighbourhood $\mathcal{U}$ of H such that every element in $\mathcal{U}$ has an extension whose associated map lies in $\mathcal{U}_1$. This proves the assertion. $\square$

The proof of the main and final theorem is now a replication of the proof of Theorem 1.2.3.6. We will use the same notation and enumeration to make the comparison of the two easier. The recipe of proof we laid out before Theorem 1.2.3.6 can be taken as a recipe for the following proof as well.

Theorem 1.2.4.5 *The set of non-degenerate homotopies of vector fields is open and dense in $h\mathfrak{X}(M,\Omega,a,b)$.*

PROOF. Openness follows immediately by taking $K \times J = \overline{\Omega} \times [0,1]$ in Lemma 1.2.4.4. For density, we now imitate the steps in the proof of Theorem 1.2.3.6.

1. Take any homotopy $H \in h\mathfrak{X}(M,\Omega,a,b)$ and let $\mathcal{U}$ be a given neighbourhood of H, φ the flow of H. Define

$$\Gamma = \{(x,\lambda) \in \Omega \times [0,1] \mid \varphi_\lambda(x,t) = x \text{ for some } t \in [a,b]\}.$$

   Γ is clearly compact. For every essential periodic orbit $\gamma \times \{\lambda\}$, choose a Poincaré system $(D_\gamma, D'_\gamma, P_\gamma, t_\gamma)$ and a neighbourhood $\mathcal{U}_\gamma$ of H such that the Poincaré maps of all elements of $\mathcal{U}_\gamma$ are defined as maps $D'_\gamma \to D_\gamma$. In particular, we find an $\varepsilon = \varepsilon(\gamma) > 0$ such that the Poincaré maps of H_μ, $\lambda - 3\varepsilon < \mu < \lambda + 3\varepsilon$, constitute a homotopy of maps

$$P : D'_\gamma \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon] \to D_\gamma$$

   and the same is true for all elements of $\mathcal{U}_\gamma$.

2. Choose open neighbourhoods $W_\gamma \subseteq V_\gamma \subseteq U_\gamma$ of the underlying geometric orbit of γ such that $\overline{W}_\gamma \subseteq V_\gamma$ and we have

$$\overline{U_\gamma} \subseteq \bigcup_{x \in D'_\gamma} \tilde{\varphi}_\mu(x, [0, \tilde{t}_\mu(x)])$$

   and

$$\tilde{\varphi}_\mu(x, [0, \tilde{t}_\mu(x)]) \subseteq U_\gamma$$

   for $x \in \overline{V_\gamma} \cap D'_\gamma$, $\tilde{\varphi}$ the flow of an element in $\mathcal{U}_\gamma$ and $\mu \in [\lambda - 3\varepsilon, \lambda + 3\varepsilon]$.

3. The sets $W_\gamma \times (\lambda - \varepsilon_\gamma, \lambda + \varepsilon_\gamma)$ cover Γ, so we find a finite subcover, corresponding to orbits $\gamma_1, \ldots, \gamma_m$ at parameters $\lambda_1, \ldots, \lambda_m$. Let

$$W_j = W_{\gamma_j} \times \left(\lambda_j - \varepsilon_{\gamma_j}, \lambda_j + \varepsilon_{\gamma_j}\right),$$

   $j = 1, \ldots, m$, $\varepsilon_j = \varepsilon_{\gamma_j}$. Then define

$$W = W_1 \cup \cdots \cup W_j,\ \mathcal{U}_1 = \mathcal{U}_{\gamma_1} \cap \cdots \cap \mathcal{U}_{\gamma_m}.$$

   The set $K = \overline{\Omega} \times [0,1] - W$ is compact and if a periodic orbit of H_μ meets a point x such that $(x,\mu) \in K$, the period of the orbit is not in $[a,b]$. Hence, the same is true in a neighbourhood $\mathcal{U}_2$ of H and we can assume $\mathcal{U}_1 \subseteq \mathcal{U}_2$. Every homotopy H' in $\mathcal{U}_1$ has the following properties.

   a) $H' \in \mathcal{U}$.

   b) All periodic orbits of H' meeting K (in the above sense) are inessential.

c) For $j = 1, \dots, m$, Lemma 1.2.4.3 is applicable to H', the sets V_{γ_j}, U_{γ_j} and the corresponding Poincaré system.

4. Assume that we have constructed a homotopy H_k for some $0 \leq k \leq m-1$, such that $H_k \in \mathcal{U}$, all periodic orbits of H_k meeting K are inessential and H_k is non-degenerate in $\overline{W_1 \cup \cdots \cup W_k}$. Then there is a neighbourhood $\mathcal{W}_k$ of H_k such that each element of $\mathcal{W}_k$ is non-degenerate in $\overline{W_1 \cup \cdots \cup W_k}$. Apply Lemma 1.2.4.3 to the sets $V_{\gamma_{k+1}}, U_{\gamma_{k+1}}$ and the corresponding Poincaré system to obtain the neighbourhood $\mathcal{V}_{k+1}$ and map $\chi_{k+1} : \mathcal{V}_{k+1} \to h\mathfrak{X}(M, \Omega, a, b)$ as stated in the lemma. Take a non-degenerate homotopy

$$\overline{W_{\gamma_{k+1}} \cap D'_{\gamma_{k+1}}} \times [\lambda_{k+1} - \varepsilon_{k+1}, \lambda_{k+1} + \varepsilon_{k+1}] \to D_{\gamma_{k+1}}$$

and extend it to a homotopy

$$Q : \Big(U(\gamma_{k+1}) \cap D'_{\gamma_{k+1}}\Big) \times [\lambda_{k+1} - 3\varepsilon_{k+1}, \lambda_{k+1} + 3\varepsilon_{k+1}] \to D_{\gamma_{k+1}}$$

that is equal to P outside of $\Big(V(\gamma_{k+1}) \cap D'_{\gamma_{k+1}}\Big) \times [\lambda_{k+1} - 2\varepsilon_{k+1}, \lambda_{k+1} + 2\varepsilon_{k+1}]$. By choosing the initial homotopy close enough to P, we can achieve that $Q \in \mathcal{V}_{k+1}$ and $H_{k+1} = \chi_{k+1}(Q) \in \mathcal{U}_1 \cap \mathcal{W}_k$. Hence, H_{k+1} has the following properties.

   a) H_{k+1} restricted to W_{k+1} is a non-degenerate homotopy.

   b) All periodic orbits of H_{k+1} meeting K are inessential.

   c) $H_{k+1} \in \mathcal{U}$.

5. Obviously, $H_m \in \mathcal{U}$ is non-degenerate, which proves the theorem.

□

So we finally reached our generic bifurcation scenario. Generically, vector fields are hyperbolic, and if they are homotopic (in $\mathfrak{X}(M, \Omega, a, b)$), then they are so via a non-degenerate homotopy. Whenever necessary, we can reduce locally to Poincaré systems and on the level of maps, we have the same generic bifurcation behaviour of fixed points.

1.3 Fuller Index Theory

In [Ful67], Fuller constructs an index for vector fields. In his approach, the rôle of the fixed points of self-maps is assigned to the periodic orbits of the field. Fuller uses differential forms in his approach, but the dynamical background of his index was made clear in the paper [CMP78] of Mallet–Paret and Chow. The general idea is to assign to an orbit γ with minimal period the fixed point index of an associated Poincaré map. One has to deal with several problems. First of all, it is not clear what the index of an orbit with non-minimal period should be. Secondly, one has to verify that the local definition yields a global object with nice properties. These questions have, of course, all been answered by Fuller, Mallet–Paret and Chow and others. However, in view of the symmetry perspective, it seems useful to give a modified proof of homotopy invariance to make it accessible to equivariant theory. It is now completely based on the use of the fixed point index and the generic bifurcation theory developed in the preceeding section.

In the first part of this section, we will carry out the construction of the Fuller index, using the density theorems for maps proven so far. In the second part, we will prove its properties, mainly homotopy invariance, using the density theorems for homotopies. The ideas will be explained as we go along.

1.3.1 Construction and Properties of the Index

The problem we want to deal with is the following. Take a vector field $\xi : M \to TM$, an open subset $\Omega \subseteq M$, $0 < a < b < \infty$ and assume $\xi \in \mathfrak{X}(M, \Omega, a, b)$. Give an algebraic count of the number of periodic orbits of f in Ω, counted with respect to multiplicity, such that this number is a topological invariant, i.e. remains unchanged under homotopies. Some other nice properties like additivity and the solution property well-known from fixed point theory should hold as well.

When it comes to construct such a number it becomes apparent that it is not enough to count periodic orbits with multiplicity, but we also have to take into account the periodicity. That is, if an orbit has local index 1, then the same geometric orbit, but run through twice, should have something like index $\frac{1}{2}$.

We briefly recall that the local fixed point index of a hyperbolic fixed point x of a smooth map f is defined to be $(-1)^s$, where s is the number of real eigenvalues of $D_x f$ larger than 1. By approximation and summation, this definition is extended to arbitrary continuous maps, giving the global fixed point index. For details, see [Nus77], or [Bre93] for its connection with the Lefschetz number.

Definition 1.3.1.1 *Let $\xi : M \to TM$ be a non-degenerate vector field, γ a periodic orbit of ξ with period $k \cdot p$, p its minimal period. Choose a Poincaré system (D, D', P, t) for γ. The (local)* index *of γ is the rational number*

$$I(\gamma) = \frac{1}{k} \cdot i(P, D')$$

where $i(P, D')$ is the fixed point index of P. The Fuller index *of ξ is the sum*

$$I_F(\xi, \Omega) = \sum_{\gamma} I(\gamma),$$

which runs over the finite set of periodic orbits of ξ in Ω.

Note that we can choose as Poincaré map in the definition the map P^k, where P is a Poincaré map for γ, considered with its minimal period. We just have to take care that all iterates P^j are defined on some subdisc of D, $1 \leq j \leq k$. The definition of the index does not depend on the choice of Poincaré system, since any two Poincaré maps are conjugate and the local index only depends on the eigenvalues of the Poincaré map. For details we refer to [CMP78]. We define the Fuller index of an arbitrary vector field in $\mathfrak{X}(M, \Omega, a, b)$ by approximation. It is not clear that the index is well-defined. This will follow from homotopy invariance.

We turn to some other properties of the index first. These properties are well-known for the fixed point index and they are the very ones that justify to call this object an index.

Proposition 1.3.1.2 *The Fuller index is additive, i.e. if $\xi \in \mathfrak{X}(M, \Omega, a, b)$, $\Omega_1, \Omega_2 \subseteq \Omega$, $\Omega_1 \cap \Omega_2 = \emptyset$ and all essential periodic orbits of ξ are contained in $\Omega_1 \cup \Omega_2$, then*

$$I_F(\xi, \Omega) = I_F(\xi, \Omega_1) + I_F(\xi, \Omega_2).$$

PROOF. In the given situation, clearly $\xi \in \mathfrak{X}(M, \Omega_i, a, b)$ for $i = 1, 2$. Assume that ξ is non-degenerate. Let E be the set of essential periodic orbits of ξ, $E_1 \subseteq E$ the set of essential periodic orbits contained in Ω_1, $E_2 \subseteq E$ the set of essential periodic orbits contained in Ω_2. Then

$$I_F(\xi, \Omega) = \sum_{\gamma \in E} i(\gamma) = \sum_{\gamma \in E_1} i(\gamma) + \sum_{\gamma \in E_2} i(\gamma) = I_F(\xi, \Omega_1) + I_F(\xi, \Omega_2),$$

since the local indices of the orbits clearly do not depend on the set they are contained in. This proves the proposition for non-degenerate fields and extends to arbitrary fields via approximation. □

With the next proposition, the proof of the homotopy invariance of Fullers index begins, so we outline the course of action at this point. The first step is the most difficult one. We show that, if two non-degenerate vector fields are homotopic via a non-degenerate homotopy, then their Fuller indices are equal. Having established this result, we proceed as follows. If ξ_0, ξ_1 are any two homotopic vector fields, we know that all fields locally around ξ_0 are homotopic. We show that being non-degenerately homotopic is the same as being arbitrarily homotopic. Thus, all non-degenerate fields locally around ξ_0 have the same Fuller index. As a byproduct, this yields the well-definedness of the index of an arbitrary field. Now we choose a non-degenerate homotopy H so close to the initial homotopy between ξ_0 and ξ_1, such that H_0 has the same index as ξ_0 and H_1 has the same index as ξ_1. But by our first result, H_0 and H_1 have equal Fuller index, which proves the invariance.

The crucial step in this plan is the invariance under non-degenerate homotopies. The idea follows our general concept: We follow the finitely many branches of periodic orbits along the homotopy and show that nothing happens to the index as long as we do not reach a bifurcation parameter. Then we show that we can also cross a bifurcation parameter without changing the index by calculating the indices of Poincaré maps on the left and on the right of the bifurcation parameter. A main point of this argument is the treatment of period multiplying bifurcations, which will show why we had to add the factor $\frac{1}{k}$ to an orbit of a period k-times its minimal period. Figure 5 indicates the central idea of reducing a homotopy of vector fields to a Poincaré homotopy locally around a bifurcation orbit.

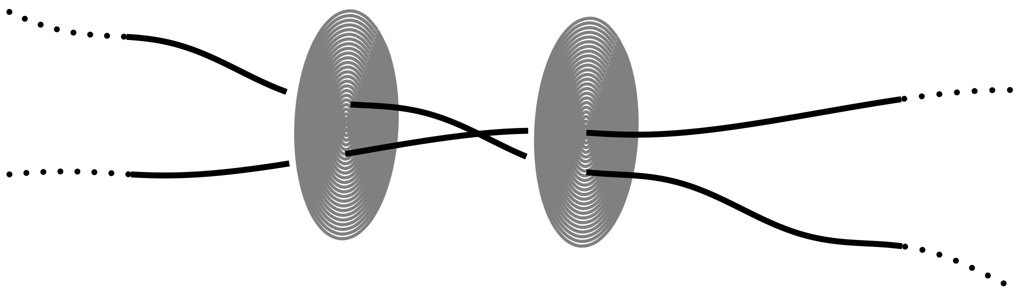


Figure 5: Pushing a Poincaré disc over a bifurcation parameter

Proposition 1.3.1.3 *If $\xi_0, \xi_1 : M \to TM$ are two non-degenerate vector fields that are non-degenerately homotopic, then $I_F(\xi_0, \Omega) = I_F(\xi_1, \Omega)$.*

Proof. Let $H \in h\mathfrak{X}(M, \Omega, a, b)$ be a non-degenerate homotopy joining ξ_0 and ξ_1. Take any two regular parameters $\lambda_1, \lambda_2 \in I$. We consider two cases.

1. There are no bifurcation parameters in $[\lambda_1, \lambda_2]$.

   In this case, let $J \subseteq [\lambda_1, \lambda_2]$ be the maximal interval containing λ_1 such that the Fuller index of H_λ, $\lambda \in J$, remains unchanged and let λ_0 be its supremum. Let $\gamma_1, \ldots, \gamma_m$ be the non-degenerate periodic orbits of H_{λ_0} of periodicities $k_1, \ldots, k_m$. Choose Poincaré systems (D_j, D'_j, P_j, t_j) around points x_j on the geometric orbit corresponding to γ_j. The Fuller index of H_λ is defined to be

   $$I_F(H_\lambda, \Omega) = \sum_{j=1}^{m} \frac{1}{k_j} \cdot i(P_j, D'_j).$$

   We find an $\varepsilon > 0$ such that each Poincaré system can be continued to a Poincaré system for $H_{\lambda+\varepsilon}$. The index here is calculated as

   $$I_F(H_{\lambda+\varepsilon}, \Omega) = \sum_{j=1}^{m} \frac{1}{k_j} \cdot i(P_j(\varepsilon), D'_j),$$

   where $P_j(\varepsilon)$ is the continuation of P_j. But this is just an admissible homotopy of P_j and $P_j(\varepsilon)$. Hence, the fixed point index does not change and so the Fuller index does not change as well. We see that $\lambda_0 = \lambda_2$, which shows that the Fuller index is unchanged during this part of the homotopy.

2. There is exactly one bifurcation parameter $\lambda \in [\lambda_1, \lambda_2]$.

In this case, H_λ is degenerate. Let γ be a periodic orbit of H_λ which is the limit of a branch of periodic orbits and p be the minimal period of its underlying geometric orbit. Let (D, D', P, t) be a Poincaré system for γ, considered with minimal period p. By choosing D small enough, the only fixed points of P_μ lying in D are those on branches converging to γ if we choose, say, μ in a 2ε-neighbourhood of λ, $\varepsilon > 0$. Denote the finitely many branches converging to γ from the left of λ by

$$\mu_1^k, \dots, \mu_{r_k}^k,$$

where k runs through the integers and indicates that the minimal period of μ_j^k approaches $k \cdot p$ for $j = 1, \dots, r_k$ as the branch approaches γ. Let $P_- = P(-\varepsilon)$, $P_+ = P(\varepsilon)$. Choose small discs $M_1^k, \dots, M_{r_k}^k$ around each orbit $\mu_1^k(0), \dots, \mu_{r_k}^k(0)$ and subdiscs ${M'}_1^k \subseteq M_1^k, \dots,$ such that P_- restricts to a map ${M_j}^{k'} \to M_j^k$, $j = 1, \dots, r_k$ for all k involved, and the iterates of P_- do so as well. We need only finitely many iterates of P_-, hence this condition can be fulfilled. We have a homotopy P between P_- and P_+ which is non-degenerate at every stage except for the parameter λ. By Corollary 1.2.2.9, we find a homotopy P' arbitrarily close to P that is non-degenerate and has no fixed points on the union of the boundaries of the discs M_j^k. In particular, P'_- has all its fixed points inside of the discs M_j^k for the various j, k and P'_- is admissibly homotopic to P_-, i.e. their fixed point index is equal. But then, also P_- and P_+ are admissibly homotopic, so we find

$$i(P_-^k, D') = i(P_+^k, D')$$

for all k. For simplicity, write $H_{\lambda-\varepsilon} = H_-$, $H_{\lambda+\varepsilon} = H_+$. We claim that the Fuller indices are given by the sums

$$I_F(H_-, \Omega) = \sum_{n \cdot p \in [a,b]} \frac{1}{n} \cdot i(P_-^n, D'),$$

$$I_F(H_+, \Omega) = \sum_{n \cdot p \in [a,b]} \frac{1}{n} \cdot i(P_+^n, D'),$$

which would immediately yield equality of the two terms. We calculate

$$I_F(H_-, \Omega) = \sum_{j \cdot k \cdot p \in [a,b]} \sum_{s=1}^{r_{jk}} \frac{1}{j} i(P_-^{jk}, M_s^k).$$

On the other hand, to calculate the fixed point index of P_-^n in D', note that the branches μ_s^k bifurcate with period $k \cdot p$ from γ, that is, a k-fold covering space of $\mathbb{S}^1$ bifurcates from the geometric orbit corresponding to γ. This corresponds to a bifurcation of k fixed points of the k-th iterate of the Poincaré map P_-, all of which have the same local index (the Poincaré systems being isotopic via the flow), namely $i(P_-^k, M_s^k)$. Hence we have a contribution of $k \cdot i(P_-^k, M_s^k)$ of these fixed points to the fixed point index of P_-^k in D'. Clearly if k divides n, then P_-^n also has these fixed points in M_s^k, and these contribute $k \cdot i(P_-^n, M_s^k)$ to the index of P_-^n. Summing all these indices up, we obtain

$$\begin{aligned} \frac{1}{n} i(P_-^n, D') &= \sum_{k \cdot j = n} \sum_{s=1}^{r_n} \frac{k}{n} \cdot i(P_-^n, M_s^k) \\ &= \sum_{k \cdot j = n} \sum_{s=1}^{r_{jk}} \frac{1}{j} \cdot i(P_-^{jk}, M_s^k) \end{aligned}$$

This finally gives

$$\sum_{n \cdot p \in [a,b]} \frac{1}{n} i(P_-^n, D') = \sum_{j \cdot k \cdot p \in [a,b]} \sum_{s=1}^{r_{jk}} \frac{1}{j} \cdot i(P_-^{jk}, M_s^k) = I_F(H_-, \Omega).$$

The whole calculation did not depend on the fact that we were working with H_- instead of H_+, and we get the same calculation on the right hand side, verifying equality of both Fuller indices.

Since the Fuller index remains unchanged in both cases and we have only finitely many bifurcation parameters, the proposition follows. ☐

We follow the general outline from above and establish next that non-degenerate homotopy and homotopy give the same relation on the set of non-degenerate fields.

Lemma 1.3.1.4 *If two non-degenerate vector fields $\xi_0, \xi_1 \in \mathfrak{X}(M, \Omega, a, b)$ are homotopic, then they are already non-degenerately homotopic.*

Proof. Let $\mathcal{U}_0$ be a neighbourhood of ξ_0 such that all elements of $\mathcal{U}_0$ are non-degenerate. Using Lemma 1.1.1.2, we can furthermore achieve that all elements of $\mathcal{U}_0$ are pairwise homotopic via a homotopy not leaving $\mathcal{U}_0$. Hence, all elements of $\mathcal{U}_0$ are non-degenerately homotopic. We can find a similar neighbourhood $\mathcal{U}_1$ of ξ_1. Now if $H \in h\mathfrak{X}(M, \Omega, a, b)$ is a homotopy joining ξ_0 and ξ_1, by Theorem 1.2.4.5 we find a non-degenerate homotopy K arbitrarily close to H. In particular we can find such a K so that $K_0 \in \mathcal{U}_0$, $K_1 \in \mathcal{U}_1$. Pasting together K and non-degenerate homotopies joining ξ_0 with K_0 and K_1 with ξ_1, respectively, we obtain a non-degenerate homotopy joining ξ_0 and ξ_1. ☐

The well-definedness of the Fuller index is now a byproduct.

Corollary 1.3.1.5 *The Fuller index is locally constant and hence well-defined.*

Proof. Any vector field $\xi \in \mathfrak{X}(M, \Omega, a, b)$ has a neighbourhood $\mathcal{U}$ such that every element of $\mathcal{U}$ is homotopic to ξ. Thus, all non-degenerate elements in $\mathcal{U}$ are pairwise homotopic. By the preceeding lemma, they are even non-degenerately homotopic, and so by Proposition 1.3.1.3, any two elements of $\mathcal{U}$ have the same Fuller index. ☐

We summarize what we have done in this section in the following theorem.

Theorem 1.3.1.6 *The Fuller index is invariant under admissible homotopies.*

PROOF. If $\xi_0, \xi_1 \in \mathfrak{X}(M, \Omega, a, b)$ are homotopic vector fields and $H \in h\mathfrak{X}(M, \Omega, a, b)$ is a homotopy between them, choose a non-degenerate homotopy $K \in h\mathfrak{X}(M, \Omega, a, b)$ such that K_0 is in a given neighbourhood $\mathcal{U}_0$ of ξ_0, K_1 is in a given neighbourhood $\mathcal{U}_1$ of ξ_1. This is possible by the density theorem 1.2.4.5. Since the Fuller index is locally constant in ξ, the theorem follows from Proposition 1.3.1.3. □

For future reference, we list all the properties of the Fuller index that are of interest to us in the subsequent theorem.

Theorem 1.3.1.7 *The Fuller index has the following properties.*

1. *It is invariant under admissible homotopies, i.e. if $H \in h\mathfrak{X}(M, \Omega, a, b)$, then*

$$I_F(H_\lambda, \Omega) \equiv const.$$

2. *It is additive, i.e. if $\Omega_1 \cap \Omega_2 = \emptyset$ and all essential periodic orbits of $\xi \in \mathfrak{X}(M, \Omega, a, b)$ are contained in $\Omega_1 \cup \Omega_2$, then*

$$I_F(\xi, \Omega) = I_F(\xi, \Omega_1) + I_F(\xi, \Omega_2).$$

3. *It is normalized, i.e. if ξ has a single non-degenerate periodic orbit in $\Omega \times (a, b)$, then*

$$I_F(\xi, \Omega) = \pm \frac{1}{k}.$$

   where k is the periodicity of the orbit.

4. *It has the solution property: If $I_F(\xi, \Omega) \neq 0$, then ξ has an essential periodic orbit in Ω.*

PROOF. Everything has been proven or is obvious except for 4. But 4. follows trivially from the fact that a field without essential periodic orbits in Ω is non-degenerate and hence has index 0. □

2 G-Transversality and Equivariant Non-Degeneracy

The second chapter has mainly two purposes. The first is to give a notion of equivariant non-degeneracy which is sufficiently generic, so we can obtain theorems similar to those of chapter one. The second is to prove these genericity theorems and therefore to establish a bifurcation scenario of relative critical elements that is identical to the scenario we developed for fixed points and periodic orbits. Equivariant non-degeneracy will depend on G-transversality, which was developed independently by Field and Bierstone in various articles and later unified by Field. Foremost to mention here is his monograph [Fie07]. Equivariant transversality is based on transversality to stratified sets, as has been studied by Thom and Mather [Mat80]. We need a slight generalization, namely we define equivariant transversality to locally semialgebraic sets. Since the established theory just uses that the preimage of zero by a polynomial map has a canonical Whitney stratification, the replacement of zero by a semialgebraic set is almost no problem.

The chapter is organized as follows. We begin with a basic introduction to the theory of group actions and establish basic facts of the topological theory. Most proofs here will be given by reference, mostly to [Bre72] and [tD87]. After a quick review of the special features of equivariant dynamical systems, with main reference [Fie80], we investigate the structure of homogeneous G-spaces, i.e. G-orbits, and the behaviour of smooth maps near fixed orbits. This culminates in the proof of the important normal decomposition lemma 2.1.3.4. This lemma is a generalization of a lemma of [Kru90], compare also [Fie91]. In the last part of the introduction, we investigate the structure of smooth maps between free G-manifolds and establish the covering homotopy theorem of Palais [Pal68], which will be important in the proof of some of the genericity results.

In the second part of the chapter we develop the theory of G-transversality as is done in [Fie07] or [Bie77a], beginning with transversality theory to stratifications. The theory of G-transversality to an invariant semialgebraic set is of special importance and is an adoption of techniques that appeared e.g. in [Fie07], chapter 7. Finally we prove the Thom-Mather Theorem for G-transverse maps as is done e.g. in [Fie07] and generalize the proof to our theory of G-transversality to semialgebraic sets.

In a short interlude, we describe the proofs of Thoms isotopy lemmas to make them equivariant, as proposed in [Bie77a]. The main point to show is the existence of invariant controlled tube systems. The main reference here is [Gib76]. The equivariant generalizations seem to be well-known but are hard to find (the author could not find any reference here).

The Thom lemma will allow us to prove equivariant isotopy theorems for equivariantly transverse homotopies, which is a major part of an development of a theory of equivariant non-degeneracy of critical elements. Instead of the ordinary diagonal, we will be using an equivariant diagonal. This is a G-subset which will in general not be a manifold but has the structure of a locally semialgebraic set. So we have to use the theory we developed earlier in the chapter. We will see that equivariant non-degeneracy will have similar implications as non-equivariant non-degeneracy. We also introduce the notion of G-hyperbolicity, so that we have two different notions of simpleness for equivariant systems. These two will suffice to derive the structural results necessary for index theory.

It would be interesting to know if G-hyperbolicity implies equivariant non-degeneracy. This is unknown so far.

Finally, we prove all the equivariant analoga of the genericity theorems of chapter one with our notion of equivariant non-degeneracy. Similar theorems may be obtained by methods developed by Field in [Fie89]. The way we present these results is more in the spirit of chapter one of this work.

2.1 Basic Facts on Group Actions

Naturally, the notion of a group action lies at the heart of this chapter. An *action* of a group G on a set X is given by a group homomorphism $G \to \mathrm{Aut}(X)$. Equivalently, there is a map $\alpha : G \times X \to X$ such that the diagram

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mathbb{1}_G \times \alpha"] \arrow[d, "\mu \times \mathbb{1}_X"'] & G \times X \arrow[d, "\alpha"] \\ G \times X \arrow[r, "\alpha"'] & X \end{tikzcd} ,$$

commutes and $\alpha(e, x) = x$ for all $x \in X$. Here, μ is the group multiplication on G. This second formulation makes it easier to define group actions on sets with additional structure. To be more precise, the diagram above defines a left action of a group. There is the obvious concept of a right action. When we define and work with twisted products, we will have to use both notions simultaneously. When it is not explicitly specified, an action will always be a left action.

If X is a topological space, we require α to be continuous (G should be a topological group). If X is a smooth manifold and G a Lie group, α should be smooth. We write $g.x$ or gx instead of $\alpha(g, x)$ and will also often regard g as an element of $\mathrm{Aut}(X)$. In the special case where V is a vector space and G acts on V via linear automorphisms, we call V a G-representation. A map that respects the group action is called equivariant. So for an equivariant map $f : X \to Y$, where G acts on X and Y, we have $f(gx) = gf(x)$ for all $g \in G$, $x \in X$. We also use the term G-map.

An action of a group divides the base space into points of several degrees of symmetry. The intuition is that points that remain fixed under a large subgroup have a lot of symmetry. The most symmetric points are those that are fixed under all of G. So the subgroups of G fixing a given element are of particular interest. They are called stabilizers or isotropy subgroups. If G acts on X and $x \in X$, denote by

$$Gx = \{gx \mid g \in G\}$$

the *orbit* of the element x and denote by

$$G_x = \{g \in G \mid gx = x\}$$

the *isotropy subgroup* of x. A well-known result is that, if G is compact, orbits are up to G-homeomorphism just the quotients of G by the stabilizer, i.e. we have a canonical G-homeomorphism $Gx \cong G/_{G_x}$, mapping x to $[e]$.

We already gave references for the general theory of continuous group actions. The theory of representations is developed, for instance, in [BtD03].

2.1.1 Equivariant Topology

We will recall some basic theorems on the structure of topological G-spaces, mainly focussing on how they are build up locally and with respect to the various fixed spaces of subgroups. At the end we will also look at some special features of G-manifolds.

Since the isotropy subgroups should indicate the symmetry of a point and points on the same orbit intuitively have the same degree of symmetry, one should be able to compare two points on the same orbit by means of their isotropy subgroups. However, if G is not abelian, in general these isotropy subgroups are not equal. This leads to the definition of the orbit type. If $y = gx$, then $G_y = gG_xg^{-1}$. So the isotropy groups of x and y are isomorphic and they are so in a special way, namely conjugation by an element of G.

Definition 2.1.1.1 *For closed subgroups $H, K \subseteq G$, denote by (H) the equivalence class of H under conjugacy, i.e. $K \in (H)$ if and only if there is a $g \in G$ such that $gKg^{-1} = H$. (H) is called the* orbit type *of H.*

If X is a G-space, denote by $\mathcal{O}_X$ the set of orbit types that have a representant which is an isotropy group of a point in X. There is a canonical partial order on $\mathcal{O}_X$ by defining $(H) \leq (K)$ if and only if K is subconjugate to H. In the cases we are interested in, there is always a maximal element of this order, called the principal orbit type (see below). We will use this partial order to give a filtration of a G-space that allows induction proofs on, e.g., the number of orbit types. Assume that there are only finitely many orbit types $(H_1), \ldots, (H_k)$ in a G-space X. We can assume that $(H_i) \leq (H_j)$ implies $i \leq j$. We can complete the partial order to a total order by saying that $i \leq j$ implies $(H_i) \leq (H_j)$ as well. Then we can define

$$X_j = \{x \in X \mid (G_x) \leq (H_j)\}.$$

Clearly, since for each $x \in X$ there is a k such that $(G_x) = (H_k)$, $X_1 \subseteq X_2 \subseteq \cdots \subseteq X_k = X$.

There are more ways to decompose X in a way compatible with symmetric reasoning. These decompositions rely on the notion of fixed subspaces of X. These are subspaces of X consisting of points that share a common interpretation of symmetry, singling them out against the other members of X. Let $H \subseteq G$ be a closed subgroup. Define

$$X^H = \{x \in X \mid hx = x \; \forall h \in H\}$$

$$X_H = \{x \in X \mid G_x = H\}$$

$$X_{(H)} = \{x \in X \mid (G_x) = (H)\}.$$

Note that in general, $X_{(H)}$ is the only G-invariant space of these three spaces. We will clarify the meaning of these spaces with the next few results.

Definition 2.1.1.2 *The action of G on X is called* free*, if $G_x = \{e\}$ for every $x \in X$. It is called* monotypic*, if there is a closed subgroup $H \subseteq G$ such that $(G_x) = (H)$ for every $x \in X$.*

If G is abelian, the fixed space X^H is G-invariant, because $hgx = ghx = gx$. This clearly fails for non-abelian groups. However, if we restrict to the normalizer $N(H)$, i.e. the largest subgroup of G containing H as a normal subgroup, this proof still works. Hence, $N(H)$ is the largest subgroup of G acting on X^H. H acts trivially on X^H, so we lose nothing by dividing out H and obtain an action of the group $W(H) = N(H)/_H$ on X^H. This last group is called the *Weyl group* of H. The action of $W(H)$ has the following interesting feature.

Proposition 2.1.1.3 *The action of $W(H)$ on X^H and X_H is well-defined. The latter action is free.*

Proof. If $n \in N(H)$, then $n^{-1}hn \in H$ for all h. So if $x \in X^H$, we have $hnx = nn^{-1}hnx = nkx = nx$, for $k = n^{-1}hn \in H$. This shows $nx \in X^H$. If $x \in X_H$, then $G_{nx} = nG_xn^{-1} = H$, hence $nx \in X_H$. We conclude that $N(H)$ acts on X^H and X_H. H fixes all elements of X^H and X_H, so the action $[n].x = nx$ of the Weyl group is well-defined. If $x \in X_H$ and $[n].x = x$, we have $nx = x$, giving $n \in H$. This yields $[n] = e$ and consequently, the action is free. □

Coming back to the filtration $X_1 \subseteq \cdots \subseteq X_k$ of X, we see that X_1 is a monotypic G-space, and so are all the spaces $X_{j+1} - X_j$. So a thorough understanding of monotypic G-spaces may in some cases be enough to track down properties of arbitrary G-spaces with finitely many orbit types, i.e. those with a finite orbit type filtration. The most well-behaved spaces of course are the free G-spaces and they deserve a special treatment in many aspects. We are going to take a short look at the structure of free G-spaces, then return to show that monotypic spaces are nicely build up from their fixed spaces.

Making it still easier, if X is any topological space, then $G \times X$ is a free G-space, where G acts on G by left translation and acts trivially on X. So it seems natural to take a look at spaces that locally look like a product, i.e. for fibre bundles with typical fibre G. Such bundles are called principal G-bundles.

Definition 2.1.1.4 *A* principal G-bundle *is a map $p : X \to B$, X, B Hausdorff spaces, together with a collection of homeomorphisms $\varphi_i : G \times U_i \to p^{-1}(U_i)$, $i \in I$, for some open sets $U_i \subseteq B$, such that the sets U_i cover B and the following holds.*

1. *The diagram*

$$\begin{tikzcd} G \times U_i \arrow[rr, "\varphi_i"] \arrow[dr, "p_2"'] & & p^{-1}(U_i) \arrow[dl, "p"] \\ & U_i & \end{tikzcd}$$

   commutes for all $i \in I$.

2. If $U_i \cap U_j \neq \emptyset$, there is a continuous map $\vartheta : U_i \cap U_j \to G$ such that

$$\varphi_i(g, u) = \varphi_j(g \circ \vartheta(u), u).$$

It is readily seen that if $p : X \to B$ is a G-principal bundle, there is a unique action of G on X such that p becomes invariant and the charts become equivariant. Similarly, every free G-space is in fact a G-principal bundle, taking p to be the quotient map. We thus note

Proposition 2.1.1.5 *If X is a free G-space, then the quotient map $X \to X/_G$ is a principal G-bundle and every principal G-bundle arises in this fashion.*

Proof. Theorem II.5.8 of [Bre72]. □

To investigate the structure of monotypic G-spaces further, we need a basic construction of transformation group theory, namely the twisted product. We will mainly be interested in the case where one of the participating spaces is a homogeneous space (meaning, an orbit $G/_H$), but we give the construction in general.

Definition 2.1.1.6 *Let $H \subseteq G$ be a closed subgroup and Y be an H-space, X a right H-space. The* twisted product *of X and Y is defined to be the G-space obtained by taking the product $X \times Y$ with H-action $h.(x, y) = (xh^{-1}, hy)$ and passing to the quotient. The resulting space is denoted with $X \times_H Y$. If X is also a left G-space and the actions are compatible, i.e. $(gx)h = g(xh)$, the twisted product becomes a G-space via $g.[x, y] = [gx, y]$.*

A monotypic G-space X has the following nice feature. If $x \in X$, then $G_x = g^{-1}Hg$ for some $g \in G$ and thus, $gx \in X^H$. We see that every orbit of X meets X^H. So if a $W(H)$-map $f : X^H \to Z^H$ is defined into any G-space Z, this extends uniquely to an equivariant map $f : X \to Z$ via $f(gx) = gf(x)$, $x \in X^H$. The subspace X^H can be thought of as a kind of fundamental domain for the action. The following result makes this precise.

Proposition 2.1.1.7 *Let X be a monotypic G-space of orbit type (H), G a compact group. Then the canonical map*

$$G/_H \times_{W(H)} X^H \to X, \ ([g], x) \mapsto gx$$

is a G-homeomorphism.

Proof. Corollary II.5.11 of [Bre72]. □

So when dealing with monotypic G-spaces, when we are searching for equivariant maps, we can restrict ourselves to the free $W(H)$-space X^H. The $W(H)$-maps $X^H \to Z^H$ are in 1-1-correspondence with the G-maps $X \to Z$.

For the remainder of this introductory chapter, we will look at G-spaces with additional structure, that is, G-manifolds. This of course requires the group to have some smooth structure as well, leading to the notion of a Lie group which marries the structure of a group and of a manifold. So a *Lie group* is a group and a manifold such that the group multiplication is a smooth map (it follows by the implicit function theorem that inversion is smooth as well). If a group acts on a manifold M, then every element of G induces a map $Tg : TM \to TM$. By the chain rule, this gives a group action of G on TM and this action clearly covers the action of G on M. Note that the tangential space T_xM is in general not a G-representation but a G_x-representation.

As one would expect, G-manifolds behave more nicely than arbitrary G-spaces. With respect to our filtration by orbit type, the following proposition is remarkable.

Proposition 2.1.1.8 *Let G be a compact Lie group. Then in every connected G-manifold there is an orbit type (H) such that $(K) \leq (H)$ (with respect to the partial order of orbit types) for every orbit type (K) of M. The set $M_{(H)}$ is open and dense in M. (H) is called the principal orbit type of the action.*

PROOF. Theorem III.3.1 of [Bre72]. □

Of fundamental importance for the theory is the existence of tubular neighbourhoods. When dealing with ordinary manifolds, one can reduce many local questions to charts and thus to euclidean space. But in G-manifolds, one cannot expect charts to be invariant. So we need another notion of invariant neighbourhoods of group orbits that are easy to handle. This are the so called tubular neighbourhoods, given by twisted products of G with a representation vector space.

Proposition 2.1.1.9 *If M is a compact G-manifold, G a compact Lie group, then every orbit $Gx \subseteq M$ has an invariant neighbourhood U such that there is a G_x-representation V and a G-diffeomorphism $\varphi : G \times_{G_x} V \to U$, satisfying $\varphi([g, 0]) = gx$.*

PROOF. Theorem II.5.4 of [Bre72]. □

As one can show, see e.g. [tD87], every G-vector bundle over an orbit of type H has the form $G \times_H V \to G/_H$ for some H-representation V. Vector bundles over a point are just euclidean spaces. This is another interpretation of tubular neighbourhoods being a substitute for manifold charts.

We occasionally will need an auxiliary result which will make some proofs in the sequel easier.

Proposition 2.1.1.10 *Let M be a smooth G-manifold, G a compact Lie group. Then there is a Riemannian metric on M that is invariant under the action of G on TM. Hence, we can always assume that G acts as a group of isometries on a smooth Riemannian manifold.*

PROOF. Just take any Riemannian metric and define a new Riemannian metric by

$$(v, w)_x = \int_G \langle T_x gv, T_x gw \rangle_{gx} \, dg,$$

where we use the existence of a unique left and right invariant normalized measure on G, called the Haar measure (compare [Bre72]). □

With this notion, the normal representation V of a tubular neighbourhood $G \times_H V$ of an orbit Gx can be taken to be the orthogonal complement $(T_xGx)^\perp$, after choosing an invariant Riemannian metric.

There are many fundamental results which can be derived from the existence of tubular neighbourhoods and invariant metrics. The former allows to do inductive proofs, either over the dimension of the manifold or over the number of orbit types. The latter allows to carry topological questions over into the metric setting, since, as is well known, any smooth manifold admits a Riemannian metric. We illustrate by the following result.

Proposition 2.1.1.11 *Let M be a compact G-manifold. Then M has only finitely many different orbit types.*

Proof. The statement is trivially true for finite M. Assume it to be proven for manifolds of dimension up to n. Let M be an $n+1$-dimensional manifold. Cover M by finitely many open sets of the form $G \times_H V$ for closed subgroups H of G. Fix an invariant Riemannian metric. Then the unit sphere $S(V)$ is a compact H-manifold of dimension less than $n+1$, hence, it has finite orbit type. But the orbit types in $G \times_H V$ are just the orbit types in $S(V)$ and in addition at most (H). So there are only finitely many orbit types meeting $G \times_H V$. The claim follows. $\square$

2.1.2 Equivariant Dynamical Systems

We begin the study of equivariant dynamical systems with discrete systems, i.e. we investigate iterates of an equivariant map $f : M \to M$, where M is a compact G-manifold. Since in the equivariant philosophy one deals with invariant objects only, we are not looking for fixed points of f alone. An equivariant "point" is the smallest invariant subspace, or in other words, a G-orbit. Thus, we should look for G-orbits that remain fixed under f. That is, we seek to solve the equation $f(x) = gx$, where $g \in G$ is any group element. If this is satisfied, then $f(Gx) \subseteq Gx$. This is a *fixed orbit* for f. In general, f is not the identity map on Gx but it acts as left multiplication by some $g \in G$. We will investigate maps between orbits in the next section.

When dealing with homotopies of equivariant maps, we cannot define bifurcation parameters as those parameters λ where the fibre map H_λ is degenerate. We have not developed an equivariant notion of non-degeneracy so far and the usual notion of non-degeneracy makes no sense for equivariant maps: There is always a trivial eigenvalue 1 for the derivative in a point on a fixed orbit, corresponding to the directions along the group action (as long $G/_{G_x}$ has positive dimension). In general, a bifurcation parameter should be a parameter where the topological structure of the set of fixed orbits changes. We will see later, however, that we can characterize bifurcation parameters by an equivariant transversality condition.

The notion of a branch of fixed orbits is straight forward.

Definition 2.1.2.1 *Let $H : M \times I \to M$ be a G-homotopy. A* prebranch of fixed orbits emanating from Gx_0 at λ *is given by a continuous map*

$$x \times \mu : (0,1) \to M \times I$$

such that $H_{\mu(s)}(x(s)) = gx(s)$ for some $g \in G$, $\mu(s) \to \lambda$, $x(s) \to x_0$ for $s \to 0$, $\mu(s)$ is regular and if $\mu(t) = \mu(s)$, $t \neq s$, then $Gx(s) \cap Gx(t) = \emptyset$.

There is an equivalence relation on the set of prebranches of fixed orbits, given by reparametrization and group multiplication. So two prebranches $x \times \mu$, $y \times \nu$ are equivalent, if there is an increasing homeomorphism $\zeta : (0,1) \to (0,1)$ and a continuous map $g : (0,1) \to G$ such that $g(s).x \circ \zeta(s) = y(s)$ and $\mu \circ \zeta(s) = \nu(s)$. An equivalence class of prebranches is called a branch of fixed orbits*.*

We will consider questions of uniqueness of branches later, when we have equivariant non-degeneracy at hand. Instead, we now turn to continuous dynamical systems. An equivariant vector field $\xi : M \to TM$ is a vector field that is equivariant with respect to the canonical action of G on the tangential bundle. So we have $\xi(gx) = T_x g \xi(x)$ for all $x \in M$, $g \in G$. An important property of an equivariant vector field is the equivariance of its flow. This follows immediately from uniqueness of solution curves. Another important feature is that an equivariant flow respects fixed spaces. If $x \in M^H_{(H)}$ for some closed subgroup $H \subseteq G$, then $\varphi(x,t) \in M^H_{(H)}$ for all $t \in \mathbb{R}$. This follows trivially, since $h\varphi(x,t) = \varphi(hx,t) = \varphi(x,t)$ for $h \in H$ and φ_t is a diffeomorphism.

The investigation of fixed points and periodic orbits was motivated by the search for points with non-trivial stabilizers with respect to the $\mathbb{R}$-action given by the flow. Having a compact symmetry group G acting, an equivariant flow is the same as an action of the group $G \times \mathbb{R}$. So we can ask instead to find points with non-compact stabilizers. Clearly this are just the points with non-trivial stabilizers under the induced $\mathbb{R}$-action on the quotient space $M/_G$. Since G is compact, the projection of a non-compact stabilizer in $G \times \mathbb{R}$ onto its second component must have non-compact image, which is a closed subgroup of $\mathbb{R}$. So the only possibilites are groups isomorphic to $\mathbb{Z}$, or $\mathbb{R}$ itself. In the case where this is $\mathbb{R}$, the stabilizer has the form $H \times \mathbb{R}$ for some closed subgroup H of G. So we have a group orbit which is fixed under the flow. In the second case, we find a smallest $T > 0$ such that $\varphi(x,T) = gx$ for some $g \in G$. The $G \times \mathbb{R}$-orbit through x is called a *relative periodic orbit*, the point $(x,T) \in M \times \mathbb{R}^+$ is called a *relative periodic point* (also if T is not minimal but a minimal positive time exists). Relative periodic orbits are compact invariant submanifolds of M, since the canonical map

$$\alpha : G \times \mathbb{R}/_{(G \times \mathbb{R})_x} \to (G \times \mathbb{R})x$$

is a continuous bijection and thus a homeomorphism, since $G \times \mathbb{R}/_{(G \times \mathbb{R})_x}$ is compact by definition. Their quotient by G is homeomorphic to $\mathbb{S}^1$. But of course, if G has positive dimension, a relative periodic orbit will carry non-trivial dynamics induced by the flow. We will deal with this issue shortly. First we want to point out a similarity with ordinary periodic orbits.

Proposition 2.1.2.2 *If γ is a relative periodic orbit of ξ of period T and p is the minimal period, then $T = k \cdot p$ for some $k \in \mathbb{Z}$.*

Proof. Let T be any period larger than p and choose $k \in \mathbb{N}$ such that $0 \leq T - kp < p$. Let $\varphi(x,T) = gx$, $\varphi(x,-kp) = hx$. We have

$$\varphi(x, T - kp) = \varphi(\varphi(x,T), -kp) = \varphi(gx, -kp) = ghx,$$

hence, $T - kp$ is a period of x. By minimality of p, we must have $T - kp = 0$. $\square$

The following proposition gives a complete classification of the dynamics on a relative periodic orbit. Again, this comes from [Fie07].

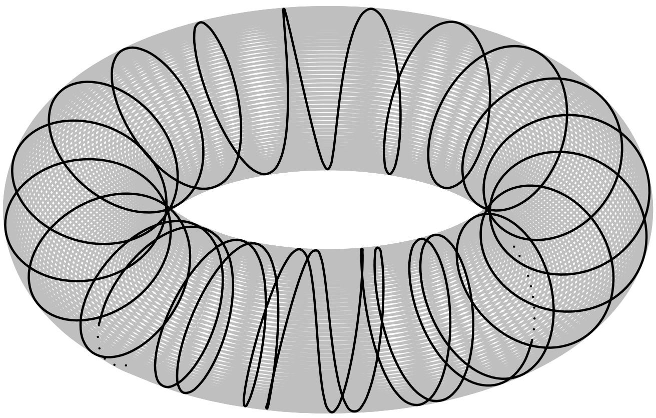


Figure 6: The irrational torus flow is an $\mathbb{S}^1$-relative periodic orbit

Proposition 2.1.2.3 *Let M be a monotypic G-space with quotient homeomorphic to $\mathbb{S}^1$. Then there is a flow invariant foliation $\{\mathcal{F}_x \mid x \in M\}$ of M such that*

1. *$\mathcal{F}_{gx} = g\mathcal{F}_x$ for all $g \in G, x \in M$.*
2. *There is an $s \in \mathbb{N}$ (which can be specified) such that each leaf $\mathcal{F}_x$ is diffeomorphic to an r-dimensional torus with $1 \leq r \leq s+1$. The restriction of the flow to a leaf is transitive and conjugate to a linear flow.*

Proof. Proposition 8.5.3 of [Fie07]. $\square$

Since fixed orbits of maps are easier to deal with than relative periodic orbits, we will try to substitute flows by Poincaré maps, if possible. This requires the definition of equivariant Poincaré systems. Such systems will be defined on equivariant discs in G-manifolds, so we have to clarify what an equivariant disc should be. In the non-equivariant case, we can interpret discs as follows. Let $x \in S \subseteq M$, where S is a 1-dimensional submanifold. Take a tubular neighbourhood of S in M. This is given by an embedding of the normal bundle N of S onto an open neighbourhood of S, sending the zero section to S, i.e. a diffeomorphism $\varphi : N \to U \subseteq M$. Now we take the pullback bundle N_x, induced by the inclusion $x \hookrightarrow S$, of N. This embeds into U and its image is a disc. We now generalize this concept taking symmetries into account.

Definition 2.1.2.4 *Let M be a G-manifold, $\gamma \subseteq M$ a G-orbit. γ has a tubular neighbourhood U G-diffeomorphic to the space $G \times_H V$, (H) the orbit type of γ, V the normal representation at some $x \in \gamma$. Assume that $\dim V^H > 0$. Let L be a 1-dimensional subspace of V^H and $L^\perp$ an invariant orthogonal complement in V. Let $p : V \to L$ be the orthogonal projection and $\pi : G \times_H V \to G \times_H L$, $[g, v] \mapsto [g, pv]$. This is well-defined since $pv \in V^H$, and we can interprete π as the bundle projection of the normal bundle of $G \times_H L$, whose fibres are given by*

$$\pi^{-1}([g, v]) = \{[g, w] \mid \ v - w \in L^\perp\}.$$

The image of the pullback of this bundle to the orbit $G/_H = G \times_H \{0\}$ in U is called an equivariant disc *in M, centered at γ. The image in U of the pullback of a disc subbundle of π is called an* equivariant subdisc *(of the disc defined above).*

As usual, we will speak of equivariant discs and subdiscs without specifying an embedding.

Note that, for example, an isolated G-fixed point in M is not contained in an equivariant disc, since it is not contained in a submanifold G-diffeomorphic to $\mathbb{R}$. However, an isolated G-fixed point must be a fixed point of any equivariant flow on M and we will never deal with fixed orbits of flows. In our applications, equivariant discs always exist.

Example 2.1.2.5

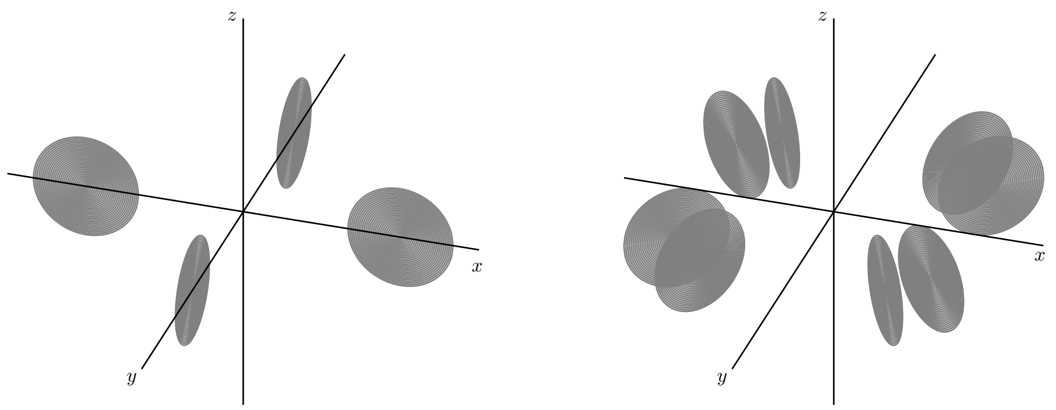
1. Let Gx be an orbit in M, $G_x = H$. We find a tubular neighbourhood $G \times_H V$ of Gx. The orbit of an equivariant flow on M through x will stay in the set $M^H_{(H)}$, and $M^H_{(H)} \cap G \times_H V = G \times_H V^H$. Hence, if Gx is not fixed under the flow, we must have $\dim V^H > 0$. This shows that equivariant discs exist around all G-orbits that are not fixed under an equivariant flow.

2. If G is finite, an equivariant disc around Gx is just an ordinary disc, centered at x, normal to the group orbit, together with all its translates by a representative system of $G/_H$, see also Figure 7.

3. Let $\mathbb{S}^1$ act on the torus $\mathbb{S}^1 \times \mathbb{S}^1$ by multiplication in the first component. A tubular neighbourhood of an $\mathbb{S}^1$-orbit $\mathbb{S}^1 \times \{\psi_0\}$ is $\mathbb{S}^1$-diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}$ and its normal bundle in the torus is the zero-dimensional bundle. Hence, an equivariant disc centered at $\mathbb{S}^1 \times \{\psi_0\}$ is just the set $\mathbb{S}^1 \times \{\psi_0\}$ itself.

We turn to the definition of equivariant Poincaré systems, which are central to the whole theory.

Definition 2.1.2.6 *Let ξ be an equivariant vector field, φ its flow. Assume γ is not a fixed orbit of the flow. Then there is an $\varepsilon > 0$ such that $S = \varphi(\gamma, (-\varepsilon, \varepsilon))$ is an equivariant embedding of $\gamma \times (-\varepsilon, \varepsilon)$ into M. If (H) is the orbit type of γ, in a suitable tubular*



(a) (b)

Figure 7: D_4-equivariant discs centered at orbits of different type

neighbourhood $G \times_H V$ of γ this can be interpreted as the embedding of the subspace $G \times_H L$ into $G \times_H V$, where $L \subseteq V^H$ is a one-dimensional subspace (corresponding to the direction of the flow). Let D be the equivariant disc defined by $G \times_H V$ and L. Since γ is not fixed by the flow, by shrinking D we can assume that there is a minimal $T > 0$ such that $\varphi(\gamma, T) \subseteq D$ and that $\xi(x) \notin T_x D$ for all $x \in D$. By continuity of φ, there is a subdisc $D' \subseteq D$ containing γ and a continuous invariant function $t : D' \to \mathbb{R}^+$, $t(\gamma) = T$, such that $\varphi(y, t(y)) \in D$ for all $y \in D'$. By the implicit function theorem, t is smooth and thus, the map

$$P : D' \to D,\ y \mapsto \varphi(y, t(y))$$

is a smooth equivariant embedding. The collection (D, D', P, t) is called an equivariant Poincaré system, *centered at γ. P is the* equivariant Poincaré map.

The disc in Figure 7 (a) can never occur in an equivariant Poincaré system. An equivariant flow starting somewhere in that disc will locally remain in it for symmetric reasons. Therefore, this disc will not be transverse to any equivariant flow. The disc in Figure 7 (b) can arise as an equivariant Poincaré disc.

We have the expected continuation result.

Proposition 2.1.2.7 *Let $\xi : M \to TM$ be an equivariant vector field, φ its flow, $\gamma \subseteq M$ a G-orbit that is not fixed by φ. A Poincaré system (D, D', P_0, t_0) around γ exists, and after possibly shrinking D, there is a neighbourhood $\mathcal{U}$ of ξ and a continuous map $t : \mathcal{U} \to \mathcal{C}^\infty_G(D', \mathbb{R}^+)$ such that $(D, D', P(\eta), t(\eta))$ is a Poincaré system for $\eta \in \mathcal{U}$, $P(\xi) = P$, $t(\xi) = t_0$, and $P(\eta)$ is defined in the obvious way.*

Proof. Since $\xi(x) \notin T_x D$, there is a neighbourhood of ξ such that for all fields η in this neighbourhood, $\eta(x) \notin T_x D$, after possibly passing to a subdisc. Hence, there is a small $\varepsilon_x > 0$ such that $\psi(x, t) \notin D$ for $0 < t < \varepsilon_x$ and $x \in D$ (ψ the flow of η). So the return time map t' is well-defined by $\psi(y, t'(y)) \in D$ and $t'(\gamma)$ close to T. Continuous dependence is obvious. $\square$

When dealing with homotopies of vector fields, we should make the standing assumptions we already did in the case without symmetries. So let $\Omega \subseteq M$ be an open invariant subset, $0 < a < b < \infty$ real numbers. Denote with $\mathfrak{X}_G(M, \Omega, a, b)$ the set of equivariant vector fields that have no relative periodic orbits meeting $\partial(\Omega \times (a, b))$. This means, there are no relative periodic points $(x, T) \in M \times \mathbb{R}^+$ such that $x \in \partial\Omega$ or $T \in \{a, b\}$. This of course excludes the existence of fixed orbits in Ω.

We extend the notion of branches of critical elements to relative periodic orbits. The definition of regular and bifurcation parameters of homotopies of equivariant vector fields is done in the same way as for maps.

Definition 2.1.2.8 *Let $H : M \times I \to TM$ be a G-homotopy of vector fields. A* prebranch of relative periodic orbits *is a map $x \times \mu \times T : (0, 1) \to M \times I \times \mathbb{R}^+$ such that $(x(s), \mu(s), T(s)) \to (x_0, \lambda, T_0)$ for $s \to 0$, $\mu(s)$ is a regular parameter, $(x(s), T(s))$ is a relative periodic point of $H_{\mu(s)}$ and if $s \neq t$, then either $\mu(s) \neq \mu(t)$ or $T(s) \neq T(t)$ or $Gx(s) \cap Gx(t) = \emptyset$.*

There is an equivalence relation on prebranches given by reparametrization and group multiplication as in the case of maps. An equivalence class is called a branch of relative periodic orbits.

We will mainly use branches to strengthen the intuition, foremost by visualizing the generic bifurcation scenario. Besides that, branches of equivariant critical elements will play a minor role in the theory.

2.1.3 Sections and Orbits

In this section we are taking a look at the structure of maps between homogeneous spaces, that is, orbits $G/_H$ for $H \subseteq G$ a closed subgroup. These maps are of interest because they describe the dynamics of a discrete dynamical system on a fixed orbit. Our investigations will lead to a normal decomposition lemma that allows us to decompose a map locally around an orbit γ that is mapped into a tubular neighbourhood of itself into a normal component and a group component. We will make use of it in the next section to prove isolatedness of G-hyperbolic fixed orbits. The material on the structure of G-maps of homogeneous spaces can be found in [Fie91] and [Bre72]. The normal decomposition lemma appeared in a less sharp form in [Fie91] and originally comes from [Kru90]. Our version drops the requirement on the map to be a diffeomorphism and also allows the orbit to be non-fixed. We will need this greater generality for the applications we have in mind.

We begin our investigations with a simple lemma which identifies maps between homogeneous spaces as elements of a certain quotient space.

Lemma 2.1.3.1 *Let $H \subseteq K \subseteq G$ be closed subgroups of G. Then the set of smooth maps $G/_H \to G/_K$ is isomorphic to the set $N(H, K)/_K$, where*

$$N(H, K) = \{g \in G \mid g^{-1}Hg \subseteq K\}.$$

PROOF. A G-map $f : G/_H \to G/_K$ is uniquely determined by its value on $[e]$. But if $f([e]) = [g]$, then $[g] = [hg]$ for all $h \in H$. Hence, $g^{-1}hg \in K$ for all $h \in H$. Clearly, an element g such that $g^{-1}Hg \subseteq K$ determines a map of the orbits and two such maps are equal if and only if their generating elements differ by an element of K. $\square$

Note the special case where $H = K$. Then the lemma identifies the equivariant self maps of $G/_H$, which are necessarily diffeomorphisms, as the elements of $W(H)$, the Weyl group of H. The identifications have the main purpose to allow us to identify G-homotopies of equivariant maps between orbits as paths in a certain space. If a G-map is equivariantly homotopic to the identity, its corresponding element must lie in the identity component of the quotient space. In case $K \neq H$, the role of the identity is taken by the map $[e] \mapsto [e]$, and the component containing this map is of special interest.

Lemma 2.1.3.2 *There is a neighbourhood U of e in $N(H,K)$ such that every element $g \in U$ has the form $g = ck$, where $c \in C(H)$, $k \in K$.*

PROOF. Let $\varphi : N(K) \to \mathrm{Hom}(K,K)$ be the conjugation homomorphism $\varphi(a)(k) = aka^{-1}$. The semidirect product $N(K) \times_\varphi K$ acts on G via $(n,k).g = ngn^{-1}k^{-1}$. Let S_e be a slice for this action at e. By Lemma 3.10.1 of [Fie07], there is an open neighbourhood W of K in G and a smooth map $\chi : W \to S_e$ such that, for $g \in W$, $gK = \chi(g)K$ and χ is $N(K)$-equivariant, where $N(K)$ acts via conjugation. The value $\chi(g)$ is defined to be the unique point in $S_e \cap gK$. Take any $h \in H$ and $g \in N(H,K) \cap W = U$. In particular, $h \in N(K)$, so by $N(K)$-equivariance of χ, we have

$$\chi(hgh^{-1}) = h\chi(g)h^{-1}.$$

Now $g^{-1}hgh^{-1} = k' \in K$, since $g \in N(H,K)$. We obtain $hgh^{-1} = gk'$. By definition of χ, $\chi(gk') = \chi(g)$. So we have

$$\chi(g) = h\chi(g)h^{-1}$$

which yields $\chi(g) \in C(H)$. The equality $gK = \chi(g)K$ then gives $g = \chi(g)k$ for some $k \in K$, proving the claim. $\square$

Corollary 2.1.3.3 *Let $f : G/_H \to G/_H$ be any G-map. Then there is an $\alpha > 0$ such that f^α is G-homotopic to the identity and is given by $f^\alpha([e]) = [c]$ for some $c \in C(H)$.*

PROOF. We have $f([e]) = [g]$ for some $g \in N(H)$. The sequence $\{g^\ell\}_{\ell\in\mathbb{N}}$ has a convergent subsequence, say, $\{g^{n_\ell}\}_{\ell\in\mathbb{N}}$. Take a neighbourhood U of e in $N(H)$ as guaranteed by Lemma 2.1.3.2. We find an $\ell > 0$ such that

$$g^{n_{\ell+1}} \circ g^{-n_\ell} = g^\alpha$$

is an element of $U \cap N(H)_0$, the lower zero indicating identity components. Hence, we can write $g^\alpha = ch$ with $c \in C(H)$, $h \in H$. Now

$$f^\alpha([e]) = [g^\alpha] = [ch] = [c]$$ $\square$

We can now state and prove the normal decomposition lemma, compare [Fie91] and [Kru90].

Lemma 2.1.3.4 (Normal Decomposition) *Let M be a compact G-manifold, $f_0 : M \to M$ a G-map and $Gx \subseteq M$ a fixed orbit of f_0, $H = G_x$. Then there is a neighbourhood $A \subseteq G/_H$ of $[e]$ and an $N(H)$-equivariant local section $\sigma : A \to G$, tubular neighbourhoods $U \subseteq U'$ of Gx and a neighbourhood $\mathcal{U}$ of f_0 such that for all $f \in \mathcal{U}$, the following holds. Let $U = G \times_H S_x$, $U' = G \times_H S'_x$, $S_x \subseteq S'_x$ and for $y \in U'$, let s_y be the unique point in $Gy \cap S'_x$.*

1. *There is an $\alpha > 0$ such that $f^\alpha : Gx \to Gf^\alpha(x)$ is equivariantly homotopic to the map $x \mapsto s_{f^\alpha(x)}$ and is given as $f^\alpha(x) = c_f.s_{f^\alpha(x)}$, $c_f \in C(H)$.*
2. *$c_f^{-1}.f^\alpha(S_x) \subseteq \sigma(A).S'_x$.*
3. *There are equivariant maps $g : U \to G$, $h : U \to U'$, where G acts on itself via conjugation, such that $f^\alpha(y) = g(y).h(y)$ for all $y \in U$ and $h(S_x) \subseteq S'_x$. g and h depend continuously on f.*

Proof. We choose suitable neighbourhoods first. Let $U = G \times_H S_x$, $U' = G \times_H S'_x$ be tubular neighbourhoods of Gx, $S_x \subseteq S'_x$ normal slices at x. Since Gx is a fixed orbit of f_0, by Corollary 2.1.3.3 there is an $\alpha > 0$ such that $f_0^\alpha\big|_{Gx}$ is equivariantly homotopic to the identity. Hence, by choosing a neighbourhood $\mathcal{U}$ of f_0 sufficiently small, the maps $f^\alpha\big|_{Gx} : Gx \to U'$ for $f \in \mathcal{U}$ are arbitrarily close to the map $x \mapsto g_0^\alpha.x$, which in turn will be homotopic to the map $g_0^\alpha.s_{f^\alpha(x)}$ for $\mathcal{U}$ properly chosen. In particular, the maps $f^\alpha\big|_{Gx}$ will all be equivariantly homotopic to the map $x \mapsto s_{f^\alpha(x)}$, since the map $x \mapsto g_0^\alpha.x$ is G-homotopic to the identity. Again by Corollary 2.1.3.3, the maps $f^\alpha\big|_{Gx}$ are given as $x \mapsto c_f.s_{f^\alpha(x)}$ for some $c_f \in C(H)$. We chose any $N(H)$-equivariant local section $\sigma : A \to G$, $A \subseteq G/_H$ a neighbourhood of $[e]$. Then by shrinking U, U' and $\mathcal{U}$, we can achieve $c_f^{-1}.f^\alpha(S_x) \subseteq \sigma(A).S'_x$ for all $f \in \mathcal{U}$. It remains to obtain the continuous decomposition.

Take any $f \in \mathcal{U}$ and assume $\alpha = 1$ for simplicity. Let $c = c_f$. Define a map

$$\tilde{f} : S_x \to M,\ y \mapsto c^{-1}f(y).$$

Then for $g \in H$, $\tilde{f}(gy) = c^{-1}gf(y) = gc^{-1}f(y) = g\tilde{f}(y)$, i.e., $\tilde{f}$ is H-equivariant. Thus, we can extend $\tilde{f}$ uniquely to a G-equivariant map $U \to M$. Then for $z = g.y \in U$, $f(z) = gf(y) = gcg^{-1}\tilde{f}(gy)$. So if we define a map $\tilde{g} : U \to G$, $g.y \mapsto gcg^{-1}$, then $\tilde{g}$ is equivariant with respect to the conjugation action on G and $f(z) = \tilde{g}(z).\tilde{f}(z)$. Let

$$\rho^\sigma : A \times S'_x \to \sigma(A).S'_x,\ (a, y) \mapsto \sigma(a).y.$$

This is a diffeomorphism with inverse $\eta = (\eta_1, \eta_2)$ and since we required that $\tilde{f}(S_x) \subseteq \sigma(A).S'_x$, we can define maps

$$h : S_x \to S'_x,\ y \mapsto \eta_2 \circ \tilde{f}(y)$$

$$\bar{g} : S_x \to G,\ \bar{g}(y) = \sigma \circ \eta_1 \circ \tilde{f}(y).$$

Extend h and $\bar{g}$ equivariantly to all of U. Then

$$f(y) = \tilde{g}(y)\bar{g}(y).h(y).$$

By letting $g(y) = \tilde{g}(y)\bar{g}(y)$, g is equivariant and $f(y) = g(y).h(y)$. The continuous dependence of g and h on f is clear from the construction. $\square$

2.1.4 Structure of Smooth Maps of Free G-Manifolds

Some of our results with symmetries will depend on the results from chapter one without symmetries. The way to make use of these results is the technique of induction over the orbit type. One proves a certain result for free G-manifolds and then uses the filtration of the manifold M by orbit type to add further orbits step by step. Therefore, if a manifold is given as the quotient of a free G-manifold by the acting group, it is important to have knowledge of the structure of the set of those smooth maps that are induced by equivariant maps in the overlying G-manifold. We will show that if the action of G is free, then the relevant set of maps of the quotient space will be open and has local sections. This is done by using Palais' theorem on covering homotopies. Palais' theorem works in a more general setting, however we only need a version for free G-spaces. The more general formulation can be found in [Bre72].

Theorem 2.1.4.1 (Covering Homotopy Theorem of Palais) *Let G be a compact Lie group and X, Y free G-spaces such that every open subset of $X/_G$ is paracompact. Let $f : X \to Y$ be equivariant and $[f] : X/_G \to Y/_G$ the induced map. Then for every homotopy $h : X/_G \times I \to Y/_G$ starting at $[f]$, there is a homotopy $H : X \times I \to Y$ inducing h and starting at f. Moreover, if K is any other homotopy with this properties, then $H = K \circ \zeta$, where $\zeta : X \times I \to X \times I$ is a G-diffeomorphism inducing the identity on $X/_G \times I$ and being the identity on $X \times \{0\}$.*

Proof. Theorem II.7.3 of [Bre72]. $\square$

Now let M, N be free G-manifolds. Denote with $\pi : \mathcal{C}^r_G(M,N) \to \mathcal{C}^r(M/_G, N/_G)$ the projection map, i.e. $\pi(f)[x] = [f(x)]$. Using Palais' theorem, we obtain the following structural result for the image of π.

Proposition 2.1.4.2 *The image $\pi(\mathcal{C}^r_G(M,N))$ is open and π has local sections, that is, for every $[f] = \pi(f)$ in $\mathcal{C}^r(M/_G, N/_G)$ there is a neighbourhood $\mathcal{U}$ of $[f]$ and a continuous map $\sigma : \mathcal{U} \to \mathcal{C}^r_G(M,N)$ such that $\pi \circ \sigma = \mathbb{1}_{\mathcal{C}^r(M/_G, N/_G)}$.*

Proof. For openness, take an element $[f] : M/_G \to N/_G$ that is induced by $f : M \to N$. Since the quotient spaces are compact manifolds, there is a neighbourhood $\mathcal{U}$ of $[f]$ such that every element h of $\mathcal{U}$ is homotopic to $[f]$ via a homotopy not leaving $\mathcal{U}$. Now lift such a homotopy to a homotopy $H : M \times I \to N$. The map $H(\,\cdot\,, 1)$ induces h, so the image of π is open.

For the local sections, take $\mathcal{U}$ as above and define a map $h : M/_G \times \mathcal{U} \times I \to N/_G$ such that $h(\,\cdot\,, k, t)$ is a homotopy joining $[f]$ and k. For example, we can take h to be the composition of a fixed retraction with the convex homotopy between $[f]$ and k (after embedding $N/_G$, of course). So we can achieve that h is continuous. By Palais' theorem, there is a lifting $H : M \times \mathcal{U} \times I \to N$ inducing h (we take $\mathcal{U}$ as a trivial G-space which is metric, hence paracompact). Define

$$\sigma : \mathcal{U} \to \mathcal{C}^r_G(M, N),\ k \mapsto H(\,\cdot\,, k, 1).$$

Then σ is a local section. $\square$

2.2 G-Transversality

In this section we are going to develop the theory of equivariant transversality as was done by Bierstone and Field in [Bie77a] and [Fie77]. As non-degeneracy is a transversality property of the graph map $\mathbb{1} \times f$, we can hope that equivariant transversality of the graph map to some suitable set can serve as an equivariant non-degeneracy condition. The complete theory of G-transversality is nicely elaborated in [Fie07]. We will nevertheless give all the necessary propositions and even proofs. We will, however, skip the various well-definedness results, or rather prove these by reference.

G-transversality depends on the theory of transversality to stratified sets, so we start with a review of the Thom-Mather theory of transversality to stratified sets. Then we define G-transversality. The basic idea is to split an equivariant map between G-representations into an invariant and an equivariant part. All the necessary informations of the map will already be contained in the invariant part. Zeros of the initial map will correspond to values of this invariant map in a certain algebraic subset, called the universal variety. This comes with a canonical stratification, and we will define equivariant transversality of f to 0 at 0 as transversality to the stratified universal variety at 0. Note that the equivariant analogue of the surjectivity of the differential is G-transversality of a map to 0 in 0. Since transversality can be expressed in terms of surjectivity of the differentials alone, compare Lemma 1.1.2.2, the generalization to equivariant transversality in manifolds is straight forward.

In the third part of this section, we will develop a theory of G-transversality to semialgebraic sets. Under some special assumptions, this theory can be made to work, so we can prove equivariant versions of the Thom-Mather theorems for G-transversality and G-transversality to semialgebraic sets in the last part.

2.2.1 Stratifications

A *stratified subset* X of a manifold M is a collection of pairwise disjoint submanifolds X_α, α in some index set, such that $X = \bigcup_\alpha X_\alpha$. We also require the union to be locally finite, so every element of X has a neighbourhood in M meeting only finitely many sets X_α. The sets X_α are called the *strata* of X.

Usually, the strata will not be closed, so in general, the set of maps transverse to a fixed stratum will not be generic (it will be dense, however). We are looking for conditions on the stratification such that simultaneous transversality to all strata will become a generic property. These are the so called Whitney conditions. To be more precise, the Whitney condition (a) will guarantee genericity. Condition (b) implies condition (a) and is much stronger, guaranteeing the isotopy lemmas. We will deal with these lemmas in a subsequent section.

Definition 2.2.1.1 *Let M be a manifold, $X \subseteq M$ a stratified subset. The stratification of X satisfies* Whitneys condition (a)*, if the following holds. Whenever $x \in X_\alpha \cap \overline{X}_\beta$, $x_n \in X_\beta$ is a sequence converging to x and the tangential spaces $T_{x_n}X_\beta$ converge (in the appropriate Grassmannian) to a subspace $E \subseteq T_xM$, then $T_xX_\alpha \subseteq E$.*

X satisfies Whitneys condition (b)*, if the following holds. Whenever $x \in X_\alpha \cap \overline{X}_\beta$, $x_n \in X_\alpha$, $y_n \in X_\beta$ are sequences converging to x, the lines L_n joining x_n and y_n (in a local chart) converge to a 1-dimensional subspace $L \subseteq T_xM$ and the tangential spaces $T_{x_n}X_\beta$ converge to a subspace $E \subseteq T_xM$, then $L \subseteq E$.*

A stratified set whose stratification fulfills the Whitney conditions is called a *Whitney stratification* and will also be called *Whitney regular* or just *regular*. As has already been said, condition (b) is stronger than condition (a).

Proposition 2.2.1.2 *A stratification that is Whitney (b) regular is Whitney (a) regular.*

PROOF. Take $x \in X_\alpha \cap \overline{X}_\beta$. Let $y_n \in X_\beta$ be a sequence converging to x and $T_{y_n}X_\beta$ converges to $E \subseteq T_xM$. Take any 1-dimensional subspace L of T_xX_α. Choose a neighbourhood of x diffeomorphic to $T_xX_\alpha \oplus T_xX_\alpha^\perp$, sending x to 0 and X_α to T_xX_α. The sequence y_n corresponds to elements $(a_n, b_n) \in T_xX_\alpha \oplus T_xX_\alpha^\perp$ converging to $(0,0)$. We find a sequence $c_n \in T_xX_\alpha$ converging to 0 such that the lines joining $(c_n, 0)$ and (a_n, b_n) converge to L. Hence, the preimages $x_n \in X_\alpha$ of $(c_n, 0)$ converge to x and by Whitneys condition (b), $L \subseteq E$. Since L was arbitrary, we obtain $T_xX_\alpha \subseteq E$. $\square$

Transversality to a stratified set is defined in the obvious way. A function $f : M \to N$ is transverse to the stratified set $X \subseteq N$, if it is transverse to every stratum of X. As indicated, Whitney (a) regularity guarantees that being transverse to X at x is an open condition.

Proposition 2.2.1.3 *Let $f : M \to N$ be transverse to the stratified subset $X \subseteq N$ at $x \in M$. Let the stratification of X be Whitney (a) regular. Then there is a neighbourhood U of x such that f is transverse to X in U.*

PROOF. The statement is clear if $f(x) = y \in X_\alpha$ is bounded away from all other strata. So we have to deal with the case $y \in X_\alpha \cap \overline{X}_\beta$. Let $y_n = f(x_n)$ be a sequence in X_β, $x_n \to x$. It suffices to show that f is transverse to X_β at x_n for n sufficiently large. By compactness of the Grassmannian, we can assume that $T_{y_n}X_\beta$ converges to a subspace $E \subseteq T_yN$. By Whitney (a) regularity, $T_yX_\alpha \subseteq E$. Since f is transverse to X_α at x,

$$T_xf(T_xM) + T_yX_\alpha = T_yN,$$

so

$$T_x f(T_x M) + E = T_y N.$$

By openness of the set of linear maps with rank larger k, $\operatorname{rank}(T_{x_n} f) \geq \operatorname{rank}(T_x f)$ for n sufficiently large. Hence, $T_{x_n} f(T_{x_n} M)$ can be assumed to converge to a subspace F of $T_{f(x)}N$ such that $T_x f(T_x M) \subseteq F$. This gives $F + E = T_{f(x)}N$, so for dimensional reasons, there must be an $N \in \mathbb{N}$ such that

$$T_{x_n} f(T_{x_n} M) + T_{f(x_n)} X_\beta = T_{f(x_n)} N$$

for all $n \geq N$, showing transversality of f to X_β at all these x_n. □

The relevance of condition (b) will become apparent later when dealing with questions of isotopy. For genericity, we have the following generalization of a result of Thom and Mather, compare [Mat80].

Theorem 2.2.1.4 *Let M, N be compact manifolds, $X \subseteq N$ a closed Whitney (a) stratified subset. Let $U \subseteq M$ be an open subset. Then if $f : M \to N$ is transverse to X in U and $V \subseteq M$ is an open subset such that $\overline{V} \subseteq U$, there is a neighbourhood $\mathcal{U}$ of f such that every element of $\mathcal{U}$ is transverse to X in V.*

Proof. Let f_n be a sequence converging to $f \in \mathcal{C}^\infty(M, N)$ and assume f_n is not transverse to X at some $x_n \in V$. By various compactness arguments and since a stratification is locally finite, we can assume that $x_n \to x \in U$, $f_n(x_n) \to f(x) \in X_\alpha$, $f_n(x_n) \in X_\beta$ and $T_{f_n(x_n)} X_\beta \to E \subseteq T_{f(x)}N$. As in the proof of the preceeding proposition, $T_{x_n} f_n(T_{x_n} M)$ can be assumed to converge to a space $F \subseteq T_{f(x)}N$ with $T_x f(T_x M) \subseteq F$. By Whitney regularity, $T_{f(x)} X_\alpha \subseteq E$. If f were transverse to X at x, then

$$T_x f(T_x M) + T_{f(x)} X_\alpha = T_{f(x)} N,$$

implying $F + E = T_{f(x)}N$ as well. But this would mean, again by dimensional reasons, that

$$T_{x_n} f_n(T_{x_n} M) + T_{f_n(x_n)} X_\beta = T_{f_n(x_n)} N$$

for n sufficiently large, contradicting the assumption. Thus, f is not transverse to X at x and the result follows. □

The original Thom-Mather Theorem is a corollary of this result.

Corollary 2.2.1.5 (Thom-Mather Theorem) *Let M, N be compact manifolds, $X \subseteq N$ a closed Whitney (a) stratified subset. Then the set of maps transverse to X is open and residual in $\mathcal{C}^\infty(M, N)$.*

Proof. For openness, we just take $U = M$ in the theorem. Residuality follows from Thom's Transversality Theorem: The set of maps transverse to any stratum of X is residual. Since the stratification is locally finite and N is compact, the stratification is finite and the set of maps transverse to every stratum of X simultaneously is the finite intersection of residual sets, hence residual. □

In the preceeding theorem we see that for density we do not even need a Whitney condition on the stratification. One can also weaken the conditions on M and N, but we will not go into detail.

2.2.2 Equivariant Transversality

In the introduction to the chapter we indicated the idea of G-transversality. One begins with maps between G-representations and defines transversality to 0 at 0. The generalizations are then obvious. To obtain the first objective, one gives a decomposition of the map into an equivariant and an invariant part, where the invariant part carries all necessary information. This decomposition depends on an interplay of analysis and algebra. Namely, the set $\mathcal{C}^\infty_G(V, W)$ of smooth maps between G-representations is a finitely generated $\mathcal{C}^\infty_G(V, \mathbb{R})$-module. Furthermore, by the equivariant Stone-Weierstrass theorem, see theorem 2.10.1 of [Fie07], generators can be assumed to be polynomial. In fact, any generating set of equivariant polynomials, as a module over invariant polynomials, already generates the smooth equivariant maps as a module over smooth invariants. Details can be found in [Fie07]. This reference also uses minimal sets of homogeneous generators in most places. This is convenient in many proofs, for example various results concerning independence of choices. The theory, however, works with arbitrary generators, compare [Bie77a]. Thus, given a smooth equivariant map $f : V \to W$ and a set of polynomial generators $F_1, \dots, F_k$ of $\mathcal{C}^\infty_G(V, W)$, we can write

$$f(v) = \sum_{i=1}^{k} f_i(v) \cdot F_i(v),$$

where the f_i are smooth invariants. This is our desired decomposition. Define a map

$$\vartheta : V \times \mathbb{R}^k \to W,\ (v, t) \mapsto \sum_{i=1}^{k} t_i F_i(v).$$

Then we have $f = \vartheta \circ \Gamma_f$, where $\Gamma_f(v) = (v, f_1(v), \dots, f_k(v))$ is the graph map of f (we somewhat ambiguously use the term "graph map" for both the maps Γ_f and $\mathbb{1} \times f$. The meaning should always be clear from the context). ϑ clearly does not depend on f and the graph map is invariant in the relevant components. The equation $f(v) = 0$ is solved if and only if $\Gamma_f(v) \in \vartheta^{-1}(0)$.

Since ϑ is a polynomial map, the set $\vartheta^{-1}(0) = \Sigma_G(V, W)$ is an algebraic subvariety of $V \times \mathbb{R}^k$. ϑ is called the *universal polynomial*, $\Sigma_G(V, W)$ is called the *universal variety*. In Figure 8, a universal variety for the canonical actions of $\mathbb{Z}_2$ on $\mathbb{R}$ and $\mathbb{R}^2$ is sketched.

The special importance of the choice of polynomial generators comes from the following result.

Proposition 2.2.2.1 *Let $A \subseteq \mathbb{R}^n$ be an algebraic variety, i.e. the zero set of a polynomial map. Then A has a unique minimal Whitney regular stratification, where minimal refers to refinement of the partition given by the stratification.*

Proof. See [Mat80]. □

This result only assures existence of a minimal stratification and no method to find it. A partial solution to this problem can be obtained as follows. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be

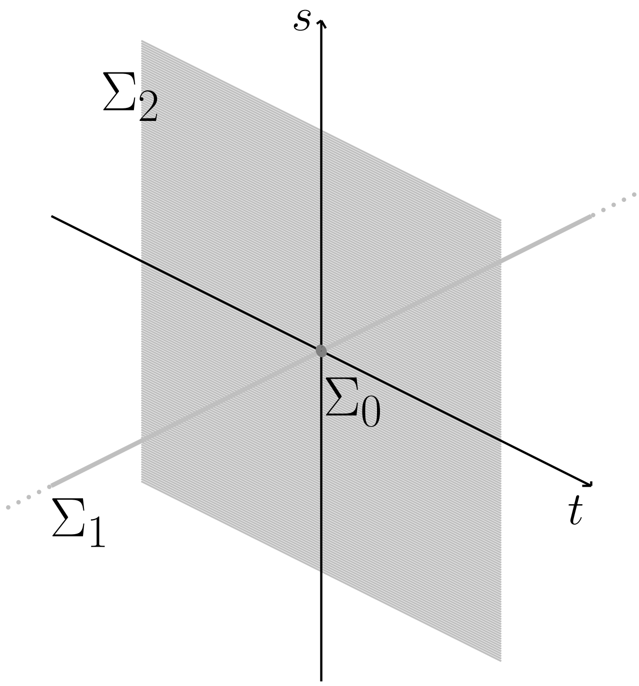


Figure 8: The universal variety $\Sigma_{\mathbb{Z}_2}(\mathbb{R}, \mathbb{R}^2)$

a smooth polynomial map. Then there is a Whitney stratification of $f^{-1}(0)$ such that Df has constant rank in each stratum. This result is proven e.g. in [Dur83]. So a first indicator for a minimal Whitney stratification is the stratification of $f^{-1}(0)$ by the sets of points where Df has constant rank. In general, this stratification will not be minimal, sometimes not even regular, but very often it is both.

We can now define equivariant transversality for maps between representations.

Definition 2.2.2.2 *Let V, W be G-representations, $f : V \to W$ smooth. f is said to be G-*transverse *to 0 at 0, if for some choice of polynomial generators of $\mathcal{C}^\infty_G(V, W)$, the graph map Γ_f is transverse to the universal variety $\Sigma_G(V, W)$.*

The verification that G-transversality is well-defined is not difficult but quite tedious, so we refer to [Fie07] or [Bie77a] instead of giving a proof. We will summarize the main results dealing with independence of choices after the general definition for G-transversality in manifolds.

In the general setting, we have a map $f : M \to N$ between two G-manifolds and we want to define when such a map is G-transverse in a point $x \in M$ to an invariant submanifold $P \subseteq N$. Of course, this is only interesting if $f(x) \in P$. First we note that, since $G_x \subseteq G_{f(x)}$, we can view $T_{f(x)}P$ as a G_x-representation. Choosing an invariant Riemannian metric on N, we find a G_x-invariant orthogonal complement $T_{f(x)}P^\perp$ to $T_{f(x)}P$. Furthermore, there is a neighbourhood U of $f(x)$ G_x-diffeomorphic via a G_x-diffeomorphism ζ to the direct sum $T_{f(x)}P \oplus T_{f(x)}P^\perp$ such that $f(x)$ maps to $(0,0)$ and $P \cap U$ maps to $T_{f(x)}P$. To get rid of the group component of f, we choose a normal slice S_x at x and a G_x-diffeomorphism $\varphi : S_x \to V_x$, where V_x is the normal representation at x, i.e. $V_x \cong T_xGx^\perp$, such that x maps to 0. Clearly we can achieve $f(S_x) \subseteq U$. Now consider the map

$$F = \pi_2 \circ \zeta \circ f \circ \varphi^{-1} : V_x \to T_{f(x)}P^\perp,$$

where π_2 is the projection to the second component. We have $F(v) = 0$ if and only if $f(\varphi^{-1}(v)) \in P$. So it is reasonable to define G-transversality of f to P at x in terms of G-transversality of F to 0 at 0.

Definition 2.2.2.3 *Let M, N be G-manifolds, $P \subseteq N$ an invariant submanifold. Take $x \in M$. A G-map $f : M \to N$ is said to be G-*transverse *to P* at *x, if either $f(x) \notin P$, or else the following holds. Choose a normal slice S_x at x and a G_x-diffeomorphism $\varphi : S_x \to V_x$, $V_x = T_xGx^\perp$, $\varphi(x) = 0$. Find a G_x-invariant neighbourhood U of $f(x) \in P$ such that $U \cong T_{f(x)}P \oplus T_{f(x)}P^\perp$ as G_x-representations via an equivariant diffeomorphism ζ, P corresponding to $T_{f(x)}P$, $f(x)$ mapping to $(0,0)$. Let $\pi_2 : T_{f(x)}P \oplus T_{f(x)}P^\perp \to T_{f(x)}P^\perp$ be the projection. Assume furthermore that $f(S_x) \subseteq U$. Then the map $F = \pi_2 \circ \zeta \circ f \circ \varphi^{-1}$ is G_x-transverse to 0 at 0.*

We will see in the remainder of this section, by the properties it has, that this is a satisfying definition of G-transversality. First, we note the results on well-definedness announced earlier.

Proposition 2.2.2.4 *Let V, W be G-representations, M, N G-manifolds, $P \subseteq N$ a smooth invariant submanifold.*

1. *If $f : V \to W$ is G-transverse to 0 at 0 with a given choice of polynomial generators of $\mathcal{C}_G^\infty(V, W)$, then f is G-transverse to 0 at 0 for any choice of generators.*
2. *If $f : M \to N$ is G-transverse to P at x with respect to a given choice S_x of normal slice, U of G_x-invariant neighbourhood of $f(x)$ and the respective equivariant diffeomorphisms, then it is G-transverse to P at x for any such choices.*
3. *If $f : M \to N$ is G-transverse to P at x, then it is G-transverse to P at gx for every $g \in G$.*

Proof. Proofs can be found in [Fie07] and [Bie77a]. □

Before pushing the theory further, we will look at some examples that may help to develop an intuition for equivariant transversality and the connection to ordinary transversality.

Example 2.2.2.5 1. Let G be any compact Lie group acting trivially on the manifolds M and N. Let P be a closed submanifold of N and $f : M \to N$ a smooth map. f is G-transverse to P, if at any $x \in f^{-1}(P)$, the map $\pi \circ \zeta \circ f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^{n-p}$ is G-transverse to 0 at 0, where φ is a chart around x, mapping x to 0, ζ is a chart around $f(x)$, mapping P to $\mathbb{R}^p \subseteq \mathbb{R}^n$ and $f(x)$ to $(0,0)$, and m, n, p are the dimensions of the respective upper case manifolds. We want to show that G-transversality in this case is nothing else than surjectivity of the differential, implying that f is G-transverse to P if and only if it is transverse to P. So specialize to the case $f : V \to W$, V, W trivial G-representations. Generators for the equivariant maps are given by $F_k(x) = e_k$, e_k the k-th unit vector in W, $k = 1, \dots, \dim W = n$. The universal variety is given as

$$\Sigma_G(V,W) = \{(x, t_1, \dots, t_n) \mid \sum_{k=1}^{n} t_k \cdot e_k = 0\} = \{(x, 0, \dots, 0) \mid x \in V\},$$

which has only a single stratum. The graph map of f is given by

$$\Gamma_f(x) = (x, f_1(x), \dots, f_k(x)) = (x, f(x))$$

and $\Gamma_f(0) = (0, f(0))$. Transversality of Γ_f to Σ means that

$$T_0\Gamma_f(V) + V \times \{0\} = V \times \mathbb{R}^n.$$

Since $T_0\Gamma_f(V) = \{(v, T_0 f(v)) \mid v \in V\}$, this is equivalent to $T_0 f$ being surjective.

2. Let $\mathbb{Z}_2$ act canonically on $V = \mathbb{R}$ and let $f : V \to V$ be an equivariant map. A generator for the equivariants is $F_1(x) = x$. The universal variety is given by

$$\Sigma = \{(x,t) \mid xt = 0\},$$

which is stratified by $\{(0,0)\}$ and $\Sigma - \{(0,0)\}$. Write $f(x) = h(x^2) \cdot x$, $h : \mathbb{R} \to \mathbb{R}$ a smooth map. f is $\mathbb{Z}_2$-transverse to 0 at 0, if the graph map

$$\Gamma_f(x) = (x, h(x^2))$$

is transverse to the stratum containing $(0, h(0))$. Since $\Gamma_f : \mathbb{R} \to \mathbb{R}^2$, this is only possible if $h(0) \neq 0$. In that case we have $T_{(0,h(0))}\Sigma = \{0\} \times \mathbb{R}$, and so,

$$T_0\Gamma_f(\mathbb{R}) + \{0\} \times \mathbb{R} = \mathbb{R} \times \{0\} + \{0\} \times \mathbb{R} = \mathbb{R}^2.$$

We see that f is transverse to 0 at 0 if and only if $h(0) \neq 0$ or equivalently, $f'(0) \neq 0$.

3. Let $\mathbb{S}^1$ act canonically on $V = \mathbb{R}^2$ and $f : V \to V$ be an equivariant map. Generators for the equivariants are

$$F_1(x,y) = (x,y), \; F_2(x,y) = (-y,x),$$

so we can write $f(x,y) = h(x^2+y^2)\cdot(x,y) + k(x^2+y^2)\cdot(-y,x)$ for smooth maps $h,k:\mathbb{R}\to\mathbb{R}$. The universal variety is given as

$$\Sigma = \Sigma_{\mathbb{S}^1}(V,V) = \{(x,y,s,t) \mid sx - ty = sy + tx = 0\}.$$

The matrix

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

is regular except for $x = y = 0$, hence,

$$\Sigma = \{(x,y,0,0) \mid (x,y) \neq (0,0)\} \cup \{(0,0,s,t) \mid (s,t) \neq (0,0)\} \cup \{(0,0,0,0)\},$$

which constitutes the canonical Whitney stratification. The graph map of f is given as

$$\Gamma_f(x,y) = (x,y,h(x^2+y^2),k(x^2+y^2))$$

and

$$\Gamma_f(0,0) = (0,0,h(0),k(0)).$$

The differential of Γ_f in $(0,0)$ is given as

$$T_{(0,0)}\Gamma_f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since this is not surjective, Γ_f will not be transverse to the stratum containing $\Gamma_f(0,0)$ if $h(0) = k(0) = 0$. If either $h(0) \neq 0$ or $k(0) \neq 0$, then $\Gamma_f(0,0) \in \{(0,0,s,t) \mid (s,t) \neq (0,0)\}$ and the tangential space of this stratum is $\{(0,0)\}\times\mathbb{R}^2$, so Γ_f is transverse to this stratum in both cases. We conclude that f is $\mathbb{S}^1$-transverse to 0 at 0 if and only if $h(0)^2 + k(0)^2 \neq 0$.

There are several important properties of transversality that should be retrieved for equivariant transversality. Foremost, we want genericity of G-transverse maps. Preimages under G-transverse maps should behave nicely. And finally, some minor results such as an equivariant analogue of the parametrized transversality lemma 1.1.2.6 should hold as well.

We begin with the following observation. Since passing to fixed spaces eliminates the special features of a group action, one would expect G-transversality to imply stratumwise transversality. The latter is defined as follows. Take an isotropy subgroup H. Then an equivariant map $f : M \to N$ induces a map $f^H : M_H \to N^H$. f is stratumwise transverse to P if and only if all the maps f^H are transverse to P^H. One can show that stratumwise transversality is a dense condition, but it will in general not be open.

Example 2.2.2.6 This example is taken from [Fie07]. Let V be the standard representation of $\mathbb{Z}_2$ on $\mathbb{R}$. Then $f : V \to V,\ x \mapsto x^5$ is equivariant and clearly is stratumwise transverse to the invariant submanifold $\{0\} \subseteq V$. Note that, by the calculations of Example 3, f is not G-transverse to 0. Define

$$f_\lambda(x) = x^5 - 2\lambda x^3 + \lambda^2 x.$$

f_λ is equivariant and $f_\lambda(\pm\sqrt{\lambda}) = 0$, $f'_\lambda(\pm\sqrt{\lambda}) = 0$. So f_λ is not stratumwise transverse to 0 for $\lambda > 0$, but converges to f for $\lambda \to 0$. Hence, stratumwise transversality cannot be an open condition.

We will now establish that equivariant transversality implies stratumwise transversality. The proof can also be found in [Bie77a].

Proposition 2.2.2.7 *If $f : M \to N$ is G-transverse to $P \subseteq N$, then it is stratumwise transverse to P.*

Proof. Working locally, we see that the set $M_H = M^H_{(H)}$ corresponds to the set S^H_x in a normal slice at x. So $f^H : M^H_{(H)} \to N^H$ is transverse to $x \in P^H$ if and only if the map $\pi_2 \circ \zeta \circ f \circ \varphi^{-1} : S^H_x \to \left(T_{f(x)}P^\perp\right)^H$ is transverse to 0 at 0. Consequently, the whole problem reduces to the following special case. Let V, W be G-representations. If $f : V \to W$ is G-transverse to 0 at 0, then $f^G : V^G \to W^G$ is transverse to 0 at 0.

Write $V = V^G \oplus A$, $W = W^G \oplus B$. Define $F_i(v,a) = e_i$, $i = 1, \ldots, \dim W^G = k$, where e_i is a basis for W^G. Add generators $F_i(v,a) = \tilde{F}_i(a)$, $i = k+1, \ldots, m$, where the $\tilde{F}$ are generators of $\mathcal{C}^\infty_G(A,B)$. Then $F_1, \ldots, F_m$ generates $\mathcal{C}^\infty_G(V,W)$. For this special choice of generators,

$$\Sigma_G(V,W) = \{(v,a,t) \in V^G \times A \times \mathbb{R}^m \mid t = (s,0) \in \mathbb{R}^{m-k} \times \mathbb{R}^k, (s,a) \in \Sigma_G(A,B)\}$$

which we can identify with $V^G \times \Sigma_G(A,B) \times \{0\} \subseteq V^G \times (A \times \mathbb{R}^{m-k}) \times \mathbb{R}^k$. The map $\pi \circ \Gamma_f \circ i$ is equal to f^G, where $i : V^G \to V$ is the inclusion, $\pi : V \times \mathbb{R}^{m-k} \times \mathbb{R}^k \to \mathbb{R}^k$ the projection. Since f is G-transverse to 0 at 0, we have

$$T_0\Gamma_f(V^G) + T_0\Gamma_f(A) + T_{\Gamma_f(0)}\Sigma_G(V,W) = V \times \mathbb{R}^m.$$

But by definition of the generators, $T_0\Gamma_f(A)$ is just $T_0\Gamma_{f|_A}(A)$ and since $f\big|_A$ is G-transverse to 0 at 0, the latter adds up with $T_{\Gamma_{f|_A}(0)}\Sigma_G(A,B)$ to $A \times \mathbb{R}^{m-k}$. Furthermore,

$$T_{\Gamma_f(0)}\Sigma_G(V,W) = V^G \times T_{\Gamma_{f|_A}(0)}\Sigma_G(A,B).$$

Thus, G-transversality of f implies that

$$T_0\Gamma_f(V^G) + V \times \mathbb{R}^{m-k} = V \times \mathbb{R}^m.$$

Composition with π gives

$$T_0 f^G(V^G) + \{0\} = \mathbb{R}^k = W^G.$$

This is transversality of f^G to 0 at 0. $\square$

Next we want to investigate the problem of preimages under G-transverse maps. For ordinary transversality, the preimage of a submanifold under a transverse map is a submanifold. For G-transverse maps, this is not true in general.

Example 2.2.2.8 Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ act on $V = \mathbb{R}^2$ as the reflections on the two diagonals, i.e. $(-1,1).(x,y) = (y,x)$ and $(1,-1).(x,y) = (-y,-x)$. G acts on $W = \mathbb{R}$ as $(\varepsilon_1, \varepsilon_2).x = \varepsilon_1 \cdot \varepsilon_2 \cdot x$. The equivariant functions $V \to W$ are generated over the invariants by $F(x,y) = x^2 - y^2$. So the universal variety $\Sigma = \Sigma_G(V,W)$ is given as

$$\Sigma = \{(x,y,t) \mid t(x^2 - y^2) = 0\}.$$

The canonical stratification of Σ is given by the strata

$$\begin{aligned} \Sigma_0 &= \{(0,0,0)\} \\ \Sigma_1 &= \{(x, \pm x, t) \mid t \neq 0,\ x \neq 0\} \\ \Sigma_2 &= \{(0,0,t) \mid t \neq 0\} \\ \Sigma_3 &= \{(x,y,0) \mid (x,y) \neq (0,0)\}. \end{aligned}$$

Consider the G-map $f : V \to W$, $f = F$. We claim that f is G-transverse to 0. We check that it is G-transverse to 0 at $(0,0)$ first. The graph map is given by

$$\Gamma_f(x,y) = (x,y,1).$$

We have $\Gamma_f(0,0) = (0,0,1) \in \Sigma_2$, hence, we have to check if Γ_f is transverse to Σ_2 in $(0,0)$. But

$$T_{(0,0)}\Gamma_f(V) + T_{(0,0,1)}\Sigma_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \mid a,b,c \in \mathbb{R} \right\} = \mathbb{R}^3.$$

This shows G-transversality of f to 0 at $(0,0)$. If $f(x,y) = 0$ and $(x,y) \neq (0,0)$, we must have $y = \pm x$. The isotropy of a point $(x, \pm x)$ is $\mathbb{Z}_2$. So to check G-transversality of f to 0 in $(x, \pm x)$ amounts to checking $\mathbb{Z}_2$-transversality of the map $h : \mathbb{R}^2 \to \mathbb{R}$, $h(w,z) = f(w + x, z \pm x) = w^2 - z^2 + 2x(w \mp z)$ to 0 at 0. Here, $\mathbb{Z}_2$ acts on $\mathbb{R}^2$ as reflection on either the diagonal or the antidiagonal $\{(x,-x) \mid x \in \mathbb{R}\}$, depending on the sign, and canonically on $\mathbb{R}$. We will assume that $(x,y) = (x,x)$. Obviously, the other case can be treated identically. Thus we deal with the map $h : \mathbb{R}^2 \to \mathbb{R}$, $h(w,z) = w^2 - z^2 + 2x(w-z)$ and have to check $\mathbb{Z}_2$-transversality to 0 at $(0,0)$. A generator for the equivariants is $F_1(w,z) = w - z$ and the graph map of h is given by

$$\Gamma_h(w,z) = (w, z, w + z + 2x).$$

The universal variety is given by

$$\Sigma = \Sigma_{\mathbb{Z}_2}(\mathbb{R}^2, \mathbb{R}) = \{(w,z,s) \mid s(w-z) = 0\},$$

which is stratified by $\Sigma_0 = \{(w,w,0) \mid w \in \mathbb{R}\}$, $\Sigma_1 = \{(w,w,s) \mid w \in \mathbb{R}, s \neq 0\}$ and $\Sigma_2 = \{(w,z,0) \mid w \neq z\}$. We have $\Gamma_h(0,0) = (0,0,2x) \in \Sigma_1$, so we have to check transversality of Γ_h to Σ_1. We obtain

$$T_{(0,0)}\Gamma_h(\mathbb{R}^2)+T_{(0,0,2x)}\Sigma_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} + c \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + d \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mid a,b,c,d \in \mathbb{R} \right\} = \mathbb{R}^3.$$

We see that h is $\mathbb{Z}_2$-transverse to 0 at $(0,0)$, implying that f is G-transverse to 0 at (x,x). As already mentioned, the case for the points $(x,-x)$ follows identically and we conclude that f is G-transverse to 0. But the preimage of the manifold 0 under f is the union of the diagonal and the antidiagonal, which is not a manifold. Obviously, however, it has a canonical Whitney stratification.

There is a general characterization of preimages due to Bierstone. He proved that the preimage of an invariant submanifold is the transverse intersection of an algebraic subset $A \subseteq \mathbb{R}^n$ with the image of an embedding $\mathbb{R}^m \subseteq \mathbb{R}^n$, compare [Bie77a]. In particular, the preimage has a minimal Whitney stratification. However, it seems more useful in our setting to give a more specialized treatment under additional hypotheses on the map.

Proposition 2.2.2.9 *Let $f : M \to N$ be a map G-transverse to the invariant submanifold $P \subseteq N$ and suppose $G_x = G_{f(x)}$ whenever $f(x) \in P$. Then $f^{-1}(P)$ is stratified by orbit type.*

Proof. By stratumwise transversality of f, the sets

$$f^{-1}(P^H) \cap M^H_{(H)} = \{x \in M \mid f(x) \in P,\ G_x = H\}$$

are submanifolds of $M^H_{(H)}$. Since $G_x = G_{f(x)}$, these sets are equal to $f^{-1}(P^H_{(H)})$. Since $f^{-1}(P)_{(H)}$ is a monotypic G-space, it decomposes as

$$f^{-1}(P)_{(H)} \cong G/_H \times_{W(H)} f^{-1}(P)^H_{(H)} = G/_H \times_{W(H)} f^{-1}(P^H_{(H)}).$$

The right hand side is a monotypic G-manifold, because $G/_H$ is a $W(H)$-principal bundle. This yields that $f^{-1}(P)$ is stratified by orbit type. □

We want to close this introductory chapter on equivariant transversality with one further result which is an equivariant version of Proposition 1.1.2.6 and was originally proven by Bierstone [Bie77a].

Proposition 2.2.2.10 *Let M, N, Λ be G-manifolds, Λ with trivial G-action, $P \subseteq N$ an invariant submanifold. Let $F : \Lambda \times M \to N$ be a G-map G-transverse to P. Then the set of parameters $\lambda \in \Lambda$ such that $F_\lambda : M \to N$ is G-transverse to P is residual in Λ. If P in addition is compact, this set is open.*

Proof. Openness in case of compactness will follow immediately from genericity of G-transverse maps. This result will be proven in a following section, compare Theorem 2.2.2.11. So we just show density here, where we follow [Bie77a]. We know that $F^{-1}(P)$ has a minimal Whitney stratification. Let $\pi : F^{-1}(P) \to \Lambda$ be the projection. We will show that F_λ is G-transverse to P if λ is a regular value for the restrictions of π to any stratum of $F^{-1}(P)$. Then Sards theorem takes care of the rest.

Clearly, if $(\lambda, x) \notin F^{-1}(P)$, then F_λ is G-transverse to P at x. So assume $F(\lambda, x) \in P$. Working locally, we can assume $M = V, N = W$ are G-representations, $\Lambda = \mathbb{R}^\ell$, $F(\lambda, 0) = 0$ and F is G-transverse to 0 at $(\lambda, 0)$. We have to check in which circumstances F_λ is G-transverse to 0 at 0. Choose generators $F_1, \dots, F_k$ of $\mathcal{C}^\infty_G(V, W)$ and write $F(\lambda, x) = \sum_{j=1}^k f_j(\lambda, x) \cdot F_j(x)$, which is possible since G acts trivially on Λ. Let Σ_λ be the stratum of $\Sigma_G(V, W)$ containing $(0, f(\lambda, 0))$. Then $\Sigma = \Lambda \times \Sigma_\lambda$ is the stratum of $\Sigma_G(\Lambda \times V, W)$ containing $(\lambda, 0, f(\lambda, 0))$. By assumption, the map $(\mu, x) \mapsto (\mu, x, f(\mu, x))$ is transverse to Σ at $(\lambda, 0)$ and we have to check under which conditions $x \mapsto (x, f(\lambda, x))$ is transverse to Σ_λ at 0.

Assume λ is a regular value for the restrictions of the projection $\pi : F^{-1}(0) \to \Lambda$ to the strata of $F^{-1}(0)$. Since $F^{-1}(0) = \Gamma_F^{-1}(\Sigma_G(\Lambda \times V, W))$, in particular the projection $p : \Gamma_F^{-1}(\Sigma) \to \Lambda$ has λ as a regular value, so

$$T_{(\lambda,0)}(\Lambda \times V) = T_{(\lambda,0)}\Gamma_F^{-1}(\Sigma) + T_0 V.$$

By this and transversality of Γ_F to Σ at $(\lambda, 0)$, we have

$$\begin{aligned} T_{\Gamma_F(\lambda,0)}(\Lambda \times V \times \mathbb{R}^k) &= T_{(\lambda,0)}\Gamma_F(T_{(\lambda,0)}(\Lambda \times V)) + T_{\Gamma_F(\lambda,0)}(\Sigma) \\ &= T_0\Gamma_{F_\lambda}(T_0 V) + T_{\Gamma_F(\lambda,0)}(\Sigma) \\ &= T_0\Gamma_{F_\lambda}(T_0 V) + T_\lambda\Lambda + T_{\Gamma_{F_\lambda}(0)}(\Sigma_\lambda). \end{aligned}$$

Since $T_{\Gamma_F(\lambda,0)}(\Lambda \times V \times \mathbb{R}^k) = T_\lambda\Lambda + T_{\Gamma_{F_\lambda}(0)}(V \times \mathbb{R}^k)$, we see that

$$T_{\Gamma_{F_\lambda}(0)}(V \times \mathbb{R}^k) = T_0\Gamma_{F_\lambda}(T_0 V) + T_{\Gamma_{F_\lambda}(0)}\Sigma_\lambda$$

which proves our claim. □

We close this introductory chapter by citing the equivariant Thom-Mather theorem which is proven, for example, in [Fie07].

Theorem 2.2.2.11 *Let M, N be compact G-manifolds, $P \subseteq N$ a smooth invariant submanifold. Then the set of G-maps G-transverse to P is residual. If P is closed, this set is open.*

Proof. Theorem 6.14.1 of [Fie07]. □

2.2.3 Equivariant Transversality to Locally Semialgebraic Sets

Equivariant transversality, as we have established it so far, deals with equivariant transversality to invariant manifolds. For our purposes, we need a more general theory of equivariant transversality to semialgebraic subsets, which shall be developed now.

Definition 2.2.3.1 *Let M be a G-manifold and $X \subseteq M$ an invariant subset. X is called locally semialgebraic, if each point $x \in X$ has a tubular neighbourhood $U \cong G \times_H V$ such that the intersection of X with the slice $e \times_H V$ is semialgebraic.*

As in the case for manifolds, we begin with the definition for maps between representations.

Definition 2.2.3.2 *Let V, W be G-representations, $f : V \to W$ a G-map, $X \subseteq W$ a locally semialgebraic, invariant set. f is said to be equivariantly transverse to X at 0, if either $f(0) \notin X$, or else the following holds. Choose polynomial generators $F_1, \dots, F_k$ of $\mathcal{C}^\infty_G(V, W)$ and write*

$$\Gamma_f : V \to V \times \mathbb{R}^k,\ v \mapsto (v, f_1(v), \dots, f_k(v)),$$

$$\vartheta : V \times \mathbb{R}^k \to W,\ (v, t) \mapsto \sum_{i=1}^{k} t_i \cdot F_i(v).$$

Then Γ_f is transverse to the canonical stratification of $\Sigma = \vartheta^{-1}(X)$ at 0.

Standard reasoning shows that equivariant transversality to a locally semialgebraic subset is defined independent of all choices made and is invariant under G-translations, compare [Bie77b], [Fie07]. Of course, this is just the basis for the general definition for G-maps of manifolds. The general setup is the following. Let M, N be G-manifolds, $f : M \to N$ equivariant, $X \subseteq N$ locally semialgebraic and invariant. Take $x \in M$ and $f(x) = y \in N$. Choose a G_x-diffeomorphism $\varphi : S_x \to V_x$, where S_x is a normal slice at x and V_x is a G_x-representation, $\varphi(x) = 0$. Let W be a G_x-representation and $\zeta : W \to N$ a G_x-diffeomorphism onto a neighbourhood of y, $\zeta(0) = y$.

Definition 2.2.3.3 *In the setup of above, f is said to be equivariantly transverse to X at x, if either $f(x) \notin X$, or else, the map $\zeta \circ f \circ \varphi^{-1} : V_x \to W$ is G_x-transverse to the canonical stratification of $\zeta^{-1}(X)$ at 0.*

Again it follows by the standard arguments from chapters five and six of [Bie77b] that the definition of G-transversality does not depend on the choice of slice or equivariant diffeomorphisms and is invariant under G-translations.

We are aiming to prove a Thom-Mather theorem in this new setting. The proof is based on the methods of [Bie77b] as well and reduces the problem to the Thom-Mather theorem for stratified spaces. The basic observation is that equivariant transversality can be reduced to transversality to stratified sets not only pointwise, but locally.

Proposition 2.2.3.4 *In the setup of the definition of G-transversality to semialgebraic sets, f is G-transverse to X at $z \in S_x$ if and only if $\zeta \circ f \circ \varphi^{-1}$ is transverse to $\zeta^{-1}(X)$ at $\varphi^{-1}(z)$.*

Proof. Let $z \in S_x$, $\varphi^{-1}(z) = v$. Choose a small disc S in $T_v G_x v^{\perp}$, centered at v. For r sufficiently small, S is a slice for the G_x-action on V_x at v. This implies that $\varphi(S)$ is a slice for the G-action on M at z. For $F_1, \ldots, F_k$ polynomial generators of $\mathcal{C}^{\infty}_{G_x}(V_x, W)$, the restrictions of the F_i to S generate $\mathcal{C}^{\infty}_{G_z}(S, W)$. The restriction of φ to S specifies a G_z-diffeomorphism $\psi : \varphi(S) \to S$ and equivariant transversality of f at z is defined by equivariant transversality of $\zeta \circ f \circ \psi^{-1}$ to $\zeta^{-1}(X)$ at v. We have

$$\zeta \circ f \circ \psi^{-1}(u) = \zeta \circ f \circ \varphi^{-1}(u) = \sum_{i=1}^{k} f_i(u) \cdot F_i(u).$$

Hence, the graph map is given as

$$\Gamma'_f : S \to S \times \mathbb{R}^k(u) = (u, f_1(u), \ldots, f_k(u))$$

and the universal polynomial is

$$\vartheta' : S \times \mathbb{R}^k \to W, \ (u, t) \mapsto \sum_{i=1}^{k} t_i \cdot F_i(u).$$

We see that f is G-transverse to X at z if and only if Γ'_f is transverse to $\Sigma' = (\zeta \circ \vartheta')^{-1}(X)$ at v. Since $\Sigma' = \Sigma \cap (S \times \mathbb{R}^k)$ and by the proof of Lemma 6.13.2 of [Fie07], Γ_f is transverse to Σ, if and only if $\Gamma_f\big|_S$ is transverse to $\Sigma \cap (V_x^{G_z} \times \mathbb{R}^k)$, we conclude that Γ'_f is transverse to Σ' if and only if Γ_f is transverse to Σ, which finishes the proof. □

This proposition allows to translate equivariant transversality locally into transversality to stratified sets. So it is no surprise that we can deduce openness of equivariant transversality and subsequently, an equivariant Thom-Mather theorem for semialgebraic sets.

Proposition 2.2.3.5 *Let $f : M \to N$ equivariant and $X \subseteq N$ locally semialgebraic. If f is equivariantly transverse to X at $x \in M$, then it is equivariantly transverse to X at y for y in a neighbourhood of x.*

Proof. This follows immediately from openness of transversality to a stratified space and Proposition 2.2.3.4. □

Theorem 2.2.3.6 *If M, N are a compact G-manifold and X is a closed semialgebraic invariant subset of N, then the set of G-maps $f : M \to N$ such that f is equivariantly transverse to X is open and residual.*

PROOF. For openness, we note that the map $(f_1, \ldots, f_k) \mapsto \Sigma_{i=1}^k f_i \cdot F_i$ is open, by the open mapping theorem in Fréchet spaces, see [Rud91]. So for every f' close to f, we find coefficients such that $f' = \sum_{i=1}^k f'_i \cdot F_i$ and the f'_i are close to the f_i's. The statement then follows from Proposition 2.2.3.5 and the openness part of the Thom-Mather theorem for ordinary Whitney stratifications, 2.2.1.4.

The density part can be seen as follows. Let $f : M \to N$ be any G-map, $x \in M$ arbitrary and assume $f(x) \in X$. Choose a tubular neighbourhood U_x of x and an invariant neighbourhood T of $f(x)$ as in the definition of equivariant transversality, such that every G-map in a neighbourhood $\mathcal{U}_x$ of f maps U into T. The graph map of f, defined with the sets U and T, need not be transverse to the universal variety Σ, but by the Thom-Mather theorem for stratified spaces, we find a graph map arbitrarily close that is transverse. Clearly, such a graph map comes from a G-map f', defined in the neighbourhood U_x of x, and this f' can be extended to a map in $\mathcal{U}_x$. Together with the openness part, this shows that the set of G-maps G-transverse to X at U_x form a residual subset. Since M is compact, we obtain that the set of G-maps G-transverse to X at all of M is residual. $\square$

Under some special circumstances, we still have the property that equivariant transversality implies stratumwise transversality.

Proposition 2.2.3.7 *Let M, N be G-manifolds, $X \subseteq N$ a locally semialgebraic subset. Assume that in a local slice S_x at x, X splits into $X = X^G \oplus Y$, $X^G \subseteq S_x^G$, $Y \subseteq \left(S_x^G\right)^\perp$. Then a map that is G-transverse to X is stratumwise transverse to X, meaning that the map*

$$f^H : M^H_{(H)} \to N^H$$

is transverse to the canonical stratification of X^H.

PROOF. In the local situation, we have to show that if $f : V \to W$ is a G-map of representations and $X \subseteq W$ is semialgebraic, satisfying the splitting condition, then $f^G : V^G \to W^G$ is transverse to X^G. Split $V = V^G \oplus A$, $W = W^G \oplus B$ and choose generators for the equivariants $\mathcal{C}^\infty_G(V, W)$ as follows. For $1 \leq i \leq \dim W^G = k$, $F_i(v) = e_i$, where the e_i are a basis for W^G. For $k+1 \leq i \leq m$, $F_i(v, a) = \tilde{F}_i(a)$, where the $\tilde{F}_i$ are generators of $\mathcal{C}^\infty_G(A, B)$. For this special set of generators, if we identify W^G with $\mathbb{R}^k$, we have

$$\vartheta : V^G \oplus A \times \mathbb{R}^k \times \mathbb{R}^{m-k} \to W^G \times B,\ (v, a, s, t) \mapsto (s, \sum_{i=k+1}^m t_i \tilde{F}_i(a)).$$

Pulling back X results in

$$\vartheta^{-1}(X) = \{(v, a, s, t) \mid s \in X^G, \sum_{i=k+1}^m \tilde{F}_i(a) \in Y\} = V^G \times X^G \times \vartheta'^{-1}(Y),$$

where ϑ' is the universal polynomial corresponding to A and B. G-transversality of f to X implies that

$$T_0\Gamma_{f^G}(V^G) + T_0\Gamma_{f\big|_A}(A) + T_{\Gamma_f(0)}\vartheta^{-1}(X) = V \times \mathbb{R}^m.$$

We have

$$T_{\Gamma_f(0)}\vartheta^{-1}(X) = V^G \times T_{\Gamma_{f^G}(0)}X^G \times T_{\Gamma_{f\big|_A}(0)}\vartheta'^{-1}(Y)$$

and the third factor of this expression sums up with $T_0\Gamma_f(A)$ to give $A \times \mathbb{R}^{m-k}$. Hence, G-transversality of f to X implies

$$T_0\Gamma_{f^G}(V^G) + V^G \times T_{\Gamma_{f^G}(0)}X^G = V^G \times \mathbb{R}^k$$

or equivalently,

$$T_0 f^G(V^G) + T_{\Gamma_{f^G}(0)}X^G = W^G,$$

where we used the fact that projecting Γ_f to the second component just gives the map f^G, by choice of the generators. This last equation proves our claim. $\square$

2.3 The Equivariant Isotopy Lemmas

One of the main properties of homotopies $H : M \times I \to N$ transverse to a submanifold $P \subseteq N$ is the invariance of preimages. If $J \subseteq I$ is a compact interval such that for all parameters $\lambda \in J$, H_λ is transverse to P, then the sets $H_\lambda^{-1}(P)$ are isotopic for λ varying in J. We proved a similar but substantially weaker result when we established uniqueness of branches of critical elements.

The more general result is still true when we replace P by a Whitney stratified subset. This is a consequence of the so called first isotopy lemma of Thom. We need an equivariant version of Thom's lemma to deduce a similar invariance result for the preimages of the fibre maps of a G-homotopy. Bierstone indicates in [Bie77a] such that an equivariant result should hold. However, there appears to be no proper reference where the whole construction is carried out in detail. Due to the lack of a reference and because it is not entirely trivial, the proof of the equivariant Thom Isotopy Lemma will be included.

We will shortly review the non-equivariant version of this result and justify the name, which might be a bit confusing when looking at the formulation of the lemma. In the second part, we prove the equivariant version by constructing controlled invariant tubular neighbourhoods and lifting equivariant vector fields in the right way. This part is a modification of [Gib76] and as mentioned above, in some cases there were just a few words to be added to make the theory work equivariantly. One can often use the non-equivariant results in a way to substantially shorten the presentation.

2.3.1 Thom's First Isotopy Lemma

The Thom Isotopy Lemma deals with maps between stratifications, that is, we have a map $f : M \to N$, stratified sets $X \subseteq M$, $Y \subseteq N$ and f maps strata of X into strata of Y. We will always assume that the stratifications of X and Y are Whitney regular. The crucial condition on f to make the isotopy lemma work is that it is a submersion when restricted to strata of X. This seems to be a quite severe condition, but it will be sufficient. We first note the following lemma.

Lemma 2.3.1.1 *Let $f : M \to N$ be transverse to the Whitney stratified subset $X \subseteq N$. Then the stratification of $f^{-1}(X)$ by strata $f^{-1}(X_\alpha)$ is a Whitney stratification.*

Proof. Corollary 8.8. of [Mat80]. □

The Thom Isotopy Lemma in a general form deals with local triviality of a stratified space. By this we mean that a stratified space locally has the structure of the product stratification $P \times F$, P a manifold, F a stratified space. To make this precise, let $f : N \to P$ be a smooth map, $X \subseteq N$ be a stratified subset. X is said to be *trivial over* P, if there is a stratified space F and a homeomorphism $h : X \to P \times F$ such that $f = \pi_P \circ h$, where π_P is the projection to P.

A stratified space $X \subseteq N$ is called *locally trivial over* P, if every point $p \in P$ has a neighbourhood V such that the stratified space $X \cap f^{-1}(V)$ is trivial over V.

Now Thom's lemma makes the following statement on local triviality.

Theorem 2.3.1.2 (Thom's First Isotopy Lemma) *Let N, P be compact manifolds, X a closed Whitney stratified subset of N and $f : N \to P$ be a smooth map such that f, restricted to each stratum, is a submersion. Then X is locally trivial over P and if $V \subseteq P$ is a trivializing set, the fibre space F can be taken to be $f^{-1}(v)$ for any $v \in V$.*

Proof. Theorem 2.5.2 of [Gib76]. □

Let $H : M \times \mathbb{R} \to N$ be a homotopy transverse to the Whitney stratified set $X \subseteq N$. The trick is now to apply the Thom lemma to the Whitney stratified set $H^{-1}(X)$ and the projection map $\pi : H^{-1}(X) \to \mathbb{R}$. As long as the fibre maps H_λ are transverse to X, π restricted to each stratum is a submersion (compare the proof of Proposition 1.1.2.6 or the equivariant version hereof, Proposition 2.2.2.10). So assume that every H_λ is transverse to X for $\lambda \in I$. The isotopy lemma is applicable and we see that every λ in I has an open neighbourhood J such that $\pi^{-1}(J)$ is homeomorphic to $\pi^{-1}(\lambda) \times J$, $\lambda \in J$ arbitrary. Since I is connected, we conclude that $\pi^{-1}(I)$ is homeomorphic to $\pi^{-1}(0) \times I$, the homeomorphism h fulfilling $\pi = \pi_2 \circ h$. But then h is just a homeomorphism $H^{-1}(X) \to H_0^{-1}(X) \times I$ such that the fibres $H_\lambda^{-1}(X)$ are homeomorphic to $H_0^{-1}(X) \times \{\lambda\}$, i.e. the fibres are pairwise isotopic.

2.3.2 Invariant Controlled Tubes and the Equivariant Isotopy Lemmas

We are now going to prove an equivariant version of Thom's isotopy lemma. That is, we will have an invariantly Whitney stratified subset X of N, a G-map $f : N \to P$ mapping strata submersively to P, and we want to conclude equivariant local triviality. By this we mean that the trivialising homeomorphism should be a G-homeomorphism and the fibres invariant stratifications. The key ingredient in the proof is the existence of a certain class of invariant tubular neighbourhoods and, connected with these neighbourhoods, a certain lifting of equivariant vector fields.

After assuming $P = \mathbb{R}^n$, which is possible since everything is local in nature, we lift the coordinate vector fields to equivariant vector fields respecting the strata of X. The flow of these lifted fields will give the desired homeomorphism. We have to show that a lifting to such a "controlled" vector field, i.e. a vector field whose flow respects strata, is possible. We begin with the notion of equivariant tubes.

Definition 2.3.2.1 *Let $P \subseteq M$ be an invariant submanifold. An* equivariant tube *around P is a quadruple $T = (E, \pi, \rho, e)$, where $\pi : E \to P$ is a smooth G-vector bundle, $\rho : E \to \mathbb{R}$ is the quadratic form of an invariant Riemannian metric on E and $e : E \to M$ is an equivariant embedding commuting with the zero section $\zeta : P \to E$, i.e. $e \circ \zeta$ is the inclusion $P \subseteq M$. We define $\pi^T = \pi \circ e^{-1} : U \to X$, $\rho^T = \rho \circ e^{-1} : U \to \mathbb{R}$, where U is an invariant open neighbourhood of P such that e^{-1} is defined on U.*

If T is a tube around P and $f : M \to N$ is a smooth G-map, we say that T is *compatible* with f, if there is an invariant neighbourhood $V \subseteq U$ of P such that $f\big|_V = f \circ \pi^T\big|_V$. So the values of f in the π^T-fibre over $x \in P$ are determined by the value of f on x.

The following theorem is basic and guarantees the existence of tubes compatible with a given map. We can even extend an existing tube under mild assumptions.

Theorem 2.3.2.2 *Let M, N be smooth G-manifolds and $P \subseteq M$ an invariant submanifold. Let $f : M \to N$ be a smooth G-map such that $f\big|_P$ is a submersion. Let $P_1 \subseteq P_0$ be open invariant subsets of P and $\overline{P}_1 \subseteq P_0$. If T_0 is an equivariant tube at P_0 and T_0 is compatible with f, then there is an equivariant tube T at P compatible with f such that $T\big|_{P_1} = T_0\big|_{P_1}$.*

Proof. Let $\xi : TM \to T^2M$ be an equivariant vector field such that ξ locally is an invariant quadratic form on M. Equivalently, if $s : TM \to TM$ is the bundle map given by multiplication by $s \in \mathbb{R}$ in the fibres, we have $\xi(sv) = Ts(s\xi(v))$. Such a vector field is called an equivariant spray on M and it is well known, see for example [Lan02], that the exponential map of ξ, restricted to a normal bundle $\pi : E \to P$ of P, determines a tube around P which clearly is an equivariant tube when we choose the normal bundle to be invariant and an invariant Riemannian metric.

Now, since f is a submersion on P, the kernel of Tf has constant rank locally around P, so we can assume for simplicity that it has constant rank on all of M. In that case $\ker Tf$ is a G-subbundle of TM and $\ker Tf + TP = TM$. Let $T_0 = (E_0, \pi_0, \rho_0, e_0)$. Since e_0 is an embedding, we can identify E_0 with its image in $T_{P_0}M$. The compatibility condition forces $E_0 \subseteq \ker Tf\big|_{P_0}$, since f is constant along fibres.

Choose an invariant Riemannian metric ρ_0 on the bundle $(\ker Tf)\big|_{P_0}$ such that this bundle splits into the orthogonal sum

$$(\ker Tf)\big|_{P_0} = \ker T(f\big|_{P_0}) \oplus E_0.$$

Clearly, by definition of P_1, we find an invariant Riemannian metric ρ on $(\ker Tf)\big|_P$ satisfying $\rho = \rho_0$ on P_1. Take E to be the orthogonal complement under ρ of $\ker T(f\big|_P)$. By abuse of notation, we write ρ for ρ restricted to E. E is a normal bundle of P and $E \subseteq (\ker Tf)\big|_P$. Furthermore, $E\big|_{P_1} = E_0\big|_{P_1}$. Hence, a spray ξ will define a tubular neighbourhood as desired, if the two conditions

- $\xi(\ker Tf) \subseteq T(\ker Tf)$
- $\exp_\xi\big|_{E\big|_{P_1}} = e_0\big|_{E\big|_{P_1}}$

are fulfilled. It is no problem to construct any spray satisfying these properties, by doing so locally and pasting together via a partition of unity. For details, consult chapter two of [Gib76]. Thus, assume $\eta : TM \to T(TM)$ is a spray fulfilling the two properties above. Define

$$\xi : TM \to T(TM),\ \xi(v) = \int_G g^{-1}.\eta(g.v)\, dg.$$

We check that ξ is a spray. Denote the action of an element g on TM by $\tilde{g}$, that is, $\tilde{g}(x,v) = (gx, T_x g(v))$. Equivalently, looking only at the vector part, $\tilde{g}(v) = T_x g(v)$. $\tilde{g}$ commutes with every map s, since

$$\tilde{g} \circ s(x,v) = T_x g(gx, sv) = (gx, sT_x g(v))$$

and

$$s \circ \tilde{g}(x,v) = s(gx, T_x g(v)) = (gx, sT_x g(v)).$$

Hence, we have $T\tilde{g} \circ Ts = Ts \circ T\tilde{g}$. Now we calculate

$$\begin{aligned}
\xi(sv) &= \int_G g^{-1}.\eta(g.sv)\, dg \\
&= \int_G T\tilde{g}^{-1}(\eta(\tilde{g}(sv)))\, dg \\
&= \int_G T\tilde{g}^{-1}(\eta(s\tilde{g}(v)))\, dg \\
&= \int_G T\tilde{g}^{-1}(Ts(s\eta(\tilde{g}v)))\, dg \\
&= \int_G TsT\tilde{g}^{-1}(s\eta(\tilde{g}v))\, dg \\
&= \int_G Ts(sT\tilde{g}^{-1}(\eta(\tilde{g}v)))\, dg \\
&= Ts(s\int_G T\tilde{g}^{-1}(\eta(\tilde{g}v))\, dg) \\
&= Ts(s\xi(v)).
\end{aligned}$$

Therefore, ξ is an equivariant spray. We have to check that ξ fulfills the two properties above. Take any $w \in \ker Tf$. Then $\eta(w) \in T(\ker Tf)$ and since $\ker Tf$ and $T(\ker Tf)$ are invariant subbundles, $g^{-1}.\eta(g.w) \in T(\ker Tf)$ for all $g \in G$. Hence, $\xi(w) \in T(\ker Tf)$, implying that ξ fulfills the first condition. But the second condition is trivially satisfied: e_0 is equivariant, so $\exp_\eta$, the exponential map of η, is equivariant when restricted to $E\big|_{P_1}$. Hence, on $E\big|_{P_1}$, $\exp_\xi = \exp_\eta$. We conclude that ξ is an equivariant spray on TM whose associated tube $T = (E, \pi, \rho, \exp_\xi)$ with E as above is an equivariant tube having all the desired properties. $\square$

Corollary 2.3.2.3 *The conclusion of the theorem remains true if f is only defined on an invariant open subset $P' \subseteq P$ such that $P - P_1 \subseteq P'$.*

Proof. Choose an invariant open subset M' of M such that $P' = M' \cap P$. We can apply the theorem to M' instead of M, obtaining an equivariant tube at P' which coincides with T_0 over $P_1 \cap P'$. Hence, we can extend this tube to P_1 by taking it to be T_0 there, which yields a tube at $P_1 \cup P' = P$. $\square$

Since submersiveness of the map in the last theorem was crucial, the next lemma is interesting because it allows us to apply the theorem to the class of maps specified below.

Lemma 2.3.2.4 *Let P, Q be two invariant submanifolds of M, $P \cap Q = \emptyset$ and the set $P \cup Q$, considered as a stratified subset of M, is Whitney regular. Let U be a neighbourhood of P. If $P \cap \overline{Q} \neq \emptyset$ and T is an equivariant tube at P, then there is a neighbourhood U of P such that $(\pi^T, \rho^T)\big|_{U \cap Q} : U \cap Q \to P \times \mathbb{R}$ is submersive.*

Proof. The statement remains true when dropping all the G-modifications. Let $F = (\pi^T, \rho^T)$. Take $x \in P$ and let $y_n \in Q$ be any sequence converging to x. We have to show that $F\big|_{U \cap Q}$ is a submersion for n sufficiently large. Working locally around x, we can assume that $M = \mathbb{R}^m$, $x = 0$, $P = \mathbb{R}^p \subseteq \mathbb{R}^p \times \mathbb{R}^{m-p} = M$ and $U = \mathbb{R}^p \times (-1, 1)$. So F is just given by

$$F(x_1, \ldots, x_n) = (x_1, \ldots, x_p, x_1^2 + \cdots + x_n^2).$$

F being a submersion at y_n yields $T_{y_n}F(\mathbb{R}^M) = \mathbb{R}^p \times \mathbb{R}$. The kernel of $T_{y_n}F$ is coincides with the orthogonal complement of $\mathbb{R}^p + \langle y_n \rangle$. Hence, $T_{y_n}F(\mathbb{R}^p + \langle y_n \rangle) = \mathbb{R}^p \times \mathbb{R}$. By compactness we can assume that $T_{y_n}Q$ converges to a subspace E, $\langle y_n \rangle$ converges to a one-dimensional subspace L of $\mathbb{R}^m$ and the linear maps $T_{y_n}F$ converge to a linear map $T : \mathbb{R}^m \to \mathbb{R}^p \times \mathbb{R}$. By Whitney regularity we have $\mathbb{R}^p + L \subseteq E$. Consequently,

$$\mathbb{R}^p \times \mathbb{R} = T_{y_n}F(\mathbb{R}^p + \langle y_n \rangle) \to T(\mathbb{R}^p + L) \subseteq T(E) = \lim T_{y_n}F(T_{y_n}Q),$$

which shows that $T_{y_n}F\big|Q$ is surjective. $\square$

We will now introduce a condition on maps between stratified subsets, similar to the Whitney conditions for sets. A map fulfilling this condition will be called a Thom stratified map. In principle, it is not necessary to look at Thom stratified maps to deduce our results and it is mainly needed to state and prove the second isotopy lemma. But it is in some cases easier to prove the results in general and then show that more special results follow by certain choices of Thom stratified maps.

Definition 2.3.2.5 *Let M, N be compact manifolds, $f : M \to N$ a map, $X \subseteq M$, $Y \subseteq N$ Whitney stratified subsets such that f maps strata of X submersively into strata of Y. Assume furthermore that f has constant rank when restricted to strata. Then f fulfills the* Thom regularity condition*, if for every sequence $y_n \in X_\beta$, converging to some $x \in X_\alpha$ such that the spaces $\ker T_{y_n} f\big|_{X_\beta}$ converge to a subspace $E \subseteq T_x M$, we have $\ker T_x f\big|_{X_\alpha} \subseteq E$. In this case, (X, Y) is called a* Thom stratification *for f.*

Henceforth X and Y will be invariantly Whitney stratified and f will be a G-map.

Let P, Q be invariant submanifolds of M and T^P, T^Q be equivariant tubes around P, Q, respectively. Denote the tubular projections by π^P, π^Q. Let Q' be an invariant submanifold of a G-manifold N and $T^{Q'}$ an equivariant tube around Q' and $f : M \to N$ an equivariant map such that $f(Q) \subseteq Q'$. Define three commutation conditions as follows:

$$\begin{array}{ll} (C\pi) & \pi^P \circ \pi^Q = \pi^P \\ (C\rho) & \rho^P \circ \pi^Q = \rho^P \\ (Cf) & f \circ \pi^Q = \pi^{Q'} \circ f, \end{array}$$

whenever both sides are defined. We are aiming at a result similar to the submersiveness result for the maps

$$(\pi^T, \rho^T)\big|_{U \cap Q} \to P \times \mathbb{R}$$

given above, but this time for Thom stratified maps.

Assume we have a G-map $f : M \to N$, invariant submanifolds $P, Q \subseteq M$, $P', Q' \subseteq N$ and f maps P, Q submersively into P', Q', respectively. Take tubes $T_P, T_{P'}, T_{Q'}$ at P, P', Q', respectively, such that $(C\pi)$ holds for $T_{P'}, T_{Q'}$ and (Cf) holds for T_P and $T_{P'}$. We can form the pullback

$$P \times_{P'} Q' = \{(x, y') \in P \times Q' \mid f(x) = \pi^{T_{Q'}} y'\}.$$

Since f is a submersion, the pullback is a smooth G-manifold and by the commutation relations, the map

$$(\pi^{T_P}, \pi^{T_{Q'}} \circ f) : M \to P \times Q'$$

has image in $P \times_{P'} Q'$. We are now in the position to state:

Lemma 2.3.2.6 *Let M, N, P, Q, P', Q' be as above and $f : M \to N$ a G-map. If Q is Thom regular over P with respect to f, then*

$$(\pi^{T_P}, f)\big|_Q : Q \to P \times_{P'} Q'$$

is a submersion locally around P, that is, there are neighbourhoods U of P and V of Q such that (π^{T_P}, f) is a submersion in $U \cap V$.

Proof. Assume the statement to be false. Then we find a sequence $y_n \in Q$ converging to $x \in P$ such that $(\pi^{T_P}, f)\big|_Q$ is not submersive at y_n. Let $x_n = \pi^{T_P}(y_n)$, $y'_n = f(y_n)$, $x'_n = f(x_n)$. The tangent space to the pullback $P \times_{P'} Q'$ at (x_n, y'_n) is given by

$$T_{x_n}P \times_{T_{x'_n}P'} T_{y'_n}Q'.$$

Since $Tf\big|_Q$ is a submersion, the map $(\pi^{T_P}, f)\big|_Q$ is not submersive at y_n if and only if the map

$$T_{y_n}\pi^{T_P} : \ker T_{y_n}f\big|_Q \to \ker T_{x_n}f\big|_P$$

is not surjective, compare the following diagram.

$$\begin{tikzcd}
\ker T_{y_n}f\big|_Q \arrow[r] \arrow[ddr] & T_{y_n}Q \arrow[dr] \arrow[drr, "T_{y_n}f"] \arrow[ddr, "T_{y_n}\pi^{T_P}"'] & & \\
& & T_{x_n}P \times_{T_{x'_n}P'} T_{y'_n}Q' \arrow[r] \arrow[d] & T_{y'_n}Q' \arrow[d] \\
& \ker T_{x_n}f\big|_P \arrow[r] & T_{x_n}P \arrow[r, "T_{x_n}f"] & T_{x'_n}P'
\end{tikzcd}$$

By compactness of the Grassmannian, we can assume that the spaces

$$T_{y_n}\pi^{T_P}\left(\ker T_{y_n}f\big|_Q\right)$$

have constant dimension and converge to a subspace S of T_xP and at the same time, the spaces $\ker T_{y_n}f\big|_Q$ converge to $E \subseteq T_xN$. Since f has constant rank when restricted to P, we must have $\ker T_{x_n}f\big|_P \to \ker T_xf\big|_P$. Due to the lack of surjectivity of the map $T_{y_n}\pi^{T_P}$,

$$T_x\pi^{T_P}(T) = S \subsetneq \ker T_xf\big|_P.$$

π^{T_P} is a retraction onto P, yielding

$$\ker T_xf\big|_P \subseteq T_x\Pi^{T_P}\left(\ker T_xf\big|_P\right).$$

So we cannot have $\ker T_xf\big|_P \subseteq T$, contrary to the definition of Thom regularity. $\square$

Our next aim is to lift equivariant vector fields under a given G-map such that the flow of the lifted field respects the stratification. For this purpose, we need equivariant tubes around each stratum and these have to be compatible in the following sense.

Definition 2.3.2.7 *Let X be an invariantly Whitney stratified subset of the G-manifold M. An* equivariant tube system *for X consists of equivariant tubes $T_\alpha = (E_\alpha, \pi_\alpha, \rho_\alpha, e_\alpha)$ for every stratum X_α of X. An equivariant tube system is said to be* weakly controlled*, if all pairs of tubes fulfill the commutation relation $(C\pi)$. It is called* controlled*, if $(C\rho)$ holds in addition.*

If $f : M \to N$ is a G-map and (X, Y) an invariant Thom stratification for $f\big|_X : X \to Y$, an equivariant tube system $\{T_i\}$ for X is said to be controlled over an equivariant tube system $\{S_i\}$ for Y, *if all relations $(C\pi)$, (Cf) are satisfied and in addition, if $f(X_\alpha) \cup f(X_\beta) \subseteq Y_\gamma$ for some γ, then $(C\rho)$ is satisfied by the tubes corresponding to X_α and X_β.*

The next theorem will guarantee the existence of an equivariant tube system on the manifold M which will provide the controls for the flow of the vector field lifted via a Thom stratified map f. It is the equivariant version of Theorem 2.2.6 of [Gib76], and the proof is a minor modification of the proof given in that reference.

Theorem 2.3.2.8 *Let M, N be smooth G-manifolds, $f : M \to N$ a G-map, (X, Y) an invariant Thom stratification for f. Then for each weakly controlled equivariant tube system $\{T'_\beta\}$ for Y there is an equivariant tube system $\{T_\alpha\}$ for X such that T is controlled over T'.*

PROOF. To simplify matters we note that, if for two strata X_α, X_β we have $X_\alpha \cap \overline{X}_\beta \neq \emptyset$, then $\dim X_\alpha < \dim X_\beta$. Hence, instead of single strata, it suffices to look at the sets X^i, where X^i is the union of all strata of dimension i. We are going to construct the tube system T inductively and assume we have already constructed a tube system $\{T^{j-1}\}$ for X^{j-1} which is controlled over T'. In the case $j = 0$, there is nothing to show. If $j > 0$, we apply Theorem 2.3.2.2 to obtain an equivariant tube T_j at X^j satisfying condition (Cf) with respect to T'. We are going to modify T_j locally around X^{j-1} such that all commutation relations are satisfied.

We do this using another induction: for $0 \leq k \leq j$ let

$$X^j_k = \bigcup_{r=k}^{j} X^r.$$

In particular, we have $X^j_j = X^j$ and X^j_0 is the union of all strata of dimension less or equal j. So we can assume inductively that the equivariant tube system $\{T^{j-1}\}$ together with T_j is controlled over T' on X^j_k for some k. We have to show that we can adjust T_j such that the induced system on X^j_{k-1} is controlled over T' as well. Let the tubes at X^r, $k \leq r < j$, be given by $T_r = (E_r, \pi_r, \rho_r, e_r)$ with tubular neighbourhood U_r. Define

$$Q_r = X^j \cap U_r.$$

We partition Q_r further. For $0 \leq s, t \leq j$, let $Q_r(s, t)$ be the set of $y \in Q_r$ with $f(y) \in Y^t$, $f(\pi_r y) \in Y^k$. Finally, let

$$Q^\rho_r = \bigcup_{s=0}^{j} Q_r(s, s).$$

This gives a partition

$$Q_r = Q^\rho_r \cup \bigcup_{s \neq t} Q_r(s, t).$$

Each member of this partition is an open set, since f maps strata into strata.

Now take the tubes of the system T', i.e. $T'_r = (E'_r, \pi'_r, \rho'_r, e'_r)$ with tubular neighbourhoods U'_r. We can shrink the U_r and U'_r such that the following conditions are satisfied:

(1) $\pi_r \circ \pi_j = \pi_r$ locally around Q_r for $k < r < j$ by downward induction hypothesis.

(2) $\pi_k \circ \pi_r = \pi_k$ locally around $Q_k \cap Q_r$ for $k < r < j$ by upward induction hypothesis.

(3) $\rho_k \circ \pi_r = \rho_k$ locally around $Q_k^\rho \cap Q_r$ for $k < r < j$ by upward induction hypothesis.

(4) $\pi'_s \circ \pi'_t = \pi'_s$ locally around $Y^t \cap U'_s$ for $0 \leq s < t \leq j$, since T' is weakly controlled.

(5) $Q_r(s,t) = \emptyset$ if $s > t$ for all r for dimensional reasons (since Y is Whitney stratified).

(6) If X_α, Y_β are strata of dimension r, s, respectively, such that $f(X_\alpha) \subseteq Y_\beta$, then f maps the restriction of U_r to X_α into the restriction of U'_s to Y_β (by continuity of f).

(7) $f \circ \pi_r = \pi'_s \circ f$ locally around $Q_r(s,t)$, $k \leq r < j$ and $0 \leq s \leq t \leq j$ by upward induction hypothesis.

(8) $(\pi_k, \rho_k)\big|_{Q_k^\rho} : Q_k^\rho \to X^k \times \mathbb{R}$ is a submersion by Whitney regularity and Lemma 2.3.2.4.

(9) $(\pi_k, f)\big|_{Q_k(s,t)} : Q_k(s,t) \to X^k \times_{Y^s} (Y^t \cap U'_s)$ is a submersion for $0 \leq s < t \leq j$ by Thom regularity and Lemma 2.3.2.6.

Now we apply Corollary 2.3.2.3 with

$$P = \bigcup_{k \leq r < j} Q_r, \quad P_0 = \bigcup_{k < r < j} Q_r, \quad T_0 = T_j\big|_{P_0}, \quad P' = Q_k$$

and the following map h, defined locally around $P' = Q_k^\rho \cup \bigcup_{s<t} Q_k(s,t)$. Around Q_k^ρ, $h = (\pi_k, \rho_k)$. Around $Q_k(s,t)$, $h = (\pi_k, \pi'_t \circ f)$. This is well-defined by the commutation relations. By properties (8) and (9), $h\big|_{P'}$ is a submersion. The tube $T_0\big|_{P'}$ is compatible with h because for every r with $k < r < j$ we have

$$\pi_k \circ \pi_j = \pi_k \circ \pi_r \circ \pi_j = \pi_k \circ \pi_r = \pi_k,$$

where we used (1) and (2), and

$$\rho_k \circ \pi_j = \rho_k \circ \pi_r \circ \pi_j = \rho_k \circ \pi_r = \rho_k,$$

where we used (1) and (3). Finally, T_j is compatible with f by construction.

To apply the corollary, it remains to specify a subset P_1. For this purpose, we shrink the tubes T_r, $k < r < j$, a bit such that the modified tubes are embedded into subsets V_r with $\overline{V_r} \subseteq U_r$. We define Q_r^1 just as Q_r, but with the modified tubes. So every Q_r^1

is an open subset of Q_r and we define $P_1 = \bigcup_{k<r<j} Q_r^1$. Clearly, P_0 is a neighbourhood of $P - P'$ and $\overline{P}_1 \subseteq P_0$. We can furthermore achieve $P - P' \subseteq P_1$. Thus, we can apply corollary 2.3.2.3 and obtain an equivariant tube $T = (E, \pi, \rho, e)$ at P which extends $T_j\big|_{P_1}$ and is compatible with h. T satisfies all necessary commutation relations by construction of h, except over the space Q_k^ρ. Here we have to check that (Cf) holds. But we have $Q_k^\rho = \bigcup_{s=0}^{j} Q_k(s,s)$ and over $Q_k(s,s)$, we have

$$f \circ \pi = \pi'_s \circ f \circ \pi = f \circ \pi_k \circ \pi = f \circ \pi_k = \pi'_s \circ f,$$

where we applied (7) repeatedly. Finally we can extend the tube T to all of X^j, making it compatible with f and keeping it fixed locally around X^{j-1}, which is possible by Theorem 2.3.2.2. The tube we obtain has all the desired properties, which completes the induction. □

To make precise what we mean by a vector field respecting a stratification, we make the next definition.

Definition 2.3.2.9 *Let X be an invariantly stratified subset of the G-manifold M. A* stratified equivariant vector field *ξ on X is given by a, probably discontinuous, equivariant vector field on each stratum X_α, giving a map $\xi : X \to TM$ such that $\xi(x) \in T_x X_\alpha$ if $x \in X_\alpha$ and the restriction to a stratum of X is smooth.*

Similar to maps, we have control conditions for a stratification and an equivariant vector field.

Let $\{T_\alpha\}$ be a tube system for X. The *control conditions* for a stratified equivariant vector field ξ on X are given by

$$\begin{array}{ll} (V\pi) & T\pi_\alpha \circ \xi = \xi \circ \pi_\alpha \\ (V\rho) & T\rho_\alpha \circ \xi = 0 \end{array}$$

which are required to hold locally around each stratum X_α. f is called weakly controlled, if $(V\pi)$ holds for all strata of X. It is called controlled, if in addition $(V\rho)$ holds for all strata in X. Let $f : M \to N$ be a G-map of G-manifolds and (X, Y) an invariant stratification of f, η a stratified equivariant vector field on $Y \subseteq N$. Then a stratified equivariant vector field ξ on X is said to lift η, if the condition

$$(Vf) \qquad Tf \circ \xi = \eta \circ f$$

holds. Finally, we have a $(V\rho)$-like condition for vector fields and maps. Let X_α be a stratum of X that is mapped to Y_β by f. Then we have the condition

$$(Vf\rho) \qquad T\rho_\alpha \circ \xi\big|_{U_\beta} = 0,$$

where U_β is some invariant neighbourhood of $f^{-1}(Y_\beta) \cap X_\alpha$. An equivariant vector field ξ is said to be controlled over the equivariant vector field η, if all the conditions $(V\pi)$, (Vf) and $(Vf\rho)$ are fulfilled.

We can now state the main auxiliary result, namely the fact that weakly controlled stratified G-vector fields can be lifted to controlled G-vector fields under a Thom stratified G-map.

Theorem 2.3.2.10 *Let $f : M \to N$ be a G-map, $X \subseteq M$, $Y \subseteq N$ invariantly stratified subsets such that (X, Y) is a Thom stratification for f. Let T, T' be equivariant tube systems for X and Y such that T is controlled over T'. Then any weakly controlled stratified equivariant vector field η on Y admits an equivariant lift ξ which is controlled over η.*

PROOF. Here we can make use of the same result for non-equivariant fields. Under the given conditions, we know by Theorem 2.3.2 of [Gib76] that there is some vector field ξ' lifting η, not necessarily equivariant. Define

$$\xi(x) = \int_G g^{-1}.\xi'(gx)\, dg.$$

We claim that ξ is an equivariant lift of η. First, we compute that ξ lifts η:

$$\begin{aligned} Tf \circ \xi(x) &= Tf(\int_G g^{-1}.\xi'(gx)\, dg) \\ &= \int_G Tf(g^{-1}.\xi'(gx))\, dg \\ &= \int_G g^{-1}.Tf \circ \xi'(gx)\, dg \\ &= \int_G g^{-1}.\eta \circ f(gx)\, dg \\ &= \int_G g^{-1}g.\eta \circ f(x)\, dg \\ &= \eta \circ f(x). \end{aligned}$$

Then we look at the control conditions. We have for each stratum X_α,

$$\begin{aligned} T\pi_\alpha \circ \xi(x) &= \int_G g^{-1}.T\pi_\alpha \circ \xi'(gx)\, dg \\ &= \int_G g^{-1}.\xi' \circ \pi_\alpha(gx)\, dg \\ &= \int_G g^{-1}.\xi'(g\pi_\alpha(x))\, dg \\ &= \xi \circ \pi_\alpha(x) \end{aligned}$$

and for x in an invariant neighbourhood U_β of $f^{-1}(Y_\beta) \cap X_\alpha$,

$$\begin{aligned} T\rho_\alpha \circ \xi(x) &= \int_G g^{-1}.T\rho_\alpha \circ \xi'(gx)\, dg \\ &= \int_G g^{-1}.0\, dg \\ &= 0. \end{aligned}$$

Hence, all control conditions are fulfilled and ξ turns out to be an equivariant lift controlled over η. □

In preparation of the equivariant first isotopy lemma, the following corollary is what we really need. The use of Thom stratifications and the much more general theorem were just a convenient shortcut to arrive at this result.

Corollary 2.3.2.11 *Let $f : M \to N$ be a G-map, X a Whitney stratified subset of M such that f maps each stratum submersively into N. Let T be a controlled tube system for X. Then each equivariant vector field on N can be lifted to a controlled equivariant vector field on X.*

PROOF. In the preceeding theorem, take $Y = N$, stratified with a single stratum. This gives an invariant Thom stratification for f. □

We have to deal with one last complication, namely, the domains of definition of stratified vector fields. Since strata are not necessarily compact or even closed, the usual global existence results fail, so we have to spare some thoughts here.

Let ξ be a stratified vector field on X. Then the restriction ξ_α to a stratum X_α defines a flow $\varphi_\alpha : D_\alpha \to X_\alpha$, where D_α is the maximal domain of definition. As is well-known, D_α is an open subset of $\mathbb{R} \times X_\alpha$ and contains $0 \times X_\alpha$. Setting $D = \bigcup_\alpha D_\alpha$, we obtain a flow $\varphi : D \to X$ of ξ. Note that this satisfies the conditions on a flow, but might be discontinuous on the edges of the strata. In addition, D need not be open in M. We say that ξ is locally integrable, if D contains an open neighbourhood of $0 \times X$ such that φ is continuous on this neighbourhood. We say that ξ is globally integrable, if $D = \mathbb{R} \times X$.

The following results for flows of stratified equivariant vector fields are proven in chapter two of [Gib76] and hold for all fields, we do not have to take symmetries into account here.

Lemma 2.3.2.12 *Let ξ be locally integrable. Then D is an open subset of $\mathbb{R} \times X$ and φ is continuous on all of D.*

Now we put together all our definitions so far to see that, given that all objects are regularly stratified or controlled, respectively, one can obtain the same integrability results for stratified fields as one has at hand for ordinary fields on manifolds.

Theorem 2.3.2.13 *Let M, N be G-manifolds, $f : M \to N$ a G-map. Let $X \subseteq M$, $Y \subseteq N$ be stratified subsets such that (X, Y) is a Thom stratification for $f : X \to Y$. Let ξ, η be stratified equivariant vector fields on X, Y, respectively, such that ξ is controlled over η with respect to a tube system T for X. If X is locally closed in M, then ξ is locally integrable.*

PROOF. Theorem 2.4.6 of [Gib76]. □

Corollary 2.3.2.14 *A controlled vector field on a locally compact stratified set is locally integrable.*

Finally we have to pass from locally integrable fields to globally integrable fields. Yet again, if the G-map f is proper, we will obtain global integrability of the lifted field from global integrability of the initial one, as was to be expected. In particular, if the participating manifolds are compact, we can conclude that globally integrable fields lift to globally integrable fields under any Thom stratified map.

Lemma 2.3.2.15 *Let $f : M \to N$ be a G-map, X, Y stratified subsets of M, N such that (X, Y) is a Thom stratification for f. Assume that the restriction of f to the closure of each stratum is proper. Let ξ, η be stratified equivariant vector fields on X, Y such that ξ lifts η and let ξ be locally integrable. Then, if η is globally integrable, so is ξ.*

Proof. Lemma 2.4.8 of [Gib76]. □

We are finally in the position to prove the equivariant version of Thom's isotopy lemma. We just have to lift the coordinate vector fields to obtain the isotopy we need and have to check that all this can be done equivariantly.

Theorem 2.3.2.16 (First Equivariant Isotopy Lemma) *Let N be a G-manifold, P a trivial G-manifold, $f : N \to P$ and $X \subseteq N$ an invariantly Whitney stratified closed subspace. Assume that f maps each stratum of X submersively to Y. Then $f\big|_X : X \to P$ is locally trivial, i.e. every $t \in P$ has a neighbourhood U in P such that there is a G-space F and a G-homeomorphism $\varphi : U \times F \to X$, making the diagram*

$$\begin{tikzcd} U \times F \arrow[r, "\varphi"] \arrow[d, "\pi_1"'] & X \arrow[dl, "f|_X"] \\ U & \end{tikzcd}$$

commutative.

Proof. Working locally, we can assume that $P = \mathbb{R}^n$. Choose an invariant tube system for X, compatible with f. By Theorem 2.3.2.10, we can lift the basic vector fields $\partial_1, \ldots, \partial_n$ on $\mathbb{R}^n$ to globally integrable vector fields $\xi_1, \ldots, \xi_n$ on X. Let ψ_i be the equivariant flow of the field ξ_i, $i = 1, \ldots, n$. Let $F = f^{-1}(0) \cap X$ and define

$$\varphi : F \times \mathbb{R}^n \to X,\ \varphi(x, t) = \psi_1(\psi_2(\ldots(\psi_n(x, t_n)), \ldots, t_2), t_1).$$

Then φ is a G-homeomorphism. □

Since we have gone so far using the help of Thom stratifications, we will also give a proof of an equivariant version of Thom's second isotopy lemma. We will not need this result in the rest of the work, however.

Theorem 2.3.2.17 (Second Equivariant Isotopy Lemma) *Let $f : M \to N$, $\pi : N \to P$ be equivariant maps of smooth compact manifolds, the action on P trivial, $X \subseteq M$, $Y \subseteq N$ invariantly Whitney stratified subsets such that (X, Y) is an invariant Thom stratification for f (in particular, f respects strata). Assume furthermore that* *each stratum of Y is mapped submersively by π into P. Then f is locally trivial as a stratified map over P. By this we mean that for every point $p \in P$ there is an open invariant neighbourhood U of p and equivariant homeomorphisms*

$$h : U \times f^{-1}(\pi^{-1}(p)) \cap X \to f^{-1}(\pi^{-1}(U)) \cap X, k : U \times \pi^{-1}(p) \cap Y \to \pi^{-1}(U) \cap Y$$

mapping strata into strata and making the following diagram commutative.

$$\begin{tikzcd}
U \times f^{-1}(\pi^{-1}(p)) \cap X \arrow[r, "\mathbb{1}\times f"] \arrow[d, "h"] & U \times \pi^{-1}(p) \cap Y \arrow[d, "k"] \arrow[dr, "p_U"] & \\
 & & U \\
f^{-1}(\pi^{-1}(U)) \cap X \arrow[r, "f"] & \pi^{-1}(U) \cap Y \arrow[ur, "\pi"] &
\end{tikzcd}$$

PROOF. As in the proof of the first isotopy lemma, we can work locally and assume $P = \mathbb{R}^n$. We lift the coordinate vector fields to controlled equivariant vector fields on Y, using Theorem 2.3.2.10. Then using 2.3.2.10 again together with Theorem 2.3.2.8, we can lift the vector fields on Y to controlled equivariant vector fields on X. By Lemma 2.3.2.15, we obtain a flow on X which covers the flow on Y, which in turn covers the standard flow on P. Then we can define the G-homeomorphisms using these flows as we did in the proof of the first isotopy lemma. □

2.4 Equivariant Critical Elements

This section will be preparatory for the genericity theorems of sets of G-maps. We will define special types of equivariant critical elements and use the results of the preceeding section to conclude that these notions are the right ones to obtain the generic bifurcation scenario.

We start with the examination of the equivariant diagonal of a G-manifold. One is tempted to think that this should be the set $\{(x, gx) \mid x \in M,\ g \in G\}$, but it turns out that this is neither philosophically the correct set to do fixed orbit theory, nor does it have a sufficiently well behaved structure. We already noticed that if Gx is a fixed orbit of a G-map $f : M \to M$, then we have $f(x) = gx$ and g is an element of the normalizer $N(G_x)$. So a better choice for the *equivariant diagonal* is the set

$$\Delta_G(M) = \overline{\{(x, gx) \mid x \in M,\ g \in N(G_x)\}} = \overline{\{(x, nx) \mid x \in M,\ n \in W(G_x)\}}.$$

We will show that this set is indeed a locally semialgebraic set, so we can do equivariant transversality theory with it and thus define a notion of equivariant non-degeneracy. But before doing so, we introduce the notion of G-hyperbolicity of critical elements. This is a more geometrically flavoured restriction on the shape of critical elements and we will prove some results for G-hyperbolic elements which are hard to obtain for equivariantly non-degenerate ones, for example, isolatedness. There will be sufficiently many interplay between these notions, by means of genericity and the equivariant isotopy lemma, to justify the use of both concepts. However, there is neither proof nor disproof of the fact that G-hyperbolicity implies equivariant non-degeneracy. It would be interesting to know if the theory of G-transversality has this intuitive implication.

Finally, we will introduce the notion of equivariant non-degeneracy, which will just be G-transversality of the graph map to the equivariant diagonal. Then we can start to harvest the fruits of our labour by using the preceeding sections to prove that in many fundamental aspects, equivariant non-degeneracy behaves just as non-degeneracy does for ordinary maps. We can then proceed to proof the equivariant genericity results, corresponding to the theorems of chapter 1. They are either corollaries of Theorem 2.2.3.6, or based on similar techniques than the non-equivariant versions.

2.4.1 The Equivariant Diagonal

We begin our investigation of the equivariant diagonal with the identification of the strata.

Proposition 2.4.1.1 *If G acts freely on M, then $\Delta_G(M)$ is a manifold of dimension $\dim M + \dim G$.*

Proof. If the action is free, $G_x = e$, so $N(G_x) = G$. Let $(x, gx) \in \Delta_G$, (φ, U) a chart around x in M, (ψ, V) a chart around g in G. Let

$$W = \{(y, hy) \mid y \in U,\, h \in V\}.$$

Clearly, W is open. Define $\Phi : W \to \mathbb{R}^n \times \mathbb{R}^m$, $\Phi(y, hy) = (\varphi(y), \psi(h))$. Since the action is free, Φ is well-defined. It is obviously a diffeomorphism onto its image, its inverse given by $(v, w) \mapsto (\varphi^{-1}(v), \psi^{-1}(w).\varphi^{-1}(v))$. Hence, (Φ, W) is a chart around (x, gx) and so Δ_G is a manifold of the stated dimension. $\square$

Corollary 2.4.1.2 *Let the action of G on M be monotypic of type (H). Then $\Delta_G(M)$ is a manifold of dimension $\dim M + \dim W(H)$.*

Proof. Take $(x, gx) \in \Delta_G$. Then $G_{(x,gx)} = \{h \in G \mid hx = x,\, hgx = gx\} = G_x \cap gG_xg^{-1} = G_x$, since $g \in N(G_x)$. So $\Delta_G(M)$ is monotypic as well. Hence, it has a decomposition

$$\Delta_G(M) = G/_H \times_{W(H)} \Delta_G(M)^H.$$

We have

$$\Delta_G(M)^H = \{(x, gx) \mid hx = x,\, hgx = gx\; \forall h \in H\}$$

so $x \in M^H, gx \in M^H$. But $N(G_x)$ acts on M^H and this action reduces to an action of the Weyl group $W(H)$. We see that

$$\Delta_G(M)^H = \Delta_{W(H)}(M^H)$$

and the right hand space is, for M^H is a free $W(H)$-space, a manifold of dimension $\dim M^H + \dim W(H)$. The twisted product

$$G/_H \times_{W(H)} \Delta_G(M)^H$$

is a manifold, since $G/_H$ is a principal $W(H)$-bundle and thus has local sections. The dimension of the twisted product thus is calculated as $\dim G/_{N(H)} + \dim M^H + \dim W(H)$. The dimension of M is given by $\dim M = \dim G/_{N(H)} + \dim M^H$ and we obtain

$$\dim \Delta_G(M) = \dim G/_{N(H)} + \dim M^H + \dim W(H) = \dim M + \dim W(H). \qquad \square$$

Example 2.4.1.3 1. Let G act trivially. Then the equivariant diagonal is the ordinary diagonal $\{(x, x) \mid x \in M\} \subseteq M \times M$.

2. Let M be the Stiefel manifold of 2-frames in $\mathbb{R}^3$, i.e.

$$M = \{(x, y) \in \mathbb{S}^2 \times \mathbb{S}^2 \mid \langle x, y \rangle = 0\}.$$

As is well-known (see e.g. [Swi02]), M is diffeomorphic to $O(3)/_{O(1)} \cong SO(3)$, so M is a 3-dimensional compact manifold. We have the canonical action of $O(3)$ on M, given by $A.(x, y) = (Ax, Ay)$ and M is a monotypic $O(3)$-manifold of orbit type $(O(1))$. The normalizer of $O(1)$ in $O(3)$ is given as $\mathbb{Z}_2 \times O(2)$, identifying the Weyl group as $W(O(1)) \cong SO(2)$. For the special choice $(x, y) = (e_2, e_3)$,

$$O(3)_{(e_2,e_3)} = \left\{ \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

and the normalizer hereof is

$$N = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & C \end{pmatrix} \mid C \in O(2) \right\}.$$

Since the action of $O(3)$ on M is doubly transitive, this suffices to describe the equivariant diagonal: it is given as the set

$$\left\{ A.(e_2, e_3, Be_2, Be_3) \mid A \in O(3),\ B = \begin{pmatrix} \pm 1 & 0 \\ 0 & C \end{pmatrix}, C \in O(2) \right\},$$

where $O(3)$ acts diagonally, and is a compact $O(3)$-manifold of dimension 4.

Denote with $\Delta^o_G(M)$ the equivariant diagonal where we do not take the closure, that is, $\Delta^o_G(M) = \{(x, gx) \mid x \in M,\ g \in N(G_x)\}$. The next result shows that the decomposition of $\Delta^o_G(M, N)$ into normal orbit strata is Whitney regular. Once we know that the equivariant diagonal is locally semialgebraic, this relates the canonical stratification and the stratification by orbit type. Recall that two points x, y in a G-manifold M

have the same normal orbit type, if $(G_x) = (G_y)$ and in addition, there is a $z \in Gy$ such that $G_z = G_x$ and the normal representations $T_xGx^\perp$, $T_zGz^\perp$ are isomorphic as G_x-representations for some invariant Riemannian metric on M. If C is a connected component of $M_{(H)}$, then the normal orbit type in $G(C)$ is constant, see Lemma 3.8.1 of [Fie07].

We need a simple lemma concerning the interaction of the equivariant diagonal and trivial subspaces.

Lemma 2.4.1.4 *Let V be an H-representation, $V = W \oplus V^H$, $W^H = \{0\}$. The equivariant diagonal of $G \times_H V$ is given as*

$$\Delta_G(G \times_H V) = \{([g, v], [hg, v]) \mid g \in G,\ v \in V,\ h \in gN(H_v)g^{-1}\}$$

and we have

$$\Delta_G(G \times_H V) \cong \Delta_G(G \times_H W) \times V^H.$$

Proof. An easy computation yields $G_{[g,v]} = gH_vg^{-1}$ and $N(gHg^{-1}) = gN(H)g^{-1}$, showing that the equivariant diagonal has the given form. So define a map

$$\Delta_G(G \times W) \times V^H \to \Delta_G(G \times_H V),\ ([g, w], [hg, w], v) \mapsto ([g, w + v], [hg, w + v]).$$

Since $H_{w+v} = H_w$ for $v \in V^H$, this is well-defined and obviously surjective. For injectivity, assume

$$[g, w + v] = [g', w' + v'],\ [hg, w + v] = [h'g', w' + v'].$$

There is a $j \in H$ such that $gj^{-1} = g'$ and $j(w + v) = w' + v'$. Since $v \in V^H$, $j(w + v) = jw + v$, so $v' - v = jw - w' \in W \cap V^H = \{0\}$. We obtain $v = v'$ and $jw = w'$, i.e.

$$[g, w] = [gj^{-1}, jw] = [g', w'].$$

Furthermore, there is an $\ell \in H$ such that $hg\ell^{-1} = h'g'$, $\ell(w + v) = w' + v'$. Substituting what we found so far, we have $h' = hg\ell^{-1}jg^{-1}$, $\ell w = jw$. This gives

$$[h'g', w'] = [hg\ell^{-1}jg^{-1}gj^{-1}, \ell w] = [hg\ell^{-1}, \ell w] = [hg, w].$$

Our map is injective and the claim follows. $\square$

An auxiliary result will allow us to skip trivial components from consideration, which will come in handy in reducing the complexity of the upcoming statement.

Lemma 2.4.1.5 *Let $X \subseteq M$ be a Whitney stratified subset of a smooth manifold M. Then $X \times \mathbb{R}^k$ with strata $X_\alpha \times \mathbb{R}^k$ is a Whitney stratified subset of $M \times \mathbb{R}^k$.*

Proof. Lemma 3.9.1 of [Fie07]. $\square$

Proposition 2.4.1.6 *Let M be a G-manifold, $\Delta_G^o(M) = \{(x, gx) \mid x \in M,\ g \in W(G_x)\}$. The stratification of $\Delta_G^o(M)$ by normal orbit type is Whitney regular.*

Proof. Take an element $(x, gx) \in \Delta_G^o(M)$. Since the Whitney conditions are local in nature, do not depend on the G-action, and x and gx have the same normal orbit type, we can assume $M = G \times_H V$, $H = G_x$, V an H-representation, and $x = [e, 0]$, $g = e$. So we have to check the Whitney conditions for the stratification of the set

$$\Delta_G^o(G \times_H V) = \{[g, v], [kg, v] \mid g \in G, v \in V, k \in gN(H_v)g^{-1}\}$$

with strata $\Delta_G^o(G \times_H V_{(K)})$. By Lemmas 2.4.1.4 and 2.4.1.5, we can assume $V^H = \{0\}$. Finally, we can assume that V has only two orbit types, namely (H), the orbit type of 0, and (K), the orbit type of all other elements.

We want to reduce the problem to checking Whitney conditions in equivariant diagonals of representations. For this purpose, let $\mathbb{R}^k \cong A \subseteq G/_H$ be a chart neighbourhood of $[e]$ such that there is a local section $\sigma : A \to G$. Similarly, let $\mathbb{R}^j \cong B \subseteq N(K)/_{N(K) \cap H}$ be a chart neighbourhood of $[e]$ and $\tau : B \to N(K)$ a local section. In a neighbourhood U of $0 \in V$, we find a smooth map $U - \{0\} \to H,\ v \mapsto h_v$, such that $h_v H_v h_v^{-1} = K$. For the sake of simplicity, we assume that this map is defined on all of $V_{(K)} = V - \{0\}$. So we can define a map

$$\Phi : A \times B \times \Delta_H(V) \to G \times_H V \times G \times_H V,\ (s, t, v, hv) \mapsto ([\sigma(s), v], [\sigma(s)h_v^{-1}\tau(t)h_v h, v])$$

with the obvious adjustments for $v = 0$. Φ is well-defined since $h \in H$. We claim that this map is injective and its image is a neighbourhood in $\Delta_G(G \times_H V)$ of $([e, 0], [e, 0])$. To see this, we define an inverse map for Φ. Take an element $([g, v], [gk, v])$ of the equivariant diagonal such that $[g] \in A$. Then $g\ell = \sigma([g])$ for some $\ell \in H$. Let $w = \ell^{-1}v$. The additional requirement that $h_w \ell^{-1} k \ell h_w^{-1}$ maps to B under the canonical projection constitutes a neighbourhood W of $([e, 0], [e, 0])$ in $\Delta_G^o(M)$.

The map

$$B \times N(K) \cap H \to N(K), \quad (b, k) \mapsto \tau(b)k$$

is an isomorphism onto its image, so we find a unique $b \in B$ and $j \in N(K) \cap H$ such that

$$h_w \ell^{-1} k \ell h_w^{-1} = \tau(b)j.$$

Let $h = h_w^{-1} j h_w \in N(H_w)$ and define a map

$$\Psi : W \to A \times B \times \Delta_H(V), \quad ([g, v], [gk, v]) \mapsto ([g], b, w, hw).$$

This is well-defined. Indeed, w is uniquely determined through g and σ and so it remains to check that $([g, v], [gkh, v])$ has the same image as $([g, v], [gk, v])$, where $h \in H_v$. But we have

$$h_w \ell^{-1} k h \ell h_w^{-1} = \tau(b) j h_w \ell^{-1} h \ell h_w^{-1}.$$

Denoting $j h_w \ell^{-1} h \ell h_w^{-1}$ by $\tilde{j}$ and letting $\tilde{h} = h_w^{-1} \tilde{j} h_w$, Ψ maps $([g, v], [gk, v])$ to the element

$$\begin{array}{rcl}([g],b,w,\tilde{h}w) & = & ([g],b,w,h_w^{-1}\tilde{j}h_w w)\\ & = & ([g],b,w,h_w^{-1}jh_w\ell^{-1}h\ell h_w^{-1}h_w w)\\ & = & ([g],b,w,h_w^{-1}jh_w\ell^{-1}h\ell\ell^{-1}v)\\ & = & ([g],b,w,h_w^{-1}jh_w\ell^{-1}v)\\ & = & ([g],b,w,hw),\end{array}$$

where we used $h \in H_v$. We conclude that Ψ is well-defined. With the notation from above, we calculate

$$\begin{array}{rcl}\Phi\circ\Psi([g,v],[gk,v]) & = & \Phi([g],b,w,hw)\\ & = & ([\sigma([g]),w],[\sigma([g])h_w^{-1}\tau(b)h_w h,w])\\ & = & ([g\ell,\ell^{-1}v],[g\ell h_w^{-1}\tau(b)h_w h_w^{-1}jh_w,w])\\ & = & ([g,v],[g\ell h_w^{-1}\tau(b)jh_w,w])\\ & = & ([g,v],[g\ell\ell^{-1}k\ell,\ell^{-1}v])\\ & = & ([g,v],[gk,v])\end{array}$$

and

$$\Psi\circ\Phi(s,t,v,hv) = \Psi([\sigma(s),v],[\sigma(s)h_v^{-1}\tau(t)h_v h,v]).$$

Using again the notation of above, we have $\ell = e$, $w = v$, $k = h_v^{-1}\tau(t)h_v h$. Thus,

$$h_v\ell^{-1}k\ell h_v^{-1} = \tau(t)h_v h h_v^{-1}.$$

By definition of Ψ, we obtain

$$\Psi([\sigma(s),v],[\sigma(s)h_v^{-1}\tau(t)h_v h,v]) = ([g],t,v,hv).$$

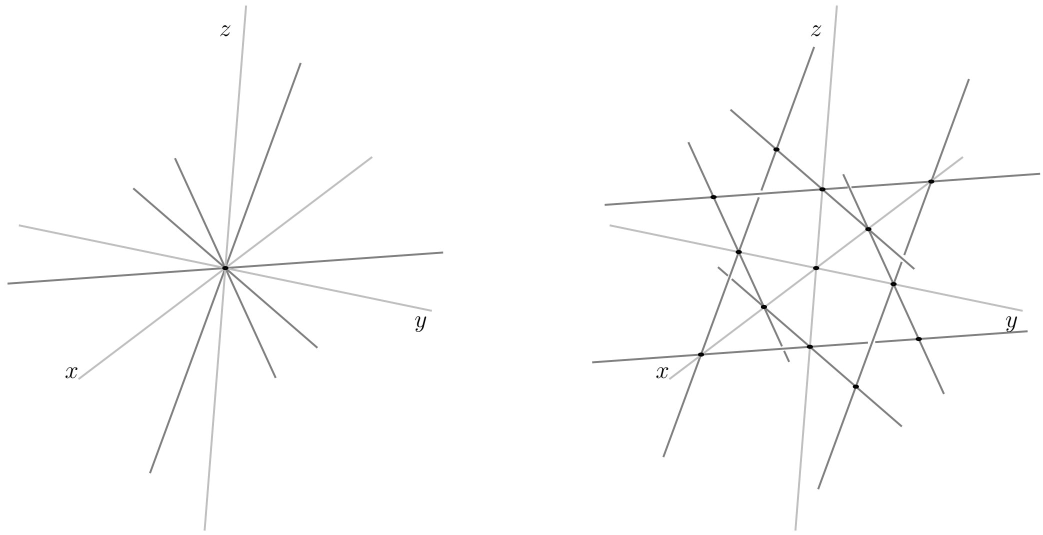
Consequently, Φ and Ψ are inverse to each other and they are smooth when restricted to a stratum. Hence, they constitute stratumwise diffeomorphisms and we can locally identify the equivariant diagonal with the stratified set $\mathbb{R}^k \times \mathbb{R}^j \times \Delta_H(V)$, the strata given by the strata of $\Delta_H(V)$ multiplied with the trivial representation $\mathbb{R}^k \times \mathbb{R}^j$.

This identification together with the conclusion of Lemma 2.4.1.5 shows that it suffices to prove the initial assertion for H-representations V with $V^H = \{0\}$ and a single non-trivial stratum. But this can be done almost in the same way as Whitney regularity of the stratification of a G-manifold by normal orbit type is proven (compare [Fie07]). Take a sequence $y_n \in \Delta_H(V_{(K)})$, (K) being the non-trivial orbit type, such that $y_n \to (0,0)$ and $T_{y_n}\Delta_H(V_{(K)})$ converges to a subspace $E \subseteq V\times V$. Define $w_n = y_n/\|y_n\| \in S(V\times V)$, the unit sphere of $V\times V$. By compactness of $S(V\times V)$, we can assume that w_n converges to a point $w \in S(V\times V)$. For Whitney (b) regularity, we have to show $w \in E$. The curve $\gamma_n : (-1,1) \to \Delta_H(V_{(K)})$, $\gamma_n(t) = (t+1)w_n$ defines an element of $T_{w_n}\Delta_G(V_{(K)})$, namely the element w_n. So we can split the tangential space as

$$T_{w_n}\Delta_H(V_{(K)}) = T_{w_n}\left(S(V\times V)\cap\Delta_H(V_{(K)})\right)\oplus\langle w_n\rangle.$$

Clearly, $T_{w_n}\Delta_H(V_{(K)}) = T_{y_n}\Delta_H(V_{(K)})$. The left hand side of the above equation converges to E, the right hand side converges to $E \cap T_w S(V \times V) \oplus \langle w \rangle$. So we see that $w \in E$, which finishes the proof. $\square$

Example 2.4.1.7 1. Let $\mathbb{Z}_2$ act on $\mathbb{R}$ canonically. The equivariant diagonal is given by $\{(x, \pm x) \mid x \in \mathbb{R}\}$, the union of the diagonal and the antidiagonal. It is stratified by the orbit strata $\{(0,0)\}$ and $\Delta_{\mathbb{Z}_2}(\mathbb{R}) - \{(0,0)\}$.



(a) $t = 0$ (b) $t \neq 0$

Figure 9: Projections of the equivariant diagonal $\Delta_{D_4}(\mathbb{R}^2, \mathbb{R}^2)$

2. Let D_4 be the dihedral group, generated by τ, σ, $\tau^2 = e$, $\sigma^4 = e$, $\tau\sigma\tau = \sigma^3$. Let D_4 act on $\mathbb{R}^2$ by letting τ be the reflection at the y-axis, σ the (clockwise) rotation by $\frac{\pi}{2}$. Points in $\mathbb{R}^2$ are divided by isotropy into six classes: The origin, the y-axis, the diagonal, the x-axis, the anti-diagonal and all other points. The isotropies are D_4, $\langle\tau\rangle$, $\langle\sigma\tau\rangle$, $\langle\sigma^2\tau\rangle$, $\langle\sigma^3\tau\rangle$ and $\{e\}$, respectively. The normalizer of $\langle\tau\rangle$ in D_4 is $N_\tau = \langle\tau, \sigma^2\rangle$. Similarly, we have
$$N_{\sigma\tau} = \left\langle \sigma\tau, \sigma^2 \right\rangle, \quad N_{\sigma^2\tau} = \left\langle \tau, \sigma^2 \right\rangle, \quad N_{\sigma^3\tau} = \left\langle \sigma\tau, \sigma^2 \right\rangle.$$
There are four orbit types, $(e), (\langle\tau\rangle), (\langle\sigma\tau\rangle)$ and (D_4). The corresponding strata of the equivariant diagonal are given by
$$\begin{aligned} \Sigma_{D_4} &= \{(0,0,0,0)\} \\ \Sigma_{\langle\tau\rangle} &= \{(0,x,0,\pm x) \mid x \neq 0\} \cup \{(x,0,\pm x,0) \mid x \neq 0\} \\ \Sigma_{\langle\sigma\tau\rangle} &= \{(x,x,\pm x,\pm x) \mid x \neq 0\} \cup \{(x,-x,\pm x,\mp x) \mid x \neq 0\}. \end{aligned}$$
and
$$\begin{aligned} \Sigma_e = &\{(x,y,\varepsilon_1 x,\varepsilon_2 y) \mid \varepsilon_1,\varepsilon_2 \in \{\pm 1\},\ xy \neq 0,\ x \neq \pm y\} \\ &\cup \{(x,y,\varepsilon_1 y,\varepsilon_2 x) \mid \varepsilon_1,\varepsilon_2 \in \{\pm 1\},\ xy \neq 0,\ x \neq \pm y\} \end{aligned}$$

We note that, for example, the point $(1,1,-1,1)$ is not contained in the equivariant diagonal, although $\tau.(1,1)=(-1,1)$. τ is not in the normalizer of the isotropy group of $(1,1)$. Figure 9 shows the projections of the equivariant D_4-diagonal to its last three components, the first component being equal to $t \in \mathbb{R}$.

We finally need a lemma that allows us to identify preimages of the equivariant diagonal under polynomial maps as being canonically stratified spaces. We essentially need the identification of the equivariant diagonal locally, which was part of the proof of Proposition 2.4.1.6.

Lemma 2.4.1.8 *The equivariant diagonal is a locally semialgebraic set.*

Proof. By the structural results obtained in the proof of Proposition 2.4.1.6, we just have to show that the set $\Delta_H(V)$ is algebraic, where V is an H-representation. Let $p_1,\dots,p_\ell$ be a minimal set of polynomial generators for the invariants $V \to \mathbb{R}$ and let $F_1,\dots,F_k$ be a set of polynomial generators for the $\mathcal{C}^\infty_G(V)$-algebra $\mathcal{C}^\infty_G(V,V)$. Let $P=(p_1,\dots,p_\ell)$ and ϑ be the universal polynomial. Define

$$\Sigma_* = \{(v,w,t) \in V \times V \times \mathbb{R}^k \mid P(v) = P \circ \vartheta(v,t),\ \vartheta(v,t)=w\}.$$

As the intersection of two algebraic sets, this set is algebraic. Furthermore, since $P(v)=P(\vartheta(v,t))$, $v = g.\vartheta(v,t)$ for some $g \in G$ and hence, by the usual reasoning, $g \in N(H_v)$. Projecting to the first two variables gives

$$\pi_{12}(\Sigma_*) = \{(v,gv) \in V \times V \mid g \in N(G_v)\},$$

since clearly for every $v \in V$ and $g \in N(H_v)$ there is a $t \in \mathbb{R}^k$ such that $\vartheta(v,t)=gv$. The image of an algebraic set under a polynomial map is semialgebraic by the Tarski-Seidenberg theorem. Since semialgebraicity is preserved under taking closures, its closure is semialgebraic as well, and this is the equivariant diagonal. $\square$

2.4.2 G-Hyperbolicity

In this section we will define G-hyperbolic critical elements as is done in [Fie07] and other works. G-hyperbolic critical elements have the advantage of a simple local dynamical behaviour. In addition, G-hyperbolicity implies isolatedness. However, the role G-hyperbolicity plays is different from the one of ordinary hyperbolicity. Since the question whether G-hyperbolicity implies equivariant non-degeneracy is open, the hyperbolicity results to be obtained later are not tightenings of the non-degeneracy results. Nonetheless, G-hyperbolicity plays an important role in the theory, also concerning implications on non-degenerate elements, as we shall see soon.

For the moment, we just develop the theory of G-hyperbolic elements, which can also be called the theory of normal hyperbolicity. Roughly, one decomposes the dynamics near a critical element into a direction normal to the element and tangential to the group orbit. G-hyperbolicity is the hyperbolicity of the normal part of this decomposition. Field proves in [Fie91] that a fixed G-orbit Gx is G-hyperbolic if and only if the normal component of the differential of $g^{-1} \circ f$ in x has no eigenvalues of absolute value 1, so this is the interpretation one should keep in mind.

The following definition comes from [HPS77] and defines normally hyperbolic elements in a much more general way than just for group actions. We specialize afterwards.

Definition 2.4.2.1 *Let M be a compact manifold, $f : M \to M$ a map and $N \subseteq M$ an f-invariant compact submanifold. N is said to be* hyperbolic*, if there is a Tf-invariant splitting of the tangential bundle*

$$T_N M = TN \oplus E^u \oplus E^s,$$

where, after fixing a Riemannian metric on M,

$$\inf_{x \in N} m(E^u_x f) > \sup_{x \in N} \left\| T_x f\big|_{T_x N} \right\|, \quad \sup_{x \in N} \|E^s_x f\| < \inf_{x \in N} m(T_x f\big|_{T_x N}),$$

with $m(A) = \inf\{\|Ax\| \mid \|x\| = 1\}$.

If, in particular, M is a G-manifold, f a G-map and $N = Gx$ is a G-orbit, Gx is called G-hyperbolic*, if it is hyperbolic in the above sense, where the splitting is G-invariant as well as the Riemannian metric.*

A relative periodic orbit is G-hyperbolic*, if the corresponding fixed orbit of an associated equivariant Poincaré map is G-hyperbolic.*

We note that by our standing assumption that G acts via isometries, the expressions in the definition of normal hyperbolicity concerning the direction of the group orbit are equal to 1. So locally around a point on a G-hyperbolic fixed orbit, we have three directions. One the trivial direction along the group orbit, complemented by an expanding and a contracting direction, varying compatible and uniformly with the point on the orbit.

Next we will prove the fundamental fact that G-hyperbolic G-orbits are isolated fixed orbits. We will need the normal decomposition lemma of section 2.1.3 in the proof.

Proposition 2.4.2.2 *Let $f_0 : M \to M$ be equivariant and $Gx \subseteq M$ a G-hyperbolic fixed orbit of f_0 of type (H). Then there is an invariant neighbourhood U of Gx such that U contains no other fixed orbits of f_0. Furthermore, there is a neighbourhood $\mathcal{U}$ of f_0 such that every element $f \in \mathcal{U}$ has a unique G-hyperbolic fixed orbit of type (H) in U which depends continuously on f.*

Proof. Let $(H) = (G_x)$ be the orbit type of Gx and assume G acts by isometries. Since Gx is fixed under f_0, we find tubular neighbourhoods $U \subseteq U'$ of Gx such that $f(\overline{U}) \subseteq U'$ for all f in a neighbourhood $\mathcal{U}_1$ of f_0. Let $U = G \times_H S_x$, $U' = G \times_H S'_x$, where we can assume $S_x \subseteq S'_x$, normal slices at x. Since Gx is a fixed orbit, by Corollary 2.1.3.3 we find an $\alpha > 0$ such that $f_0\big|^{\alpha}_{Gx}$ is equivariantly homotopic to the identity map and is given by $x \mapsto cx$ for a $c \in C(G_x)$. We find a neighbourhood $\mathcal{U}_2 \subseteq \mathcal{U}_1$ of f_0 such that f^{α} is homotopic to the map $x \mapsto s_f = G_{f(x)} \cap S_x$ and $f^{\alpha}(x) = c_f s_f$, $c_f \in C(G_x)$ for all $f \in \mathcal{U}_2$. For simplicity, we assume $\alpha = 1$. This is possible since a fixed G-orbit of f is G-hyperbolic for f if and only if it is G-hyperbolic for f^k and some $k \in \mathbb{N}$.

We have $f_0(x) = cx$, so $c^{-1}f_0(x) = x \in S'_x$. By shrinking S_x, for all sufficiently small neighbourhoods A of $[e] \in G/_H$ we can achieve $c_f^{-1}f(S_x) \subseteq \pi^{-1}(A).S'_x$ for all f in a neighbourhood $\mathcal{U}_3 \subseteq \mathcal{U}_2$ of f_0. In particular, this holds for small A, where A is the domain of an $N(H)$-equivariant local section $\sigma : A \to G$. So we are in the position to apply the normal decomposition lemma 2.1.3.4 to f_0 and $\mathcal{U}_3$, and we obtain maps $g : U \to G$, $h : U \to U'$ such that $f(y) = g(y).h(y)$ for all $y \in U$ and $h(S_y) \subseteq S'_y$ for all $y \in Gx$, where $S_{gx} = g.S_x$.

In particular we have $f_0 = g_0.h_0$ and $h_0\big|_{S_x} : S_x \to S'_x$ is an H-map which, since Gx is G-hyperbolic for f_0, has x as an H-hyperbolic fixed orbit of type (H). For simplicity, we can assume that $h_0 : \mathbb{B}_1(0) \to V$ is a self-map of an H-representation V and $h_0(0) = 0$ is an H-hyperbolic fixed orbit. But since 0 is a group fixed point, H-hyperbolicity of 0 is nothing else than ordinary hyperbolicity of h_0, so the differential of h at 0 has no eigenvalue of absolute value 1. This implies that the differential of $h_0 - k$ at 0 is invertible for all $k \in H$, since H acts via isometries. Thus, in a neighbourhood of 0, the equation $h_0(v) = kv$ is uniquely solved by $v = 0$. By eventually shrinking this neighbourhood, we can assume that this property holds in a neighbourhood of k as well, and by compactness of H we find a neighbourhood of 0 such that $h(v) = kv$ is uniquely solved by $v = 0$ for every $k \in H$. But solutions of $h_0(v) = kv$ correspond to fixed orbits of h_0 in V and hence to fixed orbits of f_0 in U. So Gx is an isolated fixed orbit for f_0.

If h is the corresponding map in the normal decomposition of an $f \in \mathcal{U}_3$, then h is close to h_0, so in a neighbourhood of $0 \in V$, there is a unique solution v to the equation $h(v) = kv$ for every $k \in H$. $h - k$ induces a map on V^H and the same reasoning yields a unique $v \in V^H$ such that $h(v) = kv = v$. v corresponds to a unique H-fixed point of h near 0 and this corresponds to a unique fixed orbit of f of type (H). □

This result shows that G-hyperbolic fixed orbits cannot bifurcate in the usual sense, but also that there cannot be a symmetry breaking bifurcation, since the orbit type remains constant.

We will further need the result that, if f has no fixed orbits in a compact subset of M, then no map in a neighbourhood of f has a fixed orbit in that compact set.

Proposition 2.4.2.3 *Let $K \subseteq M$ be an invariant compact subset and assume that f has no fixed orbit in K. Then there is a neighbourhood $\mathcal{U}$ of f such that no element of $\mathcal{U}$ has a fixed orbit in K.*

Proof. After fixing an invariant Riemannian metric on M and taking d to be the Riemannian distance, we have $d(f(x), gx) > 0$ for all $x \in K$, $g \in G$. By compactness of K and G, we find $\varepsilon > 0$ such that $d(f(x), gx) > \varepsilon$ for all $x \in K$, $g \in G$. Clearly, if f' is close to f, this inequality remains true, so f' has no fixed orbits in K. □

There are many other things to say about G-hyperbolicity, for example the whole theory of equivariant eigenvalues mentioned above. In addition, eigenvalue results are available containing the dimension of the orbits. Since we will not make use of these, we refer to [Fie07] and [Fie80] for an exhaustive treatment of G-hyperbolicity. We only need two additional facts, the first being taken from [Fie80], the second being a lemma handling the case where the action is free.

Lemma 2.4.2.4 *Let Gx be a fixed orbit of $f : M \to M$ and U any invariant neighbourhood of Gx. Then for every neighbourhood $\mathcal{U}$ of f there is $h \in \mathcal{U}$ such that $h = f$ outside of U and Gx is a G-hyperbolic fixed orbit of h.*

Proof. Lemma 6.A of [Fie80]. □

Lemma 2.4.2.5 *Let G act freely on M and let $f : M \to M$ be a G-map. Then Gx is a G-hyperbolic fixed orbit of f if and only if $[x]$ is a hyperbolic fixed point for $[f] : M/_G \to M/_G$.*

Proof. Since free G-spaces are G-principal bundles, we can work locally and assume $M = G \times M/_G$, G acting by left translations. Now if $G \times \{x\}$ is a G-hyperbolic fixed orbit of f, we have a Tf-invariant splitting

$$T_{G\times\{x\}}M \cong TG \oplus E^u \oplus E^s \cong TG \oplus T_xM/_G$$

satisfying the hyperbolicity inequalities. The above isomorphism induces a $T[f]$-invariant splitting of $T_xM/_G$ into E^u, E^s. The differential of f on $E^u \oplus E^s$ reduces to the differential of $[f]$. Since G acts via isometries, the inequalities for G-hyperbolicity imply in particular that the eigenvalues of $T_x[f]$ in E^u have absolute value larger 1, whereas the eigenvalues in E^s have absolute value less than 1, so hyperbolicity of x in $M/_G$ follows. The other direction follows by the same reasoning. □

2.4.3 Equivariant Non-Degeneracy

We are now in the position to define equivariant non-degeneracy and prove its most important properties. We will deal with maps and vector fields simultaneously. Once again, we start with the definition of critical elements.

Definition 2.4.3.1 *Let M be a smooth G-manifold.*

- *Let $f : M \to M$ be a G-map. A fixed orbit Gx of f is called* equivariantly non-degenerate, *if the map*

  $$F : M \to M \times M,\ F(y) = (y, f(y))$$

  is G-transverse to the equivariant diagonal $\Delta_G(M)$ at x (and thus at gx for all $g \in G$).

- *Let $H : M \times I \to M$ be a G-homotopy. An orbit $Gx \times \{\lambda\} \subseteq M \times I$ is called* equivariantly non-degenerate *for H, if the map*

  $$M \times I \to M \times M,\ (y, \mu) \mapsto (y, H(y, \mu))$$

  is G-transverse to the equivariant diagonal at (x, λ) (if $\lambda \in \{0, 1\}$, we require the existence of an extension of H to $M \times \mathbb{R}$ satisfying the above property).

- *Let $\xi : M \to TM$ be an equivariant vector field. A relative periodic orbit through the periodic point (x, T) of ξ is called* equivariantly non-degenerate*, if the map*

  $$F : M \times \mathbb{R} \to M \times M,\ (y, t) \mapsto (y, \varphi(y, t))$$

  is G-transverse to equivariant diagonal at (x, T), where φ is the flow of ξ.

- *Let $H : M \times I \to TM$ be a homotopy of equivariant vector fields. A relative periodic orbit $\gamma \times \{T\}$ of H_λ is called* equivariantly non-degenerate*, if for the flow φ of H, the map*

  $$M \times \mathbb{R} \times I \to M \times M,\ (y, t, \mu) \mapsto (y, \varphi_\mu(y, t))$$

  is G-transverse to the equivariant diagonal at some point (x, T, λ) with $(x, T) \in \gamma \times \{T\}$ (as above, if $\lambda \in \{0, 1\}$, we require existence of an extension).

Naturally, we define equivariantly non-degenerate maps, vector fields or homotopies thereof by the requirement that the corresponding graph maps indicated above are G-transverse to the equivariant diagonal on the whole base space, where we use extensions of the homotopies to $M \times \mathbb{R}$.

As already mentioned, it is unclear whether G-hyperbolicity of a critical element implies its non-degeneracy. However, we can still draw conclusions on the structure of the set of non-degenerate critical elements by knowledge of the set of G-hyperbolic critical elements. This depends on the following lemma.

Lemma 2.4.3.2 *Let $f : M \to N$ be a G-map of G-manifolds. Then there is a neighbourhood $\mathcal{U}$ of f such that all elements of $\mathcal{U}$ are equivariantly homotopic to f via a homotopy not leaving $\mathcal{U}$. A similar result holds for equivariant vector fields in $\mathfrak{X}_G(M, \Omega, a, b)$.*

Proof. The proof is the same as in Lemma 1.1.1.2, we just use equivariant embeddings and invariant tubular neighbourhoods. $\square$

The remainder of this section is devoted to an example of an equivariant non-degeneracy condition. For the sake of simplicity, we work on representations rather than manifolds.

Example 2.4.3.3 Denote with V the canonical representation of $\mathbb{Z}_2$ on $\mathbb{R}$ and let $f : V \to V$ be a $\mathbb{Z}_2$-map. We have $f(x) = x \cdot h(x)$ for some invariant $h : V \to \mathbb{R}$. The equivariant diagonal in $V \times V$ is given as $\Delta_{\mathbb{Z}_2}(V) = \{(t, \pm t) \mid t \in V\}$. Generators for $\mathcal{C}_{\mathbb{Z}_2}(V, V \times V)$ are given by

$$F_1(x) = (x, 0),\ F_2(x) = (0, x),$$

the universal polynomial is given by

$$\vartheta(x, s, t) = (s \cdot x, t \cdot x).$$

Hence, the preimage of the equivariant diagonal under ϑ is

$$\Sigma = \vartheta^{-1}(X) = \{(x,s,t) \mid s \cdot x = \pm t \cdot x\},$$

which is canonically stratified by the strata

$$\begin{array}{ll} \Sigma_0 = \{(0,0,0)\} & \Sigma_{0++} = \{(0,s,t) \mid s \neq \pm t\} \\ \Sigma_{0-} = \{(0,t,-t) \mid t \neq 0\} & \Sigma_{0+} = \{(0,t,t) \mid t \neq 0\} \\ \Sigma_{-0} = \{(x,0,0) \mid x < 0\} & \Sigma_{+0} = \{(x,0,0) \mid x > 0\} \\ \Sigma_{++} = \{(x,t,t) \mid x > 0,\, t \neq 0\} & \Sigma_{--} = \{(x,t,-t) \mid x < 0,\, t \neq 0\} \\ \Sigma_{+-} = \{(x,t,-t) \mid x > 0,\, t \neq 0\} & \Sigma_{-+} = \{(x,t,t) \mid x < 0,\, t \neq 0\} \end{array}.$$

Let $F = \mathbb{1} \times f$. We have $\Gamma_F(x) = (x, 1, h(x))$ and $\Gamma_F(0) = (0, 1, h(0))$. Assume $h(0) \neq \pm 1$, then $\Gamma_F(0) \in \Sigma_{0++}$. So we have

$$T_0\Gamma_F(V) + T_{\Gamma_F(0)}\Sigma_{0++} = \{(a, 0, h'(0) \cdot a) + (0, b, c) \mid a, b, c \in \mathbb{R}\} = V \times \mathbb{R}^2,$$

so Γ_F is transverse to Σ at 0. Now assume $h(0) = \pm 1$, then either $\Gamma_F(0) \in \Sigma_{0+}$ or $\Gamma_F(0) \in \Sigma_{0-}$. We calculate

$$T_0\Gamma_F(V) + T_{\Gamma_F(0)}\Sigma_{0\pm} = \{(a, 0, h'(0) \cdot a) + (0, b, \pm b) \mid a, b \in \mathbb{R}\} \neq V \times \mathbb{R}^2.$$

So in this case, Γ_F is not transverse to Σ at 0. Clearly, for fixed orbits of f different from 0, equivariant non-degeneracy is just ordinary non-degeneracy, so we must have $f'(x) \neq 1$ if $f(x) = x$, or $f'(x) \neq -1$ if $f(x) = -x$. The condition $h(0) \neq \pm 1$ is equivalent to $f'(0) \neq \pm 1$, so the global equivariant non-degeneracy condition for maps $V \to V$ is $f'(x) \neq 1$ if $f(x) = x$ and $f'(x) \neq -1$ if $f(x) = -x$.

2.4.4 The Equivariant Isotopy Theorems

Before turning to equivariant genericity theorems, we establish the isotopy theorems for equivariantly transverse maps, which allow us to conclude isolatedness of equivariantly non-degenerate critical elements and also to establish the generic bifurcation picture in this setting, this last bit in conjuction with the genericity theorems of the next section.

Proposition 2.4.4.1 *If $f : M \to N$ is equivariantly transverse to an invariant algebraic subset $X \subseteq N$, then $f^{-1}(X)$ is Whitney stratified.*

Proof. The definition of equivariant transversality to X factorizes the map f into the graph map and the universal polynomial (up to localizing diffeomorphisms). The preimage of X under f is thus, locally, given by the preimage under the graph map of the algebraic variety given by the preimage of the set X under the universal polynomial. Since the graph map, by definition, is transverse to that Whitney stratified set, its preimage is Whitney stratified as well (compare [Mat80], chapter 8.) Thus, $f^{-1}(X)$ is locally Whitney stratified and hence, Whitney stratified. □

We note another version of the parametrized transversality theorem 1.1.2.6.

Proposition 2.4.4.2 *Let M be a compact G-manifold, Λ a trivial G-manifold, $H: M \times \Lambda \to N$ and assume that H is equivariantly transverse to the locally semialgebraic subset X of N. Then the set of parameters $\lambda \in \Lambda$ such that H_λ is equivariantly transverse to X is open and residual. More precisely, it is given by the intersection of the sets of regular values of the restrictions of the projection map $p_2 : H^{-1}(X) \to \Lambda$ to the various strata.*

Proof. This is essentially the same proof as for Proposition 2.2.2.10, compare [Bie77a], [Bie77b], chapter 7. $\square$

The preceeding result allows an easy deduction of the isotopy theorem using the equivariant isotopy lemma.

Proposition 2.4.4.3 *Let M be a compact G-manifold and $H : M \times \Lambda \to N$ a G-map, Λ a trivial connected G-manifold. Assume that H_λ is equivariantly transverse to the closed semialgebraic subset $X \subseteq N$ for every $\lambda \in \Lambda$. Then every two preimages $H_\mu^{-1}(X)$, $H_\nu^{-1}(X)$ with $\mu, \nu \in \Lambda$ are equivariantly isotopic. In particular, if $\Lambda = [0, 1]$, then any two preimages of X under the fibre maps are pairwise isotopic.*

Proof. We just apply the equivariant isotopy lemma 2.3.2.16 to the projection map $p_2 : H^{-1}(X) \to \Lambda$. The lemma applies by the preceeding proposition, and we conclude that this map is locally trivial. By connectedness of Λ, it is globally trivial, which yields our claim. $\square$

2.5 Equivariant Genericity Theorems

The upcoming section is devoted to the proofs of the equivariant genericity theorems. The techniques vary substantially and unify many of the work we have done so far. For genericity of equivariantly non-degenerate G-maps and G-homotopies, we use stratified G-transversality theory. For genericity of G-hyperbolic maps and vector fields, we use induction techniques which use the genericity theorems of chapter one. Finally, for G-homotopies of vector fields, we adopt the geometric techniques we used in chapter one and make them work with symmetries. The inductive proofs for maps seem not to work in the parametrized case, so we really have to rely on the geometric method.

2.5.1 Genericity in the Space of G-Maps

The first genericity theorems for equivariant maps are not much harder to prove than the ones for arbitrary maps. They depend on the equivariant Thom-Mather Theorem for locally semialgebraic sets just as the latters depended on the Thom Transversality Theorem.

Theorem 2.5.1.1 *Let M be a compact G-manifold. Then the set of equivariantly non-degenerate maps $f : M \to M$ is open and dense in the set of all G-maps.*

Proof. By the equivariant Thom-Mather Theorem 2.2.3.6, the set of maps $M \to M \times M$ G-transverse to the equivariant diagonal is open and dense. This immediately yields the openness part. For density, let $f : M \to M$ be any G-map and $\mathcal{U}$ be any neighbourhood of f. Take any map $F : M \to M \times M$ G-transverse to the equivariant diagonal. By choosing F sufficiently close to $\mathbb{1} \times f$, we can achieve that $F_1 = \pi_1 \circ F$ is arbitrarily close to the identity, in particular we can take it to be an equivariant diffeomorphism. Furthermore, the map $F \circ F_1^{-1}$ is close to F and satisfies $\pi_1 \circ F \circ F_1^{-1} = \mathbb{1}$. Altogether, we see that the map $\pi_2 \circ F \circ F_1^{-1} : M \to M$ is arbitrarily close to f, so by the right choice of F, it will be an element of $\mathcal{U}$. The associated graph map is G-transverse to the equivariant diagonal, which proves the theorem. □

We already mentioned in the introduction to this section that we will use the technique of induction on orbit types to prove density of G-hyperbolic maps. It should be possible to derive this result using the last one by making equivariantly non-degenerate fixed orbits G-hyperbolic. But as we do not have such a result at hand, we instead make use of the results of chapter one.

Genericity of G-hyperbolic diffeomorphisms was initially proven in [Fie80]. Field in fact only proves the theorem for vector fields and asserts that the proof for diffeomorphisms is similar (which indeed is the case). We will include both proofs. The following theorem is a bit more general in the sense that it deals with arbitrary G-maps, not necessarily diffeomorphisms.

Theorem 2.5.1.2 *The set of G-hyperbolic maps is open and dense in the set of smooth equivariant maps.*

Proof. Openness: Let f be a G-hyperbolic map. Then f has only finitely many G-hyperbolic fixed orbits in M and there are invariant isolating neighbourhoods $U_1, \ldots, U_m$ of these orbits. By Proposition 2.4.2.2, there are neighbourhoods $\mathcal{U}_1, \ldots, \mathcal{U}_m$ of f such that each element of $\mathcal{U}_j$ has a unique G-hyperbolic fixed orbit in U_j, $j = 1, \ldots, m$. The set $U = U_1 \cup \cdots \cup U_m$ is open, so $K = M \setminus U$ is compact and f has no fixed orbits in K. So the same is true in a neighbourhood $\mathcal{U}_0$ of f. Let $\mathcal{U} = \mathcal{U}_0 \cap \cdots \cap \mathcal{U}_m$. Then every element of $\mathcal{U}$ is G-hyperbolic.
Density: We begin by proving the theorem for free G-actions. By Lemma 2.4.2.5, a fixed orbit of a map $f : M \to M$ of a free G-manifold is G-hyperbolic if and only if its image in $M/_G$ is hyperbolic for $[f]$. By Theorem 1.2.1.4, the set of hyperbolic maps $M/_G \to M/_G$ is open and dense, since $M/_G$ is a smooth manifold. Let

$$\mathfrak{X}_1 = \{f : M \to M \mid [f] \text{ hyperbolic } \}.$$

Then $\mathfrak{X}_1$ is the preimage of the open subset of hyperbolic self maps of $M/_G$ under the continuous projection π (see Theorem 2.1.4.1). Hence, $\mathfrak{X}_1$ is open. But again by 2.1.4.1, π has local sections, so if $f : M \to M$ is any map and $\mathcal{U}$ a given neighbourhood of f, we find a neighbourhood $\mathcal{V}$ of $[f]$ and a section mapping $\mathcal{V}$ into $\mathcal{U}$. Thus, we find a G-hyperbolic map in $\mathcal{U}$. This shows density of $\mathfrak{X}_1$.

If the action is monotypic, we have the decomposition $M = G/_H \times_{W(H)} M^H$, where M^H is a free $W(H)$-manifold. Clearly, $f : M \to M$ is G-hyperbolic if and only if the induced $W(H)$-map $M^H \to M^H$ is $W(H)$-hyperbolic. Since $W(H)$-maps $M^H \to M^H$ are in 1-1–correspondence with G-maps $M \to M$, openness and density of the set of G-hyperbolic G-maps follows from the openness and density of $W(H)$-hyperbolic $W(H)$-maps, which is already proven.

Finally let G act with arbitrarily many orbit types. By Proposition 2.1.1.11, the number of orbit types is finite and we have the orbit filtration $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_m = M$, where M_1 is a monotypic G-manifold. We can conclude that the set

$$\mathfrak{X}_2 = \{ f : M \to M \mid f\big|_{M_1} \ G\text{-hyperbolic } \}$$

is open and dense. Thus, if $f \in \mathfrak{X}_2$, f has only finitely many fixed orbits of type (H_1), say, $\gamma_1, \ldots, \gamma_m$. We find small isolating neighbourhoods $V_1, \ldots, V_m \subseteq M_1$ of these orbits such that every map $M_1 \to M_1$ in a neighbourhood $\mathcal{U}_1$ of $f\big|_{M_1}$ has a unique G-hyperbolic fixed orbit in each V_j and no other fixed orbits. We also find a neighbourhood $\mathcal{U}$ of f such that the restriction of elements of $\mathcal{U}$ to M_1 is in $\mathcal{U}_1$. In particular, we can apply Lemma 2.4.2.4 to find a map $\tilde{f} \in \mathcal{U}$ having $\gamma_1, \ldots, \gamma_m$ as G-hyperbolic fixed orbits. Since $\tilde{f}\big|_{M_1} \in \mathcal{U}_1$, $\tilde{f}$ has no other fixed orbits of orbit type (H_1). We thus have shown density of the set

$$\mathfrak{X}_3^1 = \{ f : M \to M \mid \text{all fixed orbits in } M_1 \text{ are } G\text{-hyperbolic} \}.$$

If $f \in \mathfrak{X}_3^1$, there are finitely many isolated fixed orbits $\delta_1, \ldots, \delta_n$ of type less than (H_2). Let $U_1, \ldots, U_n$ be isolating neighbourhoods. We find a neighbourhood U of M_k such that all fixed orbits of f in $\overline{U}$ are already contained in M_k and a neighbourhood $\mathcal{U}_0$ of f such that no element of $\mathcal{U}_0$ has fixed orbits in $\overline{U} - (U_1 \cup \cdots \cup U_n)$. By Proposition 2.4.2.2, we find neighbourhoods $\mathcal{U}_1 \ldots, \mathcal{U}_n$ of f such that each element of $\mathcal{U}_j$ has a unique G-hyperbolic fixed orbit in U_j. Hence, every element of $\mathcal{U}_0 \cap \mathcal{U}_1 \cap \cdots \cap \mathcal{U}_n$ is an element of $\mathfrak{X}_3^1$. We conclude that this set is open and dense.

Assume we have shown that the set $\mathfrak{X}_3^k$, consisting of G-maps all of whose fixed orbits in M_k are G-hyperbolic, is open and dense. Let $f_0 \in \mathfrak{X}_3^k$. We find an invariant neighbourhood W_k of M_k such that all fixed orbits of f_0 in W_k are already contained in M_k. Furthermore, we can assume that the boundary of W_k is smooth and contains no fixed orbits of f_0. Thus, $M_{k+1} - W_k$ is a compact monotypic G-manifold with boundary. Carrying out the first step of the proof, we see that

$$\begin{aligned} \mathfrak{X}_4 = \{ f : M \to M \quad | \quad & \text{all fixed orbits in } M_{k+1} - W_k \text{ are } G\text{-hyperbolic}, \\ & f \text{ has no fixed orbits on } \partial(M_{k+1} - W_k) \} \end{aligned}$$

is dense, so we find an element in $\mathfrak{X}_4 \cap \mathfrak{X}_3^k$ arbitrarily close to f_0. This shows density of $\mathfrak{X}_3^{k+1}$. Openness follows as for $\mathfrak{X}_3^1$. This finishes the proof by induction. $\square$

2.5.2 Genericity in the Space of G-Homotopies

It is clear that equivariantly non-degenerate homotopies exhibit the bifurcation behaviour we are aiming at and we can immediately proceed to show genericity of this set.

Theorem 2.5.2.1 *The subset of equivariantly non-degenerate homotopies is open and dense in the set of G-homotopies.*

Proof. It follows from the equivariant Thom-Mather Theorem for locally semialgebraic 2.2.3.6 that the set of G-maps $M \times \mathbb{R} \to M \times M$ G-transverse to the equivariant diagonal in any compact set $K \subseteq M \times \mathbb{R}$ is open and dense.

To prove openness of our set, let $H : M \times I \to M$ be a given non-degenerate G-homotopy and $\tilde{H} : M \times \mathbb{R} \to M$ an extension as in the definition. Then there is a neighbourhood $\mathcal{U}_1$ of $\tilde{H}$ such that all elements of $\mathcal{U}_1$ are equivariantly non-degenerate in $M \times [-1, 2]$. Clearly we find a neighbourhood $\mathcal{U}$ of H such that any $K \in \mathcal{U}$ has an element of $\mathcal{U}_1$ as an extension. We can assume that this extension is G-transverse to the diagonal on all of $M \times \mathbb{R}$ which finishes this part of the proof.

To prove density, let $H : M \times I \to M$ be any G-homotopy and $\tilde{H} : M \times \mathbb{R} \to M$ any equivariant extension, $F_{\tilde{H}} : M \times \mathbb{R} \to M \times M$ the associated graph map. We find a G-map $F : M \times \mathbb{R} \to M \times M$ arbitrarily close to $F_{\tilde{H}}$ which is equivariantly non-degenerate in $M \times [-1, 2]$ and as above, we can assume that it is so on all of $M \times \mathbb{R}$. The first component of F is close to the projection on the first factor, thus, $F_1 \times \pi_2$ is close to the identity. In particular it will ultimately be an equivariant diffeomorphism close to the identity. Then $F \circ (F_1 \times \pi_2)^{-1}$ is equivariantly non-degenerate and it is the graph map of a G-homotopy arbitrarily close to H. □

Turning to G-hyperbolic homotopies, we mention that the proof of genericity of G-hyperbolic G-maps cannot be modified (at least not trivially) to a working proof for G-homotopies. This is due to the fact that in the induction step, we cannot modify homotopies whose restriction to the set M_k is G-hyperbolic to a homotopy being G-hyperbolic locally around M_k. The bifurcation parameters prevent this method from working. One might expect that, using equivariant non-degeneracy, the proof of Proposition 1.2.2.7 can be made to work here as well. But there are still some issues concerning the relationship of equivariantly non-degenerate fixed orbits and G-hyperbolic ones and there is no equivariant version of Proposition 1.2.2.4, which would be needed to make the proof work.

So we will not prove any genericity theorem for G-hyperbolic homotopies. Anyhow, since openness failed in the case $G = \{e\}$, we could at most have expected density to hold. Moreover, genericity of equivariantly non-degenerate homotopies is enough to do equivariant index theory. It is also enough to prove the following result, which is the equivariant analogue to Proposition 1.2.2.8. We need a more special result later, which will be stated subsequently.

Theorem 2.5.2.2 *Let $S \subseteq M$ be a compact invariant submanifold such that*

$$\dim S_{(H)} + 1 + \dim W(H) < \dim M^H$$

for every isotropy group H. Then the set of equivariantly non-degenerate G-homotopies having no fixed orbits in S is open and dense.

PROOF. We again exploit the trick to look at the maps $S \times I \to S \times M$. By the equivariant Thom-Mather Theorem, the set of G-homotopies $S \times I \to S \times M$ that are G-transverse to the set

$$\{(s, gs) \mid s \in S,\ g \in W(G_s)\} \subseteq S \times S$$

is open and dense. Let $H : M \times I \to M$ be any G-homotopy. Define

$$H_S : S \times I \to S \times M,\ (s,t) \mapsto (s, H(s,t)).$$

Fix a stratum $\Delta_G(S_{(H)})$ of the S-diagonal. If a map $F : S \times I \to S \times M$ is G-transverse to the S-diagonal, it is stratumwise transverse to every set $\Delta_G(S_{(H)})$. This follows by 2.2.3.7, since diagonals satisfy the assumptions made there. Consequently, for every isotropy subgroup K with $(H) \le (K)$, the map

$$F^K_{(K)} : (S \times I)^K_{(K)} \to (S \times M)^K$$

is transverse to the set $\Delta_G(S_{(H)})^K$. Hence, the preimage of this set under $F^K_{(K)}$ is either empty, or a submanifold of dimension

$$\begin{aligned} &\dim S^K_{(K)} + 1 - \dim S^K - \dim M^K + \dim \Delta_G(S^K_{(H)}) \\ = \ &\dim S^K_{(K)} + 1 - \dim S^K - \dim M^K + \dim S^K_{(H)} + \dim W(H). \end{aligned}$$

We estimate

$$\begin{aligned} &\dim S^K_{(K)} + 1 - \dim S^K - \dim M^K + \dim S^K_{(H)} + \dim W(H) \\ \le\ & 1 - \dim M^K + \dim S^K_{(H)} + \dim W(H) \\ \le\ & 1 - \dim M^H + \dim S^K_{(H)} + \dim W(H) \\ \le\ & 1 - \dim M^H + \dim S_{(H)} + \dim W(H). \end{aligned}$$

So under the assumption of the lemma, this value is smaller than zero, i.e. the preimages are empty in any case. We see that G-transversality of H_S to $\Delta_G(S)$ implies that H has no fixed orbits in S. The theorem thus will be proven if we show that the set of $H : M \times I \to M$ such that H_S is G-transverse to the S-diagonal is open and dense.

Openness: If K is close to H, K_S is close to H_S, hence by the equivariant Thom-Mather Theorem, K_S will be G-transverse to the S-diagonal if H_S is.

Density: Let $\mathcal{U}$ be a given neighbourhood of $H : M \times I \to M$. We find a map $F : S \times I \to S \times M$ G-transverse to $\Delta_G(S)$ and arbitrarily close to H_S. In particular, the first component F_1 will be close to π_1, hence we can achieve that $F_1 \times \pi_2$ is a G-diffeomorphism close to the identity. The map $F \circ (F_1 \times \pi_2)^{-1}$ is of the form $(s,t) \mapsto (s, k(s,t))$ and we have to show that k can be extended to a map $K : M \times I \to M$ that is contained in $\mathcal{U}$. But this is done exactly in the same way as non-equivariantly (compare the proof of Proposition 1.2.2.8), choosing equivariant embeddings and invariant Urysohn functions. $\square$

Proposition 2.5.2.3 *Let S be the boundary of an equivariant disc in M. Then the set of G-homotopies without fixed orbits in S is open and dense.*

Proof. Let $\pi_1 : S \times M \to S$ be the projection and consider the set of G-maps

$$\mathcal{M} = \{F : S \times I \to S \times M \mid \pi_1 \circ F(S_{(H)} \times I) \subseteq S_{(H)} \, \forall H\}.$$

Let S be the boundary of the equivariant disc D, where D is centered around an orbit of type (H). Working locally, we can assume that $M = G \times_H V$ and D is the ball in $e \times_H L^\perp$, where $L \subseteq V^H$ is a one-dimensional subspace. By definition of equivariant discs, such an L exists. Hence, $S = e \times_H S(L^\perp)$, the sphere in $L^\perp$. We have

$$\begin{aligned} \dim S_{(H)} &= \dim G/_{N(H)} + \dim S^H_{(H)} \\ &= \dim G - \dim N(H) + \dim S^H_{(H)} \\ &= \dim G/_H - \dim W(H) + \dim S^H_{(H)}. \end{aligned}$$

In particular, we obtain $\dim S_{(H)} + 1 + \dim W(H) = 1 + \dim G/_H + \dim S^H_{(H)}$. But $\dim S^H_{(H)} = \dim S(L^\perp)^H_{(H)} = \dim V^H - 2$, which yields

$$\begin{aligned} \dim S_{(H)} + 1 + \dim W(H) &= \dim G/_H + \dim V^H - 1 \\ &= \dim(G \times_H V)^H - 1 \\ &< \dim(G \times_H V)^H \\ &= \dim M^H. \end{aligned}$$

So the condition of Theorem 2.5.2.2 is fulfilled at least for the orbit type (H). But if we take a map $F \in \mathcal{M}$, we have $F^{-1}(\Delta_G(S^K_{(H)})) \cap (S \times I)^K_{(K)} = \emptyset$ for all orbit types (K) properly larger than (H). So G-transversality of an element F of $\mathcal{M}$ to the S-diagonal implies that F has no fixed orbits in S. Clearly, every map $(x, \lambda) \mapsto (x, H(x, \lambda))$ induces an element of $\mathcal{M}$. But the proof that G-homotopies inducing maps $S \times I \to S \times M$ G-transverse to the S-diagonal form an open and dense subset did not depend on the dimension condition. So we can conclude that G-homotopies H such that $\pi_1 \times H$ is G-transverse to the S-diagonal form an open and dense subset in the space of G-homotopies and such a homotopy has no fixed orbits in S. □

Corollary 2.5.2.4 *Let M be a compact G-manifold, S the boundary of an equivariant disc in M. Then the set of equivariantly non-degenerate homotopies without fixed orbits on S is open and dense.*

Proof. Follows immediately from Theorem 2.5.2.1 and Proposition 2.5.2.3. □

2.5.3 Genericity in the Space of G-Vector Fields

The proof of genericity of G-hyperbolic vector fields depends mainly on three lemmas. The first lemma will ensure that, if a G-vector field has no essential relative periodic orbits in a compact set, then there is a neighbourhood in the set of G-vector fields having the same property. The second lemma will be the equivalent of Proposition 2.4.2.2 for vector fields. It will show that if γ is a G-hyperbolic relative periodic orbit of a field ξ, then there is a neighbourhood $\mathcal{U}$ of ξ and an invariant neighbourhood U of γ such that every element $\eta \in \mathcal{U}$ has a unique essential relative periodic orbit contained in U. The final lemma is an equivariant analogue of Lemma 1.2.3.2 and will guarantee that we can make a relative periodic orbit G-hyperbolic by slightly changing the vector field.

These are the lemmas 7.D, 5.C and 6.C of [Fie80], respectively. The proof of the second lemma is contained in [Fie80] and makes use of the theory of equivariant stable and unstable manifolds. It will also follow from Proposition 2.4.2.2 using equivariant Poincaré systems. The first lemma, however, is easy to deduce.

Lemma 2.5.3.1 *Let $\xi \in \mathfrak{X}_G(M, \Omega, a, b)$ be a G-vector field such that ξ has no essential relative periodic orbits in the compact set $K \subseteq M$. Then there is a neighbourhood $\mathcal{U}$ of ξ such that every element of $\mathcal{U}$ has the same property.*

Proof. Fix an invariant Riemannian metric on M and let d be the Riemannian distance function. Let φ be the flow of ξ and let $x \in K$ be given. By assumption we have

$$\inf_{g\in G,\, t\in[a,b]} d(\varphi(x,t), gx) > 0$$

and this condition clearly holds with φ replaced by the flow of a field in a neighbourhood $\mathcal{U}_x$ of ξ and x replaced by an element y in a neighbourhood U_x of x. The sets U_x cover K and we find a finite subcover $U_1, \dots, U_m$. Let $\mathcal{U}_1, \dots, \mathcal{U}_m$ be the corresponding neighbourhoods of ξ, then $\mathcal{U} = \mathcal{U}_1 \cap \dots \cap \mathcal{U}_m$ is a neighbourhood of ξ fulfilling all requirements. ☐

Lemma 2.5.3.2 *Let γ be an essential G-hyperbolic relative periodic orbit of a G-vector field ξ. Then there are neighbourhoods $\mathcal{U}$ of ξ and U of γ such that every element η in $\mathcal{U}$ has a unique essential relative periodic orbit in U which is G-hyperbolic.*

Proof. As mentioned above, this follows from Proposition 2.4.2.2, since if two G-vector fields are close, their equivariant Poincaré maps are close as well. ☐

The next lemma is an equivariant version of Lemma 1.2.3.2 and the proof is almost identical and very similar to the one for Lemma 6.C of [Fie80]. We include it for completeness.

Lemma 2.5.3.3 *Let γ be a relative periodic orbit of $\xi \in \mathfrak{X}_G(M, \Omega, a, b)$ and (D, D', P, t) an equivariant Poincaré system for γ. Let V, U be invariant neighbourhoods of γ such that*

$$\overline{U} \subseteq \bigcup_{x\in D'} \varphi([0, t(x)]),$$

and

$$\varphi(x, [0, t(x)]) \subseteq U,\ x \in \overline{V} \cap D',$$

φ the flow of ξ. Then there is an open neighbourhood $\mathcal{V}$ of P in the set of maps in $\mathcal{C}^r_G(D', D)$ equal to P outside of $V \cap D'$ and a continuous map $\chi : \mathcal{V} \to \mathfrak{X}_G(M, \Omega, a, b)$ such that

1. *$\chi(Q)$ has Poincaré map $Q \in \mathcal{V}$.*
2. *$\chi(Q) = \xi$ outside of U.*
3. *$\chi(P) = \xi$.*

Proof. We have

$$t_0 = \inf_{x \in D'} t(x) > 0,$$

since D' is part of an equivariant Poincaré system. Choose real numbers r, s such that $0 < r < s < t_0$. Take any equivariant map $Q \in \mathcal{C}^r_G(D', D)$ equal to P outside of $V \cap D'$. We can define a smooth map $P^{-1}Q : \overline{D'} \to D$ by taking it to be the identity close to ∂D. The resulting map will be arbitrarily close to the inclusion $\overline{D'} \hookrightarrow D$ by choosing Q close to P and extending appropriately. In particular, we can achieve that there is an equivariant isotopy

$$K : \overline{D'} \times [r, s] \to D$$

between the inclusion and $P^{-1}Q$. Extend K to an equivariant isotopy $\overline{D'} \times [0, t_0] \to D$ such that $K_0 = K_r$, $K_s = K_{t_0}$ and every K_t is equal to the inclusion close to $\partial D'$. Define

$$\psi_y(t) = \varphi(K_t(y), t)$$

for $y \in D'$, $t \in [0, t(y)]$. By definition of K, $\psi_y(t) = \varphi_y(t)$ for y close to $\partial D'$. Again by choosing Q close enough to P, none of the curves $t \mapsto \psi_y(t)$ will meet $M - U$. Moreover, by openness of the set of equivariant embeddings, $\psi\big|_{D' \times [r,s]}$ will be an embedding and so these curves will be pairwise disjoint. Thus, we can define a vector field

$$\eta(\psi_y(t)) = \dot{\psi}_y(t)$$

which we can extend smoothly to a vector field which is equal to ξ in $M - U$ and arbitrarily close to ξ. The integral curves of η meeting D' are the curves $t \mapsto \psi_y(t)$ up to reparametrization. The calculation

$$\psi_y(t(y)) = \varphi(K_t(t(y)), t(y)) = \varphi(P^{-1}Q(y), t(y)) = Q(y)$$

shows that Q is the Poincaré map of η. Clearly, all the extension processes can be carried out continuously in Q, hence, the assignment $\chi(Q) = \eta$ gives the desired result. $\square$

When inspecting the inductive proof of genericity of G-hyperbolic G-maps, the main ingredients to make it work were analoga of the three preceeding lemmas for fixed orbits. Thus we can expect that a very similar proof will work in the case of vector fields as well.

Theorem 2.5.3.4 *The set of G-hyperbolic vector fields is open and dense in the set $\mathfrak{X}_G(M, \Omega, a, b)$.*

Proof. Openness: Let $\xi : M \to TM$ be any G-hyperbolic vector field in $\mathfrak{X}_G(M, \Omega, a, b)$. Let $\gamma \subseteq \Omega$ be a relative periodic orbit of ξ. By Lemma 2.5.3.2 we find a neighbourhood $U_\gamma \subseteq \Omega \times (a, b)$ of γ and a neighbourhood $\mathcal{U}_\gamma$ of f such that every element of $\mathcal{U}_\gamma$ has a unique relative periodic orbit in U_γ which is G-hyperbolic. Furthermore, there are only finitely many essential relative periodic orbits of ξ in Ω, so we find a neighbourhood $U \subseteq \Omega \times (a, b)$ and a neighbourhood $\mathcal{U}$ of ξ such that each element of $\mathcal{U}$ has finitely many G-hyperbolic relative periodic orbits in U. The complement $\overline{\Omega \times (a, b)} - U$ is compact, so by Lemma 2.5.3.1, we find a neighbourhood $\mathcal{V}$ of ξ such that no element of $\mathcal{V}$ has relative periodic orbits in $\overline{\Omega \times (a, b)} - U$. Clearly, $\mathcal{U} \cap \mathcal{V}$ is a neighbourhood of ξ consisting of G-hyperbolic vector fields.

Density: For density, we use induction on the orbit types as in Theorem 2.5.1.2. Assume first that the action is free. Let $\xi : M \to TM$ be an equivariant vector field, φ its flow. Then G-hyperbolic relative periodic orbits of ξ correspond to hyperbolic periodic orbits of the induced field $\tilde{\xi} : M/_G \to TM/_G$. Since the set of induced maps $M/_G \to TM/_G$ is open and the projection map has local sections (see Proposition 2.1.4.2), we can conclude that the set of G-hyperbolic vector fields is open and dense, by genericity of hyperbolic vector fields on the compact manifold $M/_G$. By the same reasoning and reduction of the G-action to a $W(H)$-action, we conclude that genericity holds on monotypic G-manifolds.

Thus, let M be arbitrary and $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_k = M$ be the filtration by orbit type. M_1 is a compact monotypic manifold. So we see that the set

$$\mathfrak{X}_1 = \{\xi \in \mathfrak{X}_G(M, \Omega, a, b) \mid \xi\big|_{M_1} G - \text{hyperbolic}\}$$

is open and dense. An element of $\mathfrak{X}_1$ has finitely many G-hyperbolic relative periodic orbits $\gamma_1, \ldots, \gamma_m$. We choose small neighbourhoods $U_1, \ldots, U_m \subseteq M_1 \times [a, b]$ around these orbits such that in a neighbourhood $\mathcal{U}_j$ of $\xi\big|_{M_1}$, every element of $\mathcal{U}_j$ has a unique G-hyperbolic relative periodic orbit in U_j, $j = 1, \ldots, m$. This can be done by Lemma 2.5.3.2. We use Lemma 2.5.3.3 to replace an element $\xi \in \mathfrak{X}_1$ by an element ξ_1 arbitrarily close to ξ, $\xi_1\big|_{M_1}$ equal to $\xi\big|_{M_1}$ outside of $U = U_1 \cup \cdots \cup U_m$ and $\gamma_1, \ldots, \gamma_m$ are unique G-hyperbolic relative periodic orbits of ξ_1 in U. In particular, ξ_1 has no other relative periodic orbits of type (H_1) in $\Omega \times (a, b)$. This shows that the set

$$\mathfrak{X}_1^1 = \{\xi \in \mathfrak{X}_G(M, \Omega, a, b) \mid \text{ relative periodic orbits in } (M_1 \cap \Omega) \times [a, b] \ G\text{-hyperbolic}\}$$

is dense. It is also clear that this set is open, we just argue in the same way as in the openness part of this proof. Now assume by induction that the set $\mathfrak{X}_1^k$ of equivariant vector fields with only G-hyperbolic relative periodic orbits in $(M_k \cap \Omega) \times [a, b]$ is open and dense. Take $\xi \in \mathfrak{X}_1^k$. We find an invariant neighbourhood U_k of $M_k \times [a, b]$ such that all relative periodic orbits of f in U_k are contained in $M_k \times [a, b]$ and we can achieve that U_k has a smooth boundary containing no relative periodic orbits. Thus, $M_{k+1} \times [a, b] - U_k$ is a smooth invariant manifold with boundary and it is monotypic. So we can carry out the first step of the induction (the boundary does not matter, since there are no critical elements on it) to conclude that the set of G-maps with only G-hyperbolic relative periodic orbits in $M_{k+1} - U_k$ and no relative periodic orbits on the boundary is dense. Consequently, we find an element in this set, arbitrarily close to ξ. This shows density of the set $\mathfrak{X}_1^{k+1}$. Openness of $\mathfrak{X}_1^{k+1}$ follows as in the two cases above. This proves the theorem by induction. $\square$

Note that in the statement of Theorem 2.5.3.4, we can replace the property "G-hyperbolic" by "equivariantly non-degenerate". The proof only made use of the isolatedness property of G-hyperbolic relative periodic orbits, and this is also valid for equivariantly non-degenerate relative periodic orbits.

2.5.4 Genericity in the Space of Homotopies of G-Vector Fields

The last genericity theorem is the genericity of equivariantly non-degenerate homotopies of G-vector fields. The proof depends on the geometric construction we already used for vector fields and homotopies thereof and is a parametrized version of Lemma 2.5.3.3.

Lemma 2.5.4.1 *Let $H \in h\mathfrak{X}_G(M, \Omega, a, b)$ and γ_λ an essential relative periodic orbit of H_λ. Choose a Poincaré system $(D, D', P_\lambda, t_\lambda)$ for γ_λ such that the Poincaré maps of all fields in a neighbourhood $\mathcal{U}_1$ of H_λ are defined as maps $D' \to D$. Let V, U be invariant open neighbourhoods of γ_λ such that*

$$\overline{U} \subseteq \bigcup_{x \in D'} \varphi_\mu([0, t_\mu(x)])$$

and

$$\varphi_\mu([0, t_\mu(x)]) \subseteq U$$

for $x \in \overline{V} \cap D'$ and μ in a neighbourhood of λ, say, if $|\lambda - \mu| < 3\varepsilon$, $\varepsilon > 0$. Let P be the Poincaré homotopy

$$P : D' \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon] \to D$$

given by the Poincaré maps of the H_μ, $\mu \in [\lambda - 3\varepsilon, \lambda + 3\varepsilon]$. Then there is a neighbourhood $\mathcal{U}$ in the set of G-homotopies $D' \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon] \to D$ equal to P outside of $\overline{V} \cap D \times [\lambda - 2\varepsilon, \lambda + 2\varepsilon]$ and a continuous map $\chi : \mathcal{U} \to h\mathfrak{X}_G(M, \Omega, a, b)$ such that

1. *for $Q \in \mathcal{U}$, $\chi(Q)_\mu$ has Poincaré map Q_μ for $|\lambda - \mu| < \varepsilon$.*
2. *for $Q \in \mathcal{U}$, $\chi(Q)$ equals H outside of $U \times [\lambda - 2\varepsilon, \lambda + 2\varepsilon]$.*
3. *$\chi(P) = H$.*

Proof. The proof is once again a careful rework of the proof of Lemma 1.2.4.3. We just have to check that we can add equivariance wherever it is needed.

Let t_ξ be the period map of an element ξ of $\mathcal{U}_1$, i.e. t_ξ is the minimal $t \in \mathbb{R}^{>0}$ such that $\varphi_\xi(x, t_\xi(x)) \in D$. Since γ_λ is a relative periodic orbit and by definition of an equivariant Poincaré system, we have

$$t_0 = \inf_{x \in D'} \inf_{\xi \in \mathcal{U}_1} t_\xi(x) > 0.$$

Choose real numbers r, s with $0 < r < s < t_0$. Let $Q : D' \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon] \to D$ be any G-homotopy equal to P outside of $V \cap D' \times [\lambda - 2\varepsilon, \lambda + 2\varepsilon]$. By choosing Q close enough to P, we can assume that $P^{-1} \circ Q$ (where P^{-1} is to be taken fibrewise) is a G-embedding that is equivariantly isotopic to the inclusion $\overline{D'} \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon] \hookrightarrow D \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon]$. Let

$$K : \overline{D'} \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon] \times [r, s] \to D \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon]$$

be an isotopy joining the two maps. For the flow φ of H, define

$$\psi(y, \mu, t) = \varphi(K(y, \mu, t), t)$$

for $y \in D'$, $\mu \in [\lambda - 3\varepsilon, \lambda + 3\varepsilon]$, $t \in [0, t_\mu(y)]$, where we extend K smoothly to the interval $[0, t_0]$ such that $K_0 = K_r$, $K_s = K_{t_0}$ and every K_t is equal to the inclusion close to the boundary of $D' \times (\lambda - 3\varepsilon, \lambda + 3\varepsilon)$. We have $\psi \equiv \varphi$ in a neighbourhood of the boundary of $D' \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon]$. By choosing Q sufficiently close to P, we can achieve that none of the curves $t \mapsto \psi(y, \mu, t)$ meets $M - U$. Now φ_μ is an embedding when restricted to the interval $[r, s]$ and the set of embeddings is open. Thus, ψ_μ can be made an embedding, too. Take $\mu \in [\lambda - \varepsilon, \lambda + \varepsilon]$ and define

$$\eta_\mu(\psi(y, \mu, t)) = \frac{\mathrm{d}}{\mathrm{d}s} \psi(y, \mu, s)\big|_{s=t}.$$

η_μ is a G-vector field defined on the image of ψ_μ. Extend η to a homotopy on $M \times [\lambda - 3\varepsilon, \lambda + 3\varepsilon]$ such that $\eta_\mu = H_\mu$ outside of $U \times [\lambda - 2\varepsilon, \lambda + 2\varepsilon]$. Clearly, this extension can be done continuously in Q. The integral curves of η for $|\lambda - \mu| < \varepsilon$ are, up to reparametrization, just the curves $t \mapsto \psi(y, \mu, t)$ and we calculate

$$\psi(y, \mu, t_\mu(y)) = \varphi(K(y, \mu, t_\mu(y)), \mu, t_\mu(y)) = \varphi(P_\mu^{-1} \circ Q_\mu(y), \mu, t_\mu(y)) = Q_\mu(y).$$

So the Poincaré homotopy induced by η in $[\lambda - \varepsilon, \lambda + \varepsilon]$ is given by Q. The definition $\chi(Q) = \eta$ gives the desired result. □

Next we prove openness of equivariantly non-degenerate homotopies and a slightly more general result we will need in the proof of the upcoming genericity theorem.

Proposition 2.5.4.2 *Let $H \in h\mathfrak{X}_G(M, \Omega, a, b)$ and H is equivariantly non-degenerate in the compact subset $K \times J \subseteq \Omega \times I$. Then there is a neighbourhood $\mathcal{U}$ of H such that every element of $\mathcal{U}$ is equivariantly non-degenerate in $K \times J$.*

Proof. Since H is equivariantly non-degenerate, we have that for some extension $\tilde{H} : M \times \mathbb{R} \to M$, the map

$$F : \overline{\Omega} \times \mathbb{R} \times [a, b] \longrightarrow M \times M, \quad (x, t, \lambda) \longmapsto (x, \tilde{\varphi}_\lambda(x, t))$$

is G-transverse to the equivariant diagonal in $K \times J \times [a,b]$. By openness of G-transversality, we find a neighbourhood $\mathcal{U}_1$ of F such that all elements of $\mathcal{U}_1$ are G-transverse to the diagonal in $K \times J \times [a,b]$. Clearly, if H' is close to H, the associated map F' is close to F, so we find a neighbourhood $\mathcal{U}$ of H of equivariantly non-degenerate maps in $K \times J \times [a,b]$, i.e. all essential relative periodic orbits in $K \times J$ are equivariantly non-degenerate. □

With these two auxiliary results at hand, we can prove genericity of equivariantly non-degenerate homotopies of vector fields. The proof is another replication of the methods used for Theorem 1.2.3.6 and Theorem 1.2.4.5. We will see that there are no obstacles induced by symmetry.

Theorem 2.5.4.3 *The subset of $h\mathfrak{X}_G(M,\Omega,a,b)$ consisting of equivariantly non-degenerate G-homotopies is open and dense.*

Proof. Openness is a trivial corollary of the preceeding proposition. It therefore remains to prove density. We proceed in the 5-step system that worked for Theorem 1.2.4.5.

1. Take a homotopy $H \in h\mathfrak{X}_G(M,\Omega,a,b)$ and let $\mathcal{U}$ be any neighbourhood of H. Let $\Gamma \subseteq \Omega \times (a,b)$ be the set of essential relative geometric periodic points of H, i.e. $(x,\lambda) \in \Gamma$ if and only if there is a $t \in [a,b]$ such that $\varphi_\lambda(x,t) \in Gx$. By assumption, Γ is compact. Choose an equivariant Poincaré system $(D_\gamma, D'_\gamma, p_\gamma, t_\gamma)$ for every essential relative periodic orbit γ of H in such a way that in a neighbourhood $\mathcal{U}_\gamma$ of H, all the Poincaré maps of elements of $\mathcal{U}_\gamma$ are defined as maps $D'_\gamma \to D_\gamma$. If γ is a relative periodic orbit of H_λ, we find an $\varepsilon_\gamma > 0$ such that the Poincaré maps of H_μ for $|\lambda-\mu| \le 3\varepsilon_\gamma$ constitute a Poincaré homotopy
$$P : D'_\gamma \times [\lambda - 3\varepsilon_\gamma, \lambda + 3\varepsilon_\gamma] \to D_\gamma$$
and so do all elements of $\mathcal{U}_\gamma$.
2. Choose open invariant neighbourhoods $W_\gamma \subseteq V_\gamma \subseteq U_\gamma$ of the underlying geometric orbit of γ such that $\overline{W}_\gamma \subseteq V_\gamma$ and
$$\overline{U}_\gamma \subseteq \bigcup_{x \in D'_\gamma} \tilde{\varphi}_\mu\big([0, \tilde{t}_\mu(x)]\big),$$
$$\tilde{\varphi}_\mu\big(x, [0, \tilde{t}_\mu(x)]\big) \subseteq U_\gamma$$
for all flows $\tilde{\varphi}$ of elements in $\mathcal{U}_\gamma$ and all $x \in \overline{V}_\gamma \cap D'_\gamma$, $\mu \in [\lambda - 3\varepsilon_\gamma, \lambda + 3\varepsilon_\gamma]$.
3. The sets $W_\gamma \times (\lambda - \varepsilon_\gamma, \lambda + \varepsilon_\gamma)$ cover Γ and we find a finite subcover, corresponding to orbits $\gamma_1, \ldots, \gamma_m$ at parameters $\lambda_1, \ldots, \lambda_m$. For simplicity, let
$$W_j = W_{\gamma_j} \times \big(\lambda_j - 3\varepsilon_{\gamma_j}, \lambda_j + 3\varepsilon_{\gamma_j}\big)$$

and $\varepsilon_j = \varepsilon_{\gamma_j}$, $j = 1, \ldots, m$. Define $W = W_1 \cup \cdots \cup W_m$ and $\mathcal{U}_1 = \mathcal{U}_{\gamma_1} \cap \cdots \cap \mathcal{U}_{\gamma_m}$. Then $K = \overline{\Omega} \times I - W$ is compact and all periodic orbits of H meeting K are inessential. So the same holds in a neighbourhood of H which we can assume to contain $\mathcal{U}_1$. We conclude that every G-homotopy H' in $\mathcal{U}_1$ satisfies

- $H' \in \mathcal{U}$.
- All relative periodic orbits of H' meeting K are inessential.
- Lemma 2.5.4.1 is applicable to H', the sets $V_{\gamma_j}, U_{\gamma_j}$ and the Poincaré system $(D_{\gamma_j}, D'_{\gamma_j}, p'_{\gamma_j}, t'_{\gamma_j})$.

4. Assume that we have constructed a G-homotopy H_k, $0 \leq k \leq m-1$, such that $H_k \in \mathcal{U}$, all relative periodic orbits meeting K are inessential and H_k is equivariantly non-degenerate in $\overline{W_1 \cup \cdots \cup W_k}$. We find a neighbourhood $\mathcal{W}_k$ of H_k such that every element of $\mathcal{W}_k$ is equivariantly non-degenerate in $\overline{W_1 \cup \cdots \cup W_k}$. Now we apply Lemma 2.5.4.1 to H_k, the sets $V_{\gamma_{k+1}}, U_{\gamma_{k+1}}$ and the corresponding Poincaré system. We find a neighbourhood $\mathcal{V}_{k+1}$ of the induced Poincaré homotopy and a map $\chi_{k+1} : \mathcal{V}_{k+1} \to h\mathfrak{X}_G(M, \Omega, a, b)$ with the properties stated in the lemma. So we can use the genericity results for equivariantly non-degenerate homotopies of maps. Choose an equivariantly non-degenerate homotopy

$$\overline{W_{\gamma_{k+1}} \cap D'_{\gamma_{k+1}}} \times [\lambda_{k+1} - \varepsilon_{k+1}, \lambda_{k+1} + \varepsilon_{k+1}] \to D_{\gamma_{k+1}}$$

and extend it to a G-homotopy

$$Q : (U_{\gamma_{k+1}} \cap D'_{\gamma_{k+1}}) \times [\lambda_{k+1} - 3\varepsilon_{k+1}, \lambda_{k+1} + 3\varepsilon_{k+1}] \to D_{\gamma_{k+1}}$$

that is equal to the Poincaré homotopy of H_k outside of

$$V_{\gamma_{k+1}} \cap D'_{\gamma_{k+1}} \times [\lambda_{k+1} - 2\varepsilon_{k+1}, \lambda_{k+1} + 2\varepsilon_{k+1}]\,.$$

By choosing the initial homotopy close to the Poincaré homotopy, we can achieve that $Q \in \mathcal{V}_{k+1}$ and $\chi(Q) \in \mathcal{W}_k \cap \mathcal{U}_1$. Define $H_{k+1} = \chi(Q)$. By construction, H_{k+1} is equivariantly non-degenerate on $\overline{W_{k+1}}$. Since $H_{k+1} \in \mathcal{W}_k$, H_{k+1} is equivariantly non-degenerate on $\overline{W_1 \cap \cdots \cap W_{k+1}}$. Since $H_{k+1} \in \mathcal{U}_1$, $H_{k+1} \in \mathcal{U}$ and all relative periodic orbits meeting K are inessential.

5. Arriving at H_m, this homotopy is an element in $\mathcal{U}$, equivariantly non-degenerate in $\overline{W_1 \cup \cdots \cup W_m} \supseteq \overline{\Omega} \times I - K$ and all relative periodic orbits meeting K are inessential. So H_m is equivariantly non-degenerate, proving our claim. □

3 Equivariant Index Theory

To adopt the construction of the Fuller index from chapter one, we need a sort of equivariant fixed point index. As mentioned in the introduction, most efforts in this direction were aimed at counting group orbits of fixed points rather than fixed orbits. An equivariant Lefschetz number was constructed by Chorny in [Cho03], but it is still not exactly what we need. Chornys Lefschetz number is a homotopy invariant for self maps, whereas we want to assign such an invariant to Poincaré maps, which are maps $D' \to D$, $D' \subseteq D$. There is no simple way to adjust the equivariant Lefschetz number to be applicable to this setting. But it is possible to assign a local index to isolated fixed orbits of equivariant maps, a so called fixed orbit index, defined by Dzedzej in [Dze01]. This index is exactly what we need to do equivariant Fuller index theory. The fixed orbit index gives a global homotopy invariant as well, but the exact interplay between this index and the equivariant Lefschetz number is unclear. A sort of equivariant Lefschetz-Hopf theorem can be expected to hold, though there seem to be no results pointing in that direction. An indication may be the work of Goncalves and Weber [GW07], who gave an axiomatized treatment of an equivariant Lefschetz number for discrete groups, taking values in the Burnside ring of G. It seems possible to give a similar axiomatization of the equivariant Lefschetz number of Chorny. Then one just has to check that the global orbit index satisfies the axioms. But we will not investigate this problem any further in this work.

In the first part of the chapter we will give the definition of Dzedzejs fixed orbit index, starting with a discourse on the theory of equivariant absolute neighbourhood retracts, G-ANRs for short. This is necessary because the main idea of the fixed orbit index is that, while for a G-manifold M, $M/_G$ is rarely a manifold, it is still an absolute neighbourhood retract. Fixed point indices exist on absolute neighbourhood retracts, so we can work in the quotient to do index theory in the manifold itself. Using this procedure, we define the fixed orbit index and prove its elementary properties as well.

In the second part, we put everything together to define an equivariant Fuller index for G-vector fields, assigning the local orbit indices of Poincaré maps to periodic orbits and making the necessary adaptions coming from period space phenomena. The main result is the homotopy invariance of the equivariant Fuller index, which uses the methods of proof we already used in the case without symmetries. As stated there, the proof was developed to obtain the equivariant analog without major adjustments.

We close with some examples of calculations of fixed orbit indices, which easily extend to calculations of the equivariant Fuller index.

3.1 The Fixed Orbit Index

In classical fixed point theory, the extension of the fixed point index from self maps of manifolds to maps between absolute neighbourhood retracts, ANRs for short, is a generalization which makes it possible to do index theory in infinite dimensions. From the theoretical point of view, it is completely satisfactory to have knowledge of the local fixed point indices which can be taken ultimately to be calculated for maps in some euclidean space.

In the equivariant theory, it is essential to work with the much more general G-ANR spaces. We indicated in the introduction to the chapter that quotients of G-manifolds by the G-action are always ANRs, so it is possible to do index theory in them. By carefully exploiting the stratified structure of the quotients, it thus is possible to obtain a well-defined G-homotopy invariant on the manifold itself. The idea is to assign to a fixed orbit type (H) the difference between the fixed point index of the map induced by a G-map on the quotient of points of orbit type less or equal to (H), and the fixed point index of the induced map on the quotient of points of type strictly lesser than (H). Intuitively, this difference counts the orbits genuinely coming from the orbit type (H). Note that we can not just go into the quotient of points of type (H) directly, for on the one hand we would miss symmetry breaking bifurcation phenomena, on the other hand, this manifold is in general not compact, adding a huge amount of new obstacles.

3.1.1 G-ANRs

We quickly run through the basic definitions of the theory of ANRs and add the symmetries subsequently. Most results will be proven by reference, since details would lead us too far afield. The main reference for the theory of ANRs might still be the classical monograph of Borsuk, [Bor67]. In the equivariant direction, we mainly have to mention Murayamas work, [Mur83].

Definition 3.1.1.1 *A metric space X is said to be an* absolute neighbourhood retract *(in the category of metric spaces), if it satisfies the following universal property. Whenever M is a metric space and $i : X \to M$ is an isometric embedding of X as a closed subset, there is a neighbourhood $U \subseteq M$ of $i(X)$ and a retraction $r : U \to X$. We write ANR for the notion of an absolute neighbourhood retract.*

It is well-known, compare e.g. [Bor67], that manifolds (with boundary) are ANRs. This is essentially the tubular neighbourhood theorem. Therefore, ANRs have many properties which resemble properites of manifolds, as long as these are somehow connected to the existence of tubular neighbourhoods. For example, maps into an ANR can be extended to be defined on a neighbourhood of the initial domain under mild assumptions. We do not need these properties in the following.

There is the obvious notion of a G-ANR for a compact Lie group G. One just takes all involved spaces and maps in the definition of an ANR to be G-spaces and equivariant, respectively. Since manifolds are ANRs, G-manifolds are G-ANRs. We need a general result which identifies the usual invariant fix spaces of a G-ANR as G-ANRs as well. We denote

$$X_{\leq (H)} = \{x \in X \mid (G_x) \leq (H)\}$$
$$X_{<(H)} = \{x \in X_{\leq (H)} \mid (G_x) \neq (H)\}.$$

Then the following is true.

Proposition 3.1.1.2 *Let H be a closed subgroup of G and let X be a G-ANR. Then the spaces $X_{(H)}, X_{\leq(H)}$ and $X_{<(H)}$ are G-ANRs and the quotient $X/_G$ is an ordinary ANR.*

Proof. This is proven in [Mur83]. □

3.1.2 Definition of the Fixed Orbit Index

We illustrate the intuitive reason why index theory on ANRs is possible by taking manifolds with boundary as an example. Even more specifically, we can take closures of open subsets of $\mathbb{R}^n$. Then we have to deal with maps possibly having fixed points on the boundary. The way to solve this problem is to embed the ANR isometrically into a normed vector space. This is always possible since an ANR is metric by definition. The image will have a neighbourhood retracting onto it. Composing the map in question with this retraction, the resulting map clearly does not have fixed points on the boundary, so the fixed point index of this map is defined. One just has to go through all the problems of well-definedness to see that this defines an index for self maps of ANRs.

We come to the concrete definition of the fixed point index on ANRs. Let $X \subseteq U$ be a pair of ANRs, $f : X \to U$ a continuous map having no fixed points on ∂X. For simplicity we assume that U can be embedded into $\mathbb{R}^n$ via an embedding $i : U \to \mathbb{R}^n$, otherwise we had to put some compactness assumption on f. Now i embeds X as well and so we find a neighbourhood V of $i(X)$ and a retraction $r : V \to i(X)$. The map $i \circ f \circ i^{-1} \circ r : V \to V$ is well-defined and has no fixed points outside of $i(X)$. Hence, its fixed point index is well-defined and we write $i(f, X, U)$ for this number. It is shown in [Nus77] that this index is well-defined and enjoys all the usual properties an index should have.

Now take $X \subseteq U$ to be a pair of G-ANRs, $f : X \to U$ a G-map. Since $X/_G$ and $U/_G$ are ANRs, the fixed point index of the induced map $\overline{f} : X/_G \to U/_G$ is defined and so are the various indices of the induced maps $\overline{f_{\leq(H)}} : X_{\leq(H)}/_G \to U_{\leq(H)}/_G$ and $\overline{f_{<(H)}} : X_{<(H)}/_G \to U_{<(H)}/_G$ for closed subgroups $H \subseteq G$. So to each orbit type (H) of X, we can assign the integer

$$i_{(H)}(f, X, U) = i(\overline{f_{\leq(H)}}, X_{\leq(H)}/_G, U_{\leq(H)}/_G) - i(\overline{f_{<(H)}}, X_{<(H)}/_G, U_{<(H)}/_G).$$

To assemble all these values in a single object, we recall the definition of the tom Dieck ring $\mathbb{U}_G$ of a compact Lie group G. This is the free abelian group generated by the orbit types of closed subgroups of G, i.e.

$$\mathbb{U}_G = \bigoplus_{(H)} \mathbb{Z} \cdot (H).$$

The ring structure is of no particular interest for our purposes at this moment, roughly, it is induced by cartesian product of orbits.

Assume $X \subseteq U$ are G-ANRs with finite orbit type. Then the fixed orbit index of f is defined to be the element

$$i_G(f, X, U) = \sum_{(H)} i_{(H)}(f) \cdot (H) \in \mathbb{U}_G,$$

In particular, if $X = U$ is a compact G-manifold, the fixed orbit index of a G-map $f : X \to X$ is defined.

We have augmentation maps $\varepsilon_{(H)} = \mathbb{U}_G \to \mathbb{Z}$, $\varepsilon_{(H)}(a) = \sum_{(K)\leq(H)} \pi_{(K)}(a)$, where $\pi_{(K)} : \mathbb{U}_G \to \mathbb{Z}$ is the projection onto the (K)-th summand. Thus, $\varepsilon(H)$ takes all the coefficients of orbit types less or equal to (H) and sums them up.

We summarize the most important properties of the index.

Theorem 3.1.2.1 *Let $X \subseteq U$ be G-ANRs of finite orbit type, $f : X \to U$ a G-map such that f has no fixed orbits on the boundary of X. Then*

1. *If $V \subseteq U$ is a G-ANR such that $\overline{f(X)} \subseteq V$, then $i_G(f, X, U) = i_G(f, X, V)$.*
2. *If $X_0 \subseteq X$ is a G-ANR such that f has no fixed orbits in $X - \overline{X_0}$,*
$$i_G(f, X, U) = i_G(f, X_0, U).$$
3. *If $X_0, X_1 \subseteq X$ are G-ANRs such that $X_0 \cap X_1 = \emptyset$ and f has no fixed orbits in $X - \overline{X_0 \cup X_1}$, then*
$$i_G(f, X, U) = i_G(f, X_0, U) + i_G(f, X_1, U).$$
4. *If $H : X \times I \to U$ is a G-homotopy such that H_t has no fixed orbit on ∂X for all $t \in I$, then*
$$i_G(H_t, X, U) = i_G(H_0, X, U).$$
5. $\varepsilon_{(H)}(i_G(f, X, U)) = i(\overline{f_{\leq(H)}}, X_{\leq(H)}/G, U_{\leq(H)}/G)$.
6. *If $\varepsilon_{(H)}(i_G(f, X, U)) \neq 0$, then f has a fixed orbit of orbit type at most (H).*

Proof. 1. - 4. and 6. follow immediately from the corresponding properties of the ordinary fixed point index, 5. is a direct consequence of the definition of $\varepsilon_{(H)}$. □

Assume that M is a G-manifold and $f : M \to M$ a G-map with finitely many fixed orbits. By the properties of the fixed orbit index it is clear that the orbit index of f is the sum of local orbit indices, computed as follows. Take a fix orbit γ of f. Since γ is isolated, there is a pair of tubular neighbourhoods $U_\gamma \subseteq U'_\gamma$ such that γ is the unique fixed orbit of f in U'_γ and $f(\overline{U_\gamma}) \subseteq U'_\gamma$. The local fixed orbit index of f is the value
$$i_G(f, U_\gamma, U'_\gamma) \in \mathbb{U}_G.$$
We have
$$i_G(f, M, M) = \sum_\gamma i_G(f, U_\gamma, U'_\gamma),$$
the sum ranging over the finitely many fixed orbits of f. It is also evident, using G-homotopy invariance of the orbit index, that for general G-maps, the fixed orbit index can

be retrieved via approximation. The results of chapter two show that an approximation by maps with finitely many fixed orbits is possible.

We postpone examples to the end of chapter three, where we will calculate some fixed orbit indices and equivariant Fuller indices. Instead, we turn to the normalization of the fixed orbit index. This is an important feature that will guarantee computability and non-triviality of the index. Unfortunately, there is no completely satisfying formulation yet. The one given below is, however, a step in the proper direction.

Proposition 3.1.2.2 *Let M be a G-manifold and $f : M \to M$ a G-map, mapping all of M into a single orbit of type (H). Then the fixed orbit index of f is given as*

$$i_G(f, M, M) = (H).$$

PROOF. By Theorem 3.1.2.1, we can assume that $M = G \times_H \mathbb{B}$, where $\mathbb{B}$ is the closed unit ball in some H-representation V. Clearly, all the numbers $i_{(K)}(f, G \times_H V, G \times_H V)$ are zero for $(K) < (H)$, since the induced map has no fixed points in the respective sets. So we have to compute the index $i_{(H)}(f, G \times_H V, G \times_H V)$. We have $(G \times_H \mathbb{B})_{\leq (H)} = G \times_H \mathbb{B}$, $G \times_H \mathbb{B}/_G \cong \mathbb{B}/_H$ and f induces the constant zero map. But for self maps of compact ANRs, the fixed point index coincides with the Lefschetz number ([Fdl02] chapter 10.5), and the Lefschetz number of a constant map is 1. So we see that our index is just $1 \cdot (H)$ as claimed. $\square$

It would be interesting to compute the local index of a G-hyperbolic fixed orbit, but this seems rather difficult due to the general nature of the quotient spaces $V/_H$ for H-representations V.

3.2 The Equivariant Fuller Index

This section contains the main result of the work, which is the construction of an equivariant Fuller index, algebraically counting relative periodic orbits, distinguished by their orbit type. As before, we have a distinction of the development in two parts. In the first part we will define local indices for isolated relative periodic orbits and use them to define a global equivariant Fuller index.

In the second part, we will prove the main properties of the obtained equivariant Fuller index, foremost invariance under equivariant homotopies, but also additivity. As indicated several times, the layout of the section will strongly resemble the layout in the non-equivariant case, because it was our intention to find a method of construction that can be generalized by not much more than adding the equivariant labels.

3.2.1 Definition of the Equivariant Fuller Index

The general setup is as follows. Let M be a compact G-manifold, $\Omega \subseteq M$ an invariant open subset, $0 < a < b < \infty$ real numbers. Let ξ be a smooth vector field in $\mathfrak{X}_G(M, \Omega, a, b)$. So there are no relative periodic points on the boundary of $\Omega \times (a, b)$. We want to assign an algebraic homotopy invariant to ξ. Assume $\gamma \subseteq M \times \mathbb{R}$ is an isolated relative periodic orbit of ξ. Choose an isolating equivariant Poincaré system (D, D', P, t), centered at any point (x, T) in γ, where T is the period of γ. Let p be the minimal period, so that $T = k \cdot p$ for some $k \in \mathbb{N}$. Define the local index of γ as

$$I_G(\gamma) = \frac{1}{k} \otimes i_G(P, D', D) \in \mathbb{Q} \otimes \mathbb{U}_G,$$

where $\mathbb{U}_G$ is the tom Dieck ring

$$\mathbb{U}_G = \bigoplus_{(H)} \mathbb{Z},$$

(H) running through the orbit types of M. Note that, in principal, $i_G(P, D', D)$ lives in the tom Dieck ring associated with D instead of M. But we can naturally extend to the larger ring $\mathbb{U}_G$. The local index thus lives in the abelian group

$$\bigoplus_{(H)} \mathbb{Q}.$$

We will maintain the expression via the tensor product.

Assume that ξ has finitely many isolated essential relative periodic orbits $\gamma_1, \ldots, \gamma_n$. Let $p_1, \ldots, p_n$ be their minimal periods, $T_1, \ldots, T_n \in [a, b]$ their periods, such that $T_j = k_j \cdot p_j$ for some $k_j \in \mathbb{N}$ and every $j \in \{1, \ldots, n\}$. Choose equivariant Poincaré systems (D_j, D'_j, P_j, t_j) around any point on γ_j for each j. The *equivariant Fuller index* is defined to be

$$I_F^G(\xi, \Omega) = \sum_{j=1}^{n} I_G(\gamma_j) = \sum_{j=1}^{n} \frac{1}{k_j} i_G(P_j, D'_j, D_j).$$

Clearly, the equivariant Fuller index does not depend on the choice of Poincaré system, because any two of these for the same orbit are joined by an equivariant isotopy (namely the flow of ξ). For arbitrary G-vector fields $\xi \in \mathfrak{X}_G(M, \Omega, a, b)$, we define the equivariant Fuller index to be the limit of the Fuller indices of a sequence of equivariantly non-degenerate fields converging to ξ. We will see in the next section that this is well-defined.

3.2.2 Properties of the Index

We will now derive the three main properties one would expect of an index. Firstly, we will show additivity of the index. In principle, we just show additivity on the set of equivariantly non-degenerate fields. The general result is immediate once we have well-definedness. Secondly, we show that the index is invariant under G-homotopies. This is the hardest part and requires a careful generalization of the various lemmas used for the same result non-equivariantly. Homotopy invariance will yield well-definedness of the index, so we can proceed to prove the solution property of the index. This of course is very easy to deduce, provided the well-definedness result.

Proposition 3.2.2.1 *The equivariant Fuller index is additive, that is, if Ω_1, Ω_2 are disjoint, invariant, open subsets of Ω such that all essential relative periodic orbits of $\xi \in \mathfrak{X}_G(M, \Omega, a, b)$ are contained in $(\Omega_1 \cup \Omega_2) \times (a, b)$, then*

$$I_F^G(\xi, \Omega) = I_F^G(\xi, \Omega_1) + I_F^G(\xi, \Omega_2).$$

Proof. Assume ξ to be equivariantly non-degenerate and partition the set E of essential non-degenerate relative periodic orbits of ξ into E_1, corresponding to orbits in Ω_1, and E_2, corresponding to orbits in Ω_2. Then

$$I_F^G(\xi,\Omega) = \sum_{\gamma\in E} I_G(\gamma) = \sum_{\gamma\in E_1} I_G(\gamma) + \sum_{\gamma\in E_2} I_G(\gamma) = I_F^G(\xi,\Omega_1) + I_F^G(\xi,\Omega_2).$$

The general case follows by approximation. □

The idea to prove invariance under G-homotopies is the same as before: We reduce the question to invariance of the fixed orbit index under G-homotopies of maps. If $H : M \times I \to M$ is any equivariantly non-degenerate homotopy, we have finitely many bifurcation parameters. Let λ be such a bifurcation parameter, then we call the relative periodic orbits of H_λ which are the limit of a sequence of relative periodic orbits γ_n of H_{λ_n}, $\lambda_n \to \lambda$, $\lambda_n \neq \lambda$, the *limit periodic orbits* of H_λ. We will choose equivariant Poincaré systems around these orbits and vary them slightly to obtain Poincaré homotopies.

Proposition 3.2.2.2 *Let ξ_0, ξ_1 be two equivariantly non-degenerate G-vector fields that are G-homotopic via an equivariantly non-degenerate homotopy. Then their equivariant Fuller indices are equal.*

Proof. Since the fields are equivariantly non-degenerately homotopic, we can distinguish between two cases.

1. There are no bifurcation parameters in the interval $[\lambda_1, \lambda_2]$.

   Choose disjoint equivariant Poincaré systems around the finitely many relative periodic orbits of H_{λ_1}. These systems will be Poincaré systems around the relative periodic orbits for $\lambda_1 + \varepsilon$ for $\varepsilon > 0$ small. The Poincaré maps induce a non-degenerate homotopy of Poincaré maps, so the local fixed orbit indices of the fixed orbits corresponding to relative periodic orbits do not change. By compactness of $[\lambda_1, \lambda_2]$, the equivariant Fuller index does not change during this part of the homotopy.

2. There is precisely one bifurcation parameter $\lambda \in (\lambda_1, \lambda_2)$.

   In this case, H_λ is degenerate. Let γ be a limit relative periodic orbit of H_λ and p be its minimal period. Let (D, D', P, t) be an equivariant Poincaré system for γ, considered with minimal period p. By choosing D small enough, there is an extension of P to an equivariant Poincaré homotopy, again denoted by P, for $\mu \in [\lambda - \varepsilon, \lambda + \varepsilon]$ and some $\varepsilon > 0$, and we can achieve that the only fixed orbits of P_μ lying in D are those on branches converging to γ. Denote the finitely many branches converging to γ from the left of λ by
   $$\nu_1^k, \ldots, \nu_{r_k}^k,$$
   where k runs through the integers and indicates that the minimal period of ν_j^k approaches $k \cdot p$ for $j = 1, \ldots, r_k$ as the branch approaches γ. Let $P_- = P(-\varepsilon)$, $P_+ = P(\varepsilon)$. We choose small equivariant discs $M_1^k, \ldots, M_{r_k}^k$, centered at the fixed

orbits of P_- corresponding to the geometric orbits $\nu_1^k(-\varepsilon), \ldots, \nu_{r_k}^k(-\varepsilon)$. Furthermore, we choose equivariant subdiscs ${M'}_1^k \subseteq M_1^k, \ldots$, such that P_- restricts to a map ${M'}_j^k \to M_j^k$, $j = 1, \ldots, r_k$, for all k involved, and the iterates of P_- do so as well. We need only finitely many iterates of P_-, hence this condition can be fulfilled. We have a homotopy P between P_- and P_+ which is equivariantly non-degenerate at every stage except for the parameter λ. By Proposition 2.5.2.3, we find a homotopy P' arbitrarily close to P that is non-degenerate and has no fixed orbits on the union of the boundaries of the discs ${M'}_j^k$. In particular, P'_- has all its fixed orbits inside of the discs ${M'}_j^k$ for the various j, k and P'_- is admissibly homotopic to P_-, i.e. their fixed orbit indices are equal. But then, also P_- and P_+ are admissibly homotopic, so we find

$$i_G(P_-^k, D', D) = i_G(P_+^k, D', D)$$

for all k. For simplicity, write $H_{\lambda-\varepsilon} = H_-$, $H_{\lambda+\varepsilon} = H_+$. We claim that the equivariant Fuller indices are given by the sums

$$I_F^G(H_-, \Omega) = \sum_{n \cdot p \in [a,b]} \frac{1}{n} \cdot i_G(P_-^n, D'),$$

$$I_F^G(H_+, \Omega) = \sum_{n \cdot p \in [a,b]} \frac{1}{n} \cdot i_G(P_+^n, D'),$$

which would immediately yield equality of the two terms.

We calculate

$$I_F^G(H_-, \Omega) = \sum_{j \cdot k \cdot p \in [a,b]} \sum_{s=1}^{r_{jk}} \frac{1}{j} i_G(P_-^{jk}, M_s^k).$$

On the other hand, to calculate the fixed orbit index of P_-^n in D', note that the branches ν_s^k bifurcate with period $k \cdot p$ from γ. It is clear that Proposition 1.1.4.11 can be modified to see that such a bifurcation induces the bifurcation of a k-fold covering space in the group quotient. That is, we have k fixed orbits of the k-th iterate of the Poincaré map P_- bifurcating, all of which have the same local index, since their Poincaré systems are equivariantly isotopic via the flow. Their index is given by $i_G(P_-^k, M_s^k)$. Hence, we have a contribution of $k \cdot i_G(P_-^k, M_s^k)$ of these fixed orbits to the fixed orbit index of P_-^k in D'. Clearly, if k divides n, then P_-^n has these fixed orbits in M_s^k as well, and these contribute $k \cdot i_G(P_-^n, M_s^k)$ to the index of P_-^n. Summing all these indices up, we obtain

$$\begin{aligned} \frac{1}{n} i_G(P_-^n, D') &= \sum_{k \cdot j = n} \sum_{s=1}^{r_n} \frac{k}{n} \cdot i_G(P_-^n, M_s^k) \\ &= \sum_{k \cdot j = n} \sum_{s=1}^{r_{jk}} \frac{1}{j} \cdot i_G(P_-^{jk}, M_s^k). \end{aligned}$$

This finally gives

$$\sum_{n \cdot p \in [a,b]} \frac{1}{n} i_G(P_-^n, D') = \sum_{j \cdot k \cdot p \in [a,b]} \sum_{s=1}^{r_{jk}} \frac{1}{j} \cdot i_G(P_-^{jk}, M_s^k) = I_F^G(H_-, \Omega).$$

The whole calculation did not depend on the fact that we were working with H_- instead of H_+, and we get the same calculation on the right hand side, verifying equality of both equivariant Fuller indices.

Since the equivariant Fuller index remains unchanged in both cases and we have only finitely many bifurcation parameters, the proposition follows. □

This was the hard part of the invariance theorem. The rest follows easily by standard methods.

Lemma 3.2.2.3 *If two equivariantly non-degenerate vector fields $\xi_0, \xi_1 \in \mathfrak{X}_G(M, \Omega, a, b)$ are homotopic, then they are already equivariantly non-degenerately homotopic.*

Proof. Let $\mathcal{U}_0$ be a neighbourhood of ξ_0 such that all elements of $\mathcal{U}_0$ are equivariantly non-degenerate. Using the equivariant version of Lemma 1.1.1.2, we can furthermore achieve that all elements of $\mathcal{U}_0$ are pairwise homotopic via a homotopy not leaving $\mathcal{U}_0$. Hence, all elements of $\mathcal{U}_0$ are equivariantly non-degenerately homotopic. We can find a similar neighbourhood $\mathcal{U}_1$ of ξ_1. Now if $H \in h\mathfrak{X}_G(M, \Omega, a, b)$ is a homotopy joining ξ_0 and ξ_1, we find an equivariantly non-degenerate homotopy K arbitrarily close to H. In particular we can find such a K so that $K_0 \in \mathcal{U}_0$, $K_1 \in \mathcal{U}_1$. Pasting together K and equivariantly non-degenerate homotopies joining ξ_0 with K_0 and K_1 with ξ_1, respectively, we obtain an equivariantly non-degenerate homotopy joining ξ_0 and ξ_1. □

As before, being a homotopy invariant implies that the index is locally constant in the set of equivariantly non-degenerate fields, so it is well-defined for arbitrary G-vector fields by approximation.

Corollary 3.2.2.4 *The equivariant Fuller index is locally constant and hence well-defined.*

Proof. Any equivariant vector field $\xi \in \mathfrak{X}_G(M, \Omega, a, b)$ has a neighbourhood $\mathcal{U}$ such that every element of $\mathcal{U}$ is G-homotopic to ξ. Thus, all equivariantly non-degenerate elements in $\mathcal{U}$ are pairwise homotopic. By the preceeding lemma they are even non-degenerately homotopic, and so by Proposition 3.2.2.2, any two elements of $\mathcal{U}$ have the same equivariant Fuller index. □

We have finally arrived at the result that the equivariant Fuller index is a G-homotopy invariant.

Theorem 3.2.2.5 *The Fuller index is invariant under admissible G-homotopies.*

Proof. If $\xi_0, \xi_1 \in \mathfrak{X}_G(M, \Omega, a, b)$ are homotopic equivariant vector fields and $H \in h\mathfrak{X}_G(M, \Omega, a, b)$ is a G-homotopy between them, choose an equivariantly non-degenerate homotopy $K \in h\mathfrak{X}_G(M, \Omega, a, b)$ such that K_0 is in a given neighbourhood $\mathcal{U}_0$ of ξ_0, K_1 is in a given neighbourhood $\mathcal{U}_1$ of ξ_1. Since the Fuller index is locally constant in ξ, the theorem follows from Proposition 3.2.2.2. $\square$

We summarize all the properties of the equivariant Fuller index established so far. The normalization result is a bit unsatisfactory, since it reduces to the normalization of the fixed orbit index, which is not completely clarified.

Theorem 3.2.2.6 *The equivariant Fuller index has the following properties.*

1. *It is invariant under admissible G-homotopies, i.e. if $H \in h\mathfrak{X}_G(M, \Omega, a, b)$, then*

$$I_F^G(H_t, \Omega) \equiv const.$$

2. *It is additive, i.e. if Ω_1, Ω_2 are disjoint invariant open subsets of Ω and all essential relative periodic orbits of $\xi \in \mathfrak{X}_G(M, \Omega, a, b)$ are contained in $\Omega_1 \cup \Omega_2$, then*

$$I_F^G(\xi, \Omega) = i_F^G(\xi, \Omega_1) + I_F^G(\xi, \Omega_2).$$

3. *It is normalized. If ξ has a single relative periodic orbit in $\Omega \times (a, b)$ of periodicity k and (P, D', D, t) is an equivariant Poincaré system for the orbit, then*

$$I_F^G(\xi, \Omega) = \frac{1}{k} \otimes i_G(P, D', D).$$

4. *It has the solution property. If the projection $\pi_{(H)}$ to the (H)-component of $\mathbb{Q} \otimes \mathbb{U}_G$ of $I_F^G(\xi, \Omega)$ is not zero, then ξ has an essential periodic orbit in Ω of type at least (H).*

Proof. All statements are obvious, the last one being a trivial consequence of the corresponding result for the fixed orbit index. $\square$

3.3 Some Calculations

As is easily tested, computations of the equivariant Fuller index tend to drown in the enormous complexity provided by the interplay of the various isotropies. So we restrict our calculations to very simple cases with groups acting with few orbit types. In the case a group acts trivially, we obtain the classical Fuller index. In the case of free actions, we can give a general result.

Example 3.3.1 Let M be a free G-manifold, $\Omega \subseteq M$ open and invariant and let $\xi \in \mathfrak{X}_G(M, \Omega, a, b)$ be an equivariant vector field. Let $\bar{\xi}$ be the field induced on the quotient manifold $M/_G$. Then

$$I_F^G(\xi, \Omega) = I_F(\bar{\xi}, \Omega/_G) \cdot (e).$$

Indeed, we can assume that ξ is equivariantly non-degenerate. We choose disjoint equivariant Poincaré systems around the finitely many essential relative periodic orbits of ξ. The equivariant discs in this case are of the form $G \times D$, where D is an ordinary disc of the proper dimension. So our Poincaré maps are given as G-maps $P : G \times D' \to G \times D$. The fixed orbit index of such a map is calculated to be

$$i_G(P, G \times D', G \times D) = i_{(e)}(\overline{P}, G \times D'/_G, G \times D/_G) \cdot (e)$$

and the index $i_{(e)}(\overline{P}, G \times D'/_G, G \times D/_G)$ is just the ordinary fixed point index of the quotient map $\overline{P} : D' \to D$. But it is obvious that, if $(P, G \times D, G \times D', t)$ is an equivariant Poincaré system for ξ, then $(\overline{P}, D, D', \overline{t})$ is an ordinary Poincaré system for $\overline{\xi}$. This implies the result.

Finally, we will present a non-trivial and very concrete example.

Example 3.3.2 We take the manifold $\mathbb{S}^1$ with acting group $\mathbb{Z}_2$, acting as reflection at the y-axis. If we embed $\mathbb{S}^1 \subseteq \mathbb{C}$, then the action is given as $z \mapsto -\overline{z}$. The canonical maps to investigate on $\mathbb{S}^1$ are the maps $z \mapsto z^n$. To be equivariant under the $\mathbb{Z}_2$-action requires n to be odd. So we want to calculate the fixed orbit index of the maps $p_{2n+1} : \mathbb{S}^1 \to \mathbb{S}^1,\ z \mapsto z^{2n+1}$, $n = 0, 1, \ldots$.

- There are two orbit types in $\mathbb{S}^1$, namely the orbit type $(\mathbb{Z}_2)$ of the points i and $-i$, and the orbit type (e) of all other points. We have

$$\mathbb{S}^1_{\leq(\mathbb{Z}_2)} = \{i, -i\} \qquad \mathbb{S}^1_{\leq(e)} = \mathbb{S}^1$$

$$\mathbb{S}^1_{<(\mathbb{Z}_2)} = \emptyset \qquad \mathbb{S}^1_{<(e)} = \mathbb{S}^1_{(\mathbb{Z}_2)} = \{i, -i\}$$

- We have to calculate the numbers

$$i_{(H)}(f, \mathbb{S}^1, \mathbb{S}^1) = i_{(H)}(f) = i(\overline{f}, \mathbb{S}^1_{\leq(H)}/_G, \mathbb{S}^1_{\leq(H)}/_G) - i(\overline{f}, \mathbb{S}^1_{<(H)}/_G, \mathbb{S}^1_{<(H)}/_G)$$

  for $(H) \in \{(e), (\mathbb{Z}_2)\}$. The case $(H) = (\mathbb{Z}_2)$ is simple. The second summand is necessarily zero, and the first summand is the index of a map on the two point space $\{i, -i\}$. Our maps p_{2n+1} either fix both points or switch them, according to n being even or odd, respectively. If n is even, the fixed point index of the induced map is 2, if n is odd, the fixed point index is 0. We note

$$i_{(\mathbb{Z}_2)}(p_{2n+1}) = \begin{cases} 2 & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd.} \end{cases}$$

- For the index $i_{(e)}(p_{2n+1})$, we have to go into the quotient $\mathbb{S}^1/_{\mathbb{Z}_2}$, which can be identified with the interval $[-1, 1]$. A homeomorphism is given by $\mathbb{S}^1 \to [-1, 1]\,,\ (x + iy) \mapsto y$. We can embed this interval into $[-2, 2]$ and take $r : [-2, 2] \to [-1, 1]$ to be the obvious retraction.

A fixed orbit of p_{2n+1} is the orbit of an element $z \in \mathbb{S}^1$ such that $z^{2n+1} = z$ or $z^{2n+1} = -\overline{z}$, that is, z is either a $2n$-th root of unity or a $(2n+2)$-th root of -1. p_{2n+1} induces a polynomial map $q_{2n+1} : [-1, 1] \to [-1, 1]$ of degree $2n + 1$ which has $2n + 1$ pairwise distinct fixed points. The leading coefficient of q_{2n+1} is easily seen, using the binomial theorem, to be

$$\sum_{k=0}^{n} (-1)^n \binom{2n+1}{2k+1} = (-1)^n 2^{2n}.$$

The sign of the derivative of $\mathbb{1} - q_{2n+1}$ in the fixed points changes from each fixed point to the consecutive one. If n is odd, i and $-i$ are no fixed orbits of p_{2n+1}, hence -1 and 1 are no fixed points of q_{2n+1}. Since the leading coefficient of $\mathbb{1} - q_{2n+1}$ is positive, we start with a fixed point with index 1 so that the fixed point index of q_{2n+1} will be 1.

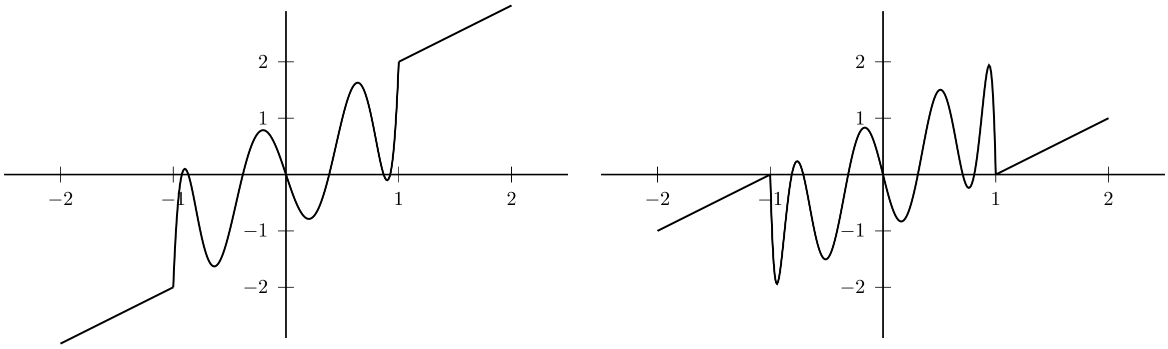
If n is even, i and $-i$ are fixed orbits of p_{2n+1} and -1 and 1 are fixed points of q_{2n+1}. To calculate the fixed point index of this map, concatenated with the retraction r, we have to approximate it by a smooth map. As is indicated in Figure 10, the indices of the fixed points -1 and 1 resolve to 0 during this approximation. But the leading coefficient of $\mathbb{1} - q_{2n+1}$ is now negative and since we have to discard the first and the last fixed point, the rest sums up to 1 again.

We conclude that

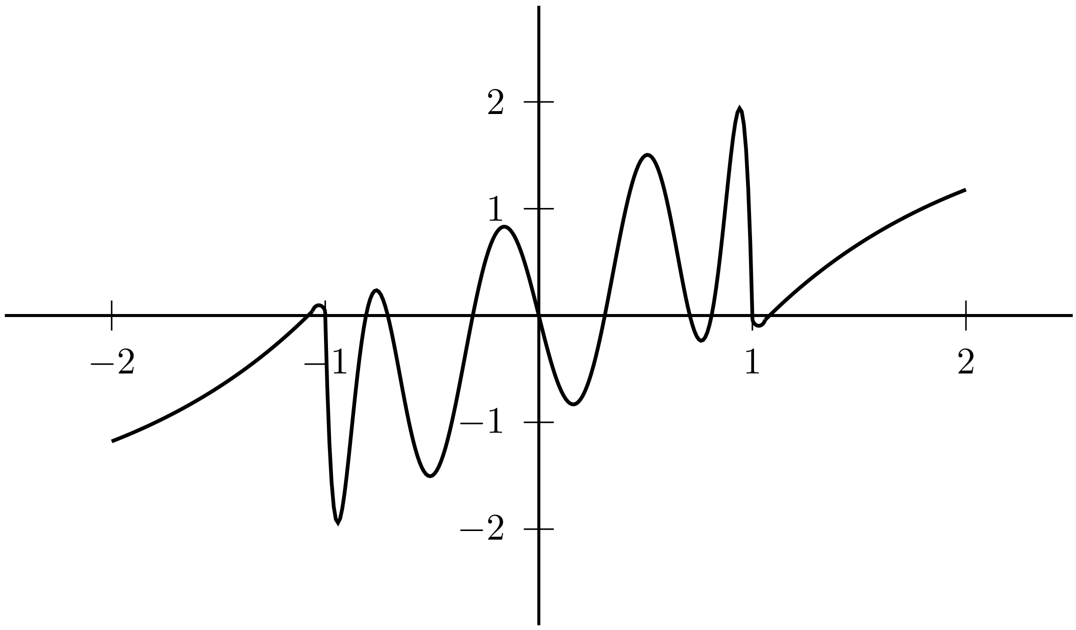
$$i_{(e)}(p_{2n+1}) = i(q_{2n+1}, [-1, 1], [-2, 2]) - i_{(\mathbb{Z}_2)}(p_{2n+1}) = \begin{cases} -1 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd.} \end{cases}$$

- We put the results together to obtain the global fixed orbit index of the map p_{2n+1} to be

$$i_G(p_{2n+1}, \mathbb{S}^1, \mathbb{S}^1) = \begin{cases} -1 \cdot (e) + 2 \cdot (\mathbb{Z}_2) & \text{if } n \text{ even} \\ (e) & \text{if } n \text{ odd.} \end{cases}$$



(a) $n = 3$ (b) $n = 4$



(c) $n = 4$ smoothed

Figure 10: The extensions of the functions $\mathbb{1} - q_{2n+1}$

4 Prospects and Comments

The work shall close with a short summary of what we have done and, in particular, what we have not done. Our aim was to construct a G-homotopy invariant, giving information on the relative periodic orbits of an equivariant vector field. We used a bifurcation theoretic approach to solve this problem. That is, we defined the invariant only for maps with finitely many critical elements, assigned a local index to each critical element, and summed them up to obtain the global invariant. The main difficulty was to prove that this definition could be extended to all maps via approximation. The solution came in two ways. First, we developed the theory of equivariant transversality to locally semialgebraic sets to obtain a notion of equivariant non-degeneracy. This allowed us to deduce that the classes we defined our invariant on are indeed generic. Then we proved homotopy invariance of the index, showing that it is well-defined through approximation. In some cases, we just merely managed to solve the underlying problems, where a more general solution is desirable and most probably true. The parts of the work where generalizations seem possible or other general comments seem in order will be sketched below.

1. Non-equivariantly, hyperbolicity is a stronger form of non-degeneracy, giving information on the dynamics around a critical element as well. As indicated several times, it is unclear whether G-hyperbolic critical elements are equivariantly non-degenerate. Such a result might help to a deeper geometric understanding of equivariant transversality.

2. Since it was not necessary, we did not contemplate long on genericity properties of hyperbolic homotopies. Using eigenvalue crossing conditions it is possible to show that the set of hyperbolic homotopies contains an open and dense subset, as is done in [Bru70]. Does a similar result hold true for G-hyperbolic homotopies? Can appropriate conditions be formulated using the reduced spectrum and equivariant eigenvalues of [Fie80]?

3. We still lack a proper normalization result for the fixed orbit index in the sense that we do not know the local index of an arbitrary G-hyperbolic fixed orbit. It might be conjectured that we should obtain something like $(-1)^s \cdot (H)$ if the orbit is of type (H), where s is the number of real eigenvalues larger one of the normal map, induced on the component of the tangential bundle normal to the group orbit. There is some, evidence for this fact by calculating simple examples.

4. The gap between the fixed orbit index and the equivariant Lefschetz number was already mentioned. The conjecture that, for self maps of compact G-manifolds, the global fixed orbit index and the equivariant Lefschetz number coincide is supported by examples, but remains to be proven. An approach using an axiomatic description of the equivariant Lefschetz number might work.

5. We defined the equivariant Fuller index in purely bifurcation theoretic terms. The non-equivariant Fuller index can be defined in homological terms, or rather there

is an even stronger invariant, a homological Fuller index, projecting to the usual one under an augmentation map, see [Fra90]. The construction of the homological index relies on intersection theoretic arguments and fundamentally on Poincaré duality. The equivariant cellular homology theories we studied do not exhibit Poincaré duality, so a generalization of the homological index to the case with symmetries seems hard in these theories. There is, however, a very interesting approach due to Costenoble, Waner and others, see [CW07], who defined non-integer graded equivariant homology and cohomology theories which restore Poincaré duality. These theories are hard to work with, especially since one has to deal with four theories instead of two: There are two dual theories, namely dual homology and dual cohomology, associated with the ordinary theories. A naive approach to an equivariant Lefschetz number would locate it in a dual homology group, and these seem very hard to compute.

6. Fuller index theory has long been applied to infinite dimensional dynamical systems, e.g. in [CMP78], as well as systems less regular than smooth flows on manifolds, e.g. in [Fen03]. It is highly expectable that similar generalizations of the equivariant Fuller index should be possible without further ado.

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