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0 Introduction

The theory of group actions lies at the heart of the mathematical interpretation of symmetry. The knowledge of the automorphisms of an object often provides a deep insight into the object itself. The automorphism group tells us under which transformations a certain object remains invariant. The 2-sphere $\mathbb{S}^2$ for example is invariant under all elements of the orthogonal group $O(3)$. Embedding a cube C into $\mathbb{S}^2$, this cube is invariant under much less linear automorphisms (in fact only finitely many). Of course, there are many more general homeomorphisms of $\mathbb{R}^3$ that leave the cube invariant than the linear ones. We see that it is important to specify what kind of transformations we want to allow. Furthermore, it is often hard to give a precise characterization of the group of all automorphisms. It is more convenient to find subgroups of automorphisms, and this amounts to the theory of group actions. At the beginning, we will take a look at some probably well-known examples, where the student might not be completely aware of the role of the acting groups.

These examples will come informally and just should provide a motivation as well as a background for the intuition on the subject. The formal background on group actions will be developed in the first chapter. We will have a quick review of basic category theory which will allow us to define a common framework of various kinds of group actions, unifying for example the two cases from the beginning. We will then turn to the main topic of this course, namely continuous group actions on topological spaces. Our main attention will be on the decomposition of spaces with group actions related to orbits and fixed spaces.

In the second chapter, we will develop the theory of fibre bundles and of G-principal bundles, which will lead to interesting classification theorems for these specific bundles related to group actions.

In the final chapter we will turn to smooth actions on manifolds, derive some of the classical results like the tubular neighbourhood theorem, and finally consider the structural decomposition of G-manifolds into orbit types.

0.1 Some Informal Examples

In this introductory section, we will recall two classical constructions where group actions play a major role, even though this point may not have been emphasized when developing these theories.

Covering Space Theory

Let X be a connected topological space. A covering space of X is a connected topological space E together with a map $p : E \to X$ such that the following holds. There is a discrete space F, called the fibre, and an open cover of X by open sets U_i with homeomorphisms $\varphi_i : p^{-1}(U_i) \to U_i \times F$ such that the diagrams

$$\begin{tikzcd} p^{-1}(U_i) \arrow[rr, "\varphi_i"] \arrow[dr, "p"'] & & U_i \times F \arrow[dl, "\pi_1"] \\ & U_i & \end{tikzcd}$$

commute. Usually one also requires the involved spaces to be Hausdorff. Here are some classical examples. The map $e : \mathbb{R} \to \mathbb{S}^1$, $t \mapsto \exp(2\pi i t)$ is a covering with fibre $\mathbb{Z}$. The maps $z_n : \mathbb{S}^1 \to \mathbb{S}^1$, $z \mapsto z^n$ are coverings with fibre $\{1, \ldots, n\}$. The maps $q_n : \mathbb{S}^n \to \mathbb{P}^n$ from the sphere to projective space are coverings with fibre $\mathbb{Z}_2$.

A fundamental property of covering spaces is the path lifting property. This property ensures that, given any path γ in X with starting point x_0 and any point $y_0 \in E$ mapping to x_0 under p, there is a unique path covering γ. This property also extends to homotopies and shows that homotopic paths lift to homotopic paths. Summarizing, one sees that the map p induces a monomorphism $p_* : \pi_1(E, y_0) \to \pi_1(X, x_0)$. The image of p_* consists of classes of loops at x_0 that lift to loops at y_0.

In the theory of covering spaces, one notices the following fact. Fixing any element in the normalizer of the image of p_*, which is a subgroup of $\pi_1(X, x_0)$, one obtains a self-map of E. Representing the group element by a loop α, the value of this map on a point $y \in E$ is constructed as follows. α can be lifted to a loop $L_{y_0}(\alpha)$ starting at y_0. Let λ be any path from y_0 to y. Then there is a lift γ of the path $p \circ \lambda$ to a path starting at $L_{y_0}(\alpha)(1)$. The lifting is possible since α is in the normalizer of the image of p_*. The value of our function on y is defined to be $\gamma(1)$. This construction specifies a map

$$\Phi : N(p_*(\pi_1(E, y_0))) \times E \to E$$

such that $\Phi(e, y) = y$ and $\Phi(\alpha \circ \beta, y) = \Phi(\alpha, \Phi(\beta, y))$. This is what we later will define to be a group action. If the normalizer $N(p_*(\pi_1(E, y_0)))$ happens to be all of $\pi_1(X, x_0)$, we call the covering regular and we have a group action of $\pi_1(X, x_0)$ on E. Identifying the point $y \in E$ with all the points $\alpha.y \in E$ yields a relation and the quotient space is easily seen to be homeomorphic to X. Under some mild assumptions on the base space X, one can conclude with the following result.

Theorem 0.1.1 *Let X be a connected, locally arcwise connected space with a simply connected covering space $\tilde{X}$. Then there is a one-to-one correspondence between isomorphism classes of covering spaces of X and subgroups of $\pi_1(X, x_0)$. The correspondence is given by*

$$H \mapsto \tilde{X}/_H, \quad E \mapsto p_*(\pi_1(E, y_0)).$$

Galois Theory

Let k be a field, K an algebraic extension of k, that is, K is a field containing k as a subfield and every element of K is a zero of a polynomial in $k[X]$. We make two assumptions on the extension.

(1) Every irreducible polynomial in $k[X]$ which has a zero in K splits into linear factors in K.

(2) The minimal polynomial in $k[X]$ of every element of K has pairwise distinct zeros in an algebraic closure.

An extension satisfying these two conditions is called a Galois extension of k and the group G of field automorphisms $\varphi : K \to K$ such that $\varphi(a) = a$ for every $a \in k$ is called the Galois group of K over k.

Theorem 0.1.2 *If K is a finite extension, then for every subfield $E \subseteq K$ of K containing k, there is a subgroup H of G such that*

$$E = K^H = \{a \in K \mid h(a) = a \,\forall\, h \in H\}.$$

E is a Galois extension if and only if H is normal in G. In that case, the restriction map $Aut_k(K) \to Aut_k(E)$ induces an isomorphism of $G/_H$ with the Galois group of E.

If we consider a separable polynomial in $k[X]$, a splitting field K of f is a Galois extension of k. The Galois group of such a splitting field is also called the Galois group of f. It has been shown that a polynomial equation is solvable in terms of roots of elements of k if and only if its Galois group G is solvable (hence the name). The action of the Galois group on K is given by permuting the roots of f. It is therefore clear that the same group will act in many different ways. For example, for any separable polynomial of prime degree and exactly two non-real roots, the Galois group can be shown to be isomorphic to S_p, the permutation group on p elements.

It is one of the aims of the theory of group actions to work out the differences and similarities of actions of the same abstract group in different situations.

1 Categorical Background

1.1 Categories and Functors

Definition 1.1.1 A category $\mathcal{C}$ is given by a class of objects, often denoted by $ob\,\mathcal{C}$, and for any two objects A, B of $\mathcal{C}$ a proper set of morphisms $\mathcal{C}(A, B)$, such that

1. there is an associative composition of morphisms

$$\circ : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C),\ (f, g) \mapsto g \circ f$$

2. for any object A of $\mathcal{C}$, there is an identity morphism $\mathrm{id}_A \in \mathcal{C}(A, A)$ which is a unit for the composition law.

A subcategory $\mathcal{C}'$ of $\mathcal{C}$ consists of a subclass of objects of $\mathcal{C}$, for every $A, B \in ob\,\mathcal{C}'$ we have $\mathcal{C}'(A, B) \subseteq \mathcal{C}(A, B)$, and composition law and identity element are inherited from $\mathcal{C}$.

This definition comprises the basic mathematical philosophy that mathematical objects can be understood by investigating their transformations. Almost every mathematical structure a student will encounter during his first years will be rooted in a category. Some examples are given below.

Example 1.1.2

1. The category $\mathcal{SET}$ of sets, together with maps between sets as morphisms. This category is of particular importance, since many categories are defined by adding extra structure to sets and morphisms of sets.

2. The category $\mathcal{VECT}_k$ of k vector spaces and linear maps between vector spaces as morphisms. More generally, for a ring R, there is the category $\mathcal{MOD}_R$ of modules over R together with R-linear maps as morphisms.

3. The category $\mathcal{TOP}$ of topological spaces and continuous maps as morphisms. This is one of the main categories we will be working with.

4. The category $\mathcal{MAN}$ of smooth finite dimensional real manifolds and smooth maps between them. Of course, there are similar categories for complex manifolds, infinite-dimensional manifolds etc. We will be working exclusively with finite-dimensional real manifolds.

5. The category $h\mathcal{TOP}$ of topological spaces and homotopy classes of continuous maps between them as morphisms. That is, for any two topological spaces X, Y, a morphism $[f] \in h\mathcal{TOP}(X, Y)$ is an equivalence class of continuous maps from X to Y, where two such maps f_0, f_1 are equivalent, if there is a continuous homotopy $H : X \times [0, 1] \to Y$ such that $H_0 = f_0$, $H_1 = f_1$. This is the first example of a category whose morphisms are not given as concrete maps.

6. A group G can be regarded as a category with a single object G with morphisms the elements of G.

Applying the basic philosophy to the definition of a category itself, we have to specify morphisms of categories. These are so called functors.

Definition 1.1.3 Let $\mathcal{C}, \mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is given by

(i) a map $ob\,\mathcal{C} \to ob\,\mathcal{D}$ (denoted by F as well)

(ii) for any pair A, B of objects of $\mathcal{C}$ a map

$$\mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))$$

(again denoted by F).

Furthermore, the map on morphism level is required to be compatible with identity and composition, that is, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and $F(f \circ g) = F(f) \circ F(g)$.

Again there are many examples that will be well known to the student, whereas the abstract interpretation may be unfamiliar.

Example 1.1.4

1. The functor $\mathcal{P} : \mathcal{SET} \to \mathcal{SET}$ assigns to a set X its power set $\mathcal{P}(X)$. On morphisms, if $f : X \to Y$ is a map, $\mathcal{P}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$ maps a subset $A \subseteq X$ to the subset $f(A) \subseteq Y$.

2. The functor $T : \mathcal{MAN} \to \mathcal{MAN}$ of the category of (smooth) manifolds assigns to a manifold M its tangential bundle TM. For a smooth map $f : M \to N$, Tf is the usual tangential map.

3. The functor $\pi_0 : \mathcal{TOP} \to \mathcal{SET}$ assigns to a topological space X the set $\pi_0(X)$ of its connected components. For a continuous map $f : X \to Y$, $\pi_0(f)$ maps a connected component C of X to the connected component of Y which contains $f(C)$ (since images of connected subsets are connected, this is well defined).

One can carry on and ask for transformations of functors and transformations of these transformations etc. We will only need the concept of a natural transformation between functors.

Definition 1.1.5 Let $\mathcal{C}, \mathcal{D}$ be categories and $F, G : \mathcal{C} \to \mathcal{D}$ functors. A natural transformation $\eta : F \to G$ is defined by a collection of morphisms $\eta_A : F(A) \to G(A)$ for every object A of $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\eta_A"'] & F(B) \arrow[d, "\eta_B"'] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

commutes for any two objects A, B of $\mathcal{C}$ and every morphism $f : A \to B$. A natural equivalence is a natural transformation which is an isomorphism for every object of $\mathcal{C}$.

Example 1.1.6 The above definition clarifies the meaning of the word natural in many mathematical contexts. As an example, let V be a k-vector space and let V'' be its second dual space. This specifies a functor

$$D : \mathcal{VECT}_k \to \mathcal{VECT}_k,\ V \mapsto V'',\ (f : V \to W) \mapsto (f'' : V'' \to W'').$$

Recall that the dual map of $f : V \to W$ is defined by $f' : W' \to V',\ f'(w')(v) = w'(f(v))$, so the double dual is defined as $f'' : V'' \to W'',\ f''(v'')(w') = v''(f'(w'))$. We now define a natural transformation between the identity functor and D, namely we define

$$\eta_V : V \to V'',\ v \mapsto (v' \mapsto v'(v)).$$

It is easy to see that the diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\eta_V"'] & W \arrow[d, "\eta_W"'] \\ V'' \arrow[r, "f''"] & W'' \end{tikzcd}$$

commutes, so indeed η is a natural transformation. It is well-known that η_V is a bijection if and only if V is finite dimensional.

1.2 Adjoint Functors

When one is interested in a certain functor $F : \mathcal{C} \to \mathcal{D}$ between two categories, from an abstract point of view there is not much else one can do than to investigate the morphism sets $\mathcal{D}(F(A), B)$ for objects A of $\mathcal{C}$ and B of $\mathcal{D}$. Under this aspect, it is often extremely useful, and in general has severe implications on the structure of the functor, that this morphism set can be identified with a morphism set $\mathcal{C}(A, G(B))$, where $G : \mathcal{D} \to \mathcal{C}$ is another functor. This amounts to the following definition.

Definition 1.2.1 Let $\mathcal{C}, \mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. If there is a natural equivalence

$$\mathcal{C}(G(A), B) \to \mathcal{D}(A, F(B))$$

for any objects A of $\mathcal{D}$, B of $\mathcal{C}$, then F is called a right adjoint functor of G and G is called a left adjoint functor of F.

As we already pointed out, being an adjoint functor has severe categorical implications, but it would lead us too much afar to discuss these here. Instead, we will give some examples.

Example 1.2.2 1. A very important case of adjoint functors arises in connection with forgetful functors. A forgetful functor is a functor $f : \mathcal{C} \to \mathcal{D}$ which, rather imprecisely, "forgets" some of the structure of the category $\mathcal{C}$. The functor $F : \mathcal{TOP} \to \mathcal{SET}$, assigning to a topological space its underlying set and to a continuous map its underlying set map is a typical example of a forgetful functor. Similar examples arise as forgetful functors $\mathcal{VECT}_k \to \mathcal{SET}$, $\mathcal{MAN} \to \mathcal{SET}$, $\mathcal{MAN} \to \mathcal{TOP}$.

We claim that the functor $F : \mathcal{TOP} \to \mathcal{SET}$ has a left adjoint $i : \mathcal{SET} \to \mathcal{TOP}$, assinging to the set X the topological space X with the discrete topology. Indeed, if we define

$$\eta : \mathcal{TOP}(i(X), Y) \to \mathcal{SET}(X, F(Y)),\ f \mapsto f,$$

this is a natural equivalence. Every set map $X \to F(Y)$ is continuous when X carries the discrete topology, no matter which topology is imposed on $F(Y)$. F also has a right adjoint, given by equipping a set with the indiscrete topology.

There is also a left adjoint funtor for the forgetful functor $F : \mathcal{VECT}_k \to \mathcal{SET}$. For a set X, we define $i(X)$ to be the free vector space with basis X, that is,

$$i(X) = \{\sum_{i \in I} \lambda_i x_i \mid I \text{ finite},\ \lambda_i \in k,\ x_i \in X\}.$$

For a map $f : X \to Y$, we define

$$i(f) : i(X) \to i(Y),\ \sum \lambda_i x_i \mapsto \sum \lambda_i f(x_i),$$

which is obviously well-defined and linear. To see that this functor is left adjoint to F, we define

$$\eta : \mathcal{SET}(X, F(V)) \to \mathcal{VECT}_k(i(X), V),\ f \mapsto \left(\sum \lambda_i x_i \mapsto \sum \lambda_i f(x_i)\right).$$

This is a natural equivalence, since every linear map $i(X) \to V$ is completely determined by the images of a base of $i(X)$, which is X, and the images of elements of the base carry no further restriction.

2. To see that adjoint functors can also occur in different contexts than with forgetful functors, we give another example. For a set X, consider the functor

$$\mathcal{SET}(X, \cdot) : \mathcal{SET} \to \mathcal{SET},\ Y \mapsto \mathcal{SET}(X, Y).$$

For a morphism $f : Y \to Z$, we define

$$\mathcal{SET}(X, f) : \mathcal{SET}(X, Y) \to \mathcal{SET}(X, Z),\ h \mapsto f \circ h.$$

We claim that this functor has a left adjoint, namely the functor

$$\cdot \times X : \mathcal{SET} \to \mathcal{SET},\ Y \mapsto Y \times X$$

and for $f : Y \to Z$,

$$f \times X : Y \times X \to Z \times X,\ (y, x) \mapsto (f(y), x).$$

We define a map

$$\eta : \mathcal{SET}(Y, \mathcal{SET}(X, Z)) \to \mathcal{SET}(Y \times X, Z),\ \eta(f)(y, x) = f(y)(x).$$

This map has an inverse, assigning to $g : Y \times X \to Z$ the map $\hat{g}$ such that $\hat{g}(y)(x) = g(y, x)$, hence, η is bijective. Naturality is easily checked, so the two functors are indeed adjoints.

Adjoint functors are unique up to natural equivalence, which we will prove now.

Proposition 1.2.3 *Let $\mathcal{C}, \mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$, $G, G' : \mathcal{D} \to \mathcal{C}$ be functors such that both G and G' are left adjoints of F. Then G and G' are naturally equivalent. The same holds for right adjoints.*

PROOF. For any object A of $\mathcal{D}$ and B of $\mathcal{C}$, let

$$\eta_{A,B} : \mathcal{C}(G(A), B) \to \mathcal{D}(A, F(B))$$

be an isomorphism such that η is natural in A and B, and similarly $\xi_{A,B}$ for G'. Let $\vartheta_A : G(A) \to G'(A)$ be the morphism specified by the image of the identity of $G'(A)$

under $\xi_{A,G'(A)}$ and $\eta^{-1}_{A,G'(A)}$. We claim that ϑ is a natural equivalence. Hence, let $f : A \to B$ be a morphism in $\mathcal{D}$. Consider the diagram

$$\begin{tikzcd}
\mathcal{C}(G'(A), G'(A)) \arrow[r, "\xi_{A,G'(A)}"] \arrow[d, "G'(f)\circ"'] & \mathcal{D}(A, F \circ G'(A)) \arrow[r, "\eta^{-1}_{A,G'(A)}"] \arrow[d, "F\circ G'(f)\circ"'] & \mathcal{C}(G(A), G'(A)) \arrow[d, "G'(f)\circ"'] \\
\mathcal{C}(G'(A), G'(B)) \arrow[r, "\xi_{A,G'(B)}"] & \mathcal{D}(A, F \circ G'(B)) \arrow[r, "\eta^{-1}_{A,G'(B)}"] & \mathcal{C}(G(A), G'(B)) \\
\mathcal{C}(G'(B), G'(B)) \arrow[r, "\xi_{B,G'(B)}"] \arrow[u, "\circ G'(f)"'] & \mathcal{D}(B, F \circ G'(B)) \arrow[r, "\eta^{-1}_{B,G'(B)}"] \arrow[u, "\circ f"'] & \mathcal{C}(G(B), G'(B)) \arrow[u, "\circ G(f)"']
\end{tikzcd}$$

By naturality, the diagram commutes. Starting with $\mathrm{id}_{G'(A)}$ in the upper left, going down gives $G'(f)$ and going to the middle right gives $G'(f) \circ \vartheta_A$. Similarly, starting with $\mathrm{id}_{G'(B)}$ in the lower left, going up gives $G'(f)$, and going to the middle right gives $\vartheta_B \circ G(f)$. Hence, $\vartheta_B \circ G(f) = G'(f) \circ \vartheta_A$ and the transformation ϑ is natural. It is obvious that the same reasoning holds for the transformation $\psi_A : G'(A) \to G(A)$, where the roles of $G'(A)$ and $G(A)$ in the definition of ϑ_A are reversed. To see that ψ_A is inverse to ϑ_A, consider the commutative diagram

$$\begin{tikzcd}
\mathcal{C}(G(A), G(A)) \arrow[r, "\eta_{A,G(A)}"] \arrow[d, "\vartheta_A\circ"'] & \mathcal{D}(A, F \circ G(A)) \arrow[r, "\xi^{-1}_{A,G(A)}"] \arrow[d, "F(\vartheta_A)\circ"'] & \mathcal{C}(G'(A), G(A)) \arrow[d, "\vartheta_A\circ"'] \\
\mathcal{C}(G(A), G'(A)) \arrow[r, "\eta_{A,G'(A)}"] & \mathcal{D}(A, F \circ G'(A)) \arrow[r, "\xi^{-1}_{A,G'(A)}"] & \mathcal{C}(G'(A), G'(A))
\end{tikzcd}$$

Starting with $\mathrm{id}_{G(A)}$ in the upper left, going down yields ϑ_A. By definition of ϑ_A, this maps to $\mathrm{id}_{G'(A)}$ through the lower row. But sending $\mathrm{id}_{G(A)}$ through the upper row and going down yields $\vartheta_A \circ \psi_A$, so this composition is the identity. The other direction follows in exactly the same manner and we conclude that ϑ is a natural equivalence. $\square$

2 Basic Theory of Transformation Groups

2.1 Definition and Examples of Group Actions

Definition 2.1.1 Let G be a group, X a set. An action of G on X is given by a map $\alpha : G \times X \to X$ such that $\alpha(e, x) = x$ and the diagram

$$\begin{tikzcd}
G \times G \times X \arrow[r, "\mathrm{id}_G \times \alpha"] \arrow[d, "\mu \times \mathrm{id}_X"'] & G \times X \arrow[d, "\alpha"] \\
G \times X \arrow[r, "\alpha"] & X
\end{tikzcd}$$

commutes.

An action of G on X induces a map $\rho : G \to Aut(X)$ into the bijective self maps of X. The conditions on α imply that this map is a group homomorphism. Conversely, any group homomorphism $\rho : G \to Aut(X)$ induces a group action by $\alpha(g, x) = \rho(g)(x)$. This can alternatively be seen by the natural equivalence

$$\mathcal{SET}(X \times Y, Z) \cong \mathcal{SET}(X, \mathcal{SET}(Y, Z)).$$

Taking the first set to be G, the two remaining sets to be X, the image of α under this map is ρ and vice versa. We can now generalize the definition in several directions. First, assume that $\mathcal{C}$ is a category whose objects are sets and whose morphisms are induced by set maps. We will call such a category a concrete category. Then we can require of a group action of G on an object C of $\mathcal{C}$ that the induced morphism $G \to Aut(C)$ actually maps into the $\mathcal{C}$-automorphisms of C, i.e. the map $c \mapsto \alpha(g, c)$ is a $\mathcal{C}$-morphism for every $g \in G$.

Secondly, let G be an object of $\mathcal{C}$ together with multiplication and inversion maps that turn it into a group when considered as a set. We call G a $\mathcal{C}$-group, if these multiplication and inversion maps are actual morphisms in the category $\mathcal{C}$. Then we can require the action map $\alpha : G \times C \to C$ to be a $\mathcal{C}$-morphism itself in addition to the requirement that ρ maps into $Aut_{\mathcal{C}}(C)$. In this case, we speak of a $\mathcal{C}$-action of the $\mathcal{C}$-group G on C.

Of course, we can turn the collection of $\mathcal{C}$-groups into a category by taking morphisms to be group homomorphisms that are $\mathcal{C}$-morphisms as well.

Example 2.1.2

1. Let G be a group. An action of G on a k vector space V is given by a map $\alpha : G \times V \to V$ such that each map $\alpha(g, \cdot)$ is a linear isomorphism, $\alpha(e, v) = v$ and the action diagram commutes. The adjoint map ρ in this case is a group homomorphism $G \to GL_k(V)$. A k vector space with a linear action of G is also called a representation of G. Representation spaces are of fundamental importance also for the theory of group actions in other categories.

2. Let k be a field. An action of a group G on k is given by an action map $\alpha : G \times k \to k$ such that the adjoint $\rho : G \to Aut(k)$ is a map into the field automorphisms of k.

3. Let G be a group with a topology such that multiplication and inversion are continuous maps. Then G is called a topological group. A continuous action of G on a topological space is given by a continuous map $G \times X \to X$ that is an action when considered as a $\mathcal{SET}$-map. The adjoint map is now a map $\rho : G \to Homeo(X)$. It is in general false that this map is continuous, even with reasonable topologies on $Homeo(X)$. It will be continuous if X is locally compact Hausdorff, as we will see in the exercises.

4. Let G be a group with a manifold structure such that multiplication is smooth. By the implicit function theorem it can be shown that inversion is automatically smooth in this case. G is called a Lie group. A smooth action of G on a manifold M is a smooth map $G \times M \to M$ which is an action when considered as a $\mathcal{SET}$-map. The adjoint map is a map $\rho : G \to Diffeo(M)$. If M is finite dimensional, it is possible to define a structure of a smooth manifold on $Diffeo(M)$, but it will

be infinite dimensional in general. In this case, the formulation via the action map α is more convenient.

5. Let V be a real or complex Banach space. Then we can consider linear actions of topological groups or Lie groups on V that are topological or smooth in addition, respectively. In general, representation theory for finite groups will not involve topological notions, whereas representation theory of infinite groups will require a topology on G.

We can proceed to define the category of symmetric objects in a category $\mathcal{C}$.

Definition 2.1.3 Let $\mathcal{C}$ be a concrete category. The category *Sym* $\mathcal{C}$ has as objects triples (G, A, α), where G is a group and α an action of G on A. A morphism between (G, A, α) and (H, B, β) is given by a pair (φ, f), with a group homomorphism $\varphi : G \to H$ and a morphism $f : A \to B$. These morphisms are required to fit into the commutative diagram

$$\begin{tikzcd} G \times A \arrow[r, "\alpha"] \arrow[d, "\varphi\times f"'] & A \arrow[d, "f"] \\ H \times B \arrow[r, "\beta"] & B \end{tikzcd} .$$

We have the subcategory of *Sym* $\mathcal{C}$ of G-objects in $\mathcal{C}$, objects being triples (G, A, α) as above with G fixed and morphisms of the form (id_G, f). In this case, the morphism $f : A \to B$ is called G-equivariant.

One can generalize this definition further by considering the subcategory where G is required to be a $\mathcal{C}$-group and $\varphi : G \to H$ is required to be a $\mathcal{C}$-group homomorphism. It will be clear from the context which of these symmetric categories we will be working in.

Our most prominent objects of study will be the symmetric objects in the categories $\mathcal{TOP}$ and $\mathcal{MAN}$. For a fixed group G, we will call these objects G-spaces and G-manifolds, respectively. It is also important to consider G-representations, but they will not be our main object of study.

There are some special properties of group actions which are easier to understand than the general case and nevertheless of interest.

Definition 2.1.4 A group action G on a set X is called

1. free, if $g.x = x$ implies $g = e$
2. effective, if the kernel of the action homomorphism ρ is trivial, equivalently, $g.x = x$ for all $x \in X$ implies $g = e$
3. transitive, if for any two elements $x, y \in X$ there is a $g \in G$ such that $g.y = x$.

Any action can be turned into an effective action by factoring out the kernel of the action homomorphism. So in some cases it suffices and will be easier to consider only effective actions. We will discuss the importance of free actions in the second chapter on fibre bundles.

Example 2.1.5 1. Let V be a real vector space. The general linear group $GL(V)$ of invertible linear maps $V \to V$ acts on V by evaluation. Many other interesting group actions are given as restriction of this action to subgroups. If V is finite dimensional, there is a unique topology on V, induced by any norm, that turns V into a locally compact Hausdorff space. So the evaluation action is continuous. More generally, it is continuous for any normed vector space V.

2. Let V be an inner product space. The group $O(V)$ of linear maps preserving the inner product acts on V by evaluation. By restriction, $O(V)$ acts on the unit sphere $S(V) \subseteq V$ as well, and so do all subgroups of $O(V)$. As above, these actions are continuous if V is finite dimensional or a normed vector space.

3. For $n \geq 3$, let D_n be the symmetry group of a regular n-gon. That is, the subgroup of linear maps of $\mathbb{R}^2$ leaving a regular n-gon centered at the origin fixed. D_n is generated by two elements τ and σ, where $\tau^2 = e$, $\sigma^n = e$ and $\sigma\tau\sigma = \tau$. We can realize this group concretely by letting τ be the reflection at the y-axis and σ be the element represented by the matrix

$$\begin{pmatrix} \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) \\ -\sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) \end{pmatrix}.$$

The groups D_n are called the dihedral groups.

4. The dihedral groups are examples of a useful construction from group theory, the semi-direct product. Let G, H be groups and assume that G acts on H via group automorphisms. That is, we have an action homomorphism $\rho : G \to Aut(H)$. The semidirect product of G and H is defined as the group $G \times H$ with group operation given by

$$(g, h) \circ (g', h') = (g \circ g', h \circ \rho(g)(h')).$$

To avoid confusion, this group is denoted by $G \times_\rho H$. The dihedral group D_n is isomorphic to the semidirect product $\mathbb{Z}_2 \times_\rho \mathbb{Z}_n$, where ρ is the canonical action of $\mathbb{Z}_2$ on $\mathbb{Z}_n$.

5. Let M be a compact manifold and $f : M \to TM$ a smooth vector field. From the theory of ordinary differential equations it is well known that f induces a global flow, that is, there are solution curves $\varphi(\cdot, x) : \mathbb{R} \to M$ such that $\varphi(0, x) = 0$ and $\dot{\varphi}(t, x) = f(\varphi(t, x))$. Uniqueness of solutions implies that $\varphi(t+s, x) = \varphi(t, \varphi(s, x))$. So we see that the flow of a smooth vector field on M is nothing but an action of the group $\mathbb{R}$ on M.

2.2 Topologies on Mapping Spaces

When we are dealing with topological actions $\alpha : G \times X \to X$ it is useful to know whether the adjoint map $\rho : G \to Homeo(X)$ is continuous with a suitable topology on $Homeo(X)$. Even more, one can ask if the group $Homeo(X)$ acts topologically on X in the obvious way. In this case, $Homeo(X)$ must be identified as a topological group and evaluation $Homeo(X) \times X \to X$ must be continuous.

The most basic topology on sets of continuous maps is the compact-open topology, which we will describe now. Let X, Y be topological spaces. Let $K \subseteq X$ be compact and $U \subseteq Y$ be open. Then we define

$$\mathcal{U}(K, U) = \{f : X \to Y \mid f \text{ continuous, } f(K) \subseteq U\}.$$

These sets, with K, U ranging over all compact subsets of X and open subsets of Y, respectively, form a subbasis of a topology on $\mathcal{TOP}(X, Y)$, the compact-open topology or CO-topology for short. We summarize the most interesting properties of this topology in the next proposition.

Proposition 2.2.1 *Let X, Y, Z be topological spaces, X, Y Hausdorff and Y locally compact.*

(i) The exponential law holds: The canonical map

$$\Phi : \mathcal{TOP}(X, \mathcal{TOP}(Y, Z)) \to \mathcal{TOP}(X \times Y, Z),\ \Phi(f)(x, y) = f(x)(y)$$

is a homeomorphism.

(ii) The evaluation map

$$ev : \mathcal{TOP}(Y, Z) \times Y \to Z,\ (f, y) \mapsto f(y)$$

is continuous in the CO-topology.

(iii) If X is locally compact as well, the composition map

$$\mathcal{TOP}(Y, Z) \times \mathcal{TOP}(X, Y) \to \mathcal{TOP}(X, Z),\ (f, g) \mapsto f \circ g$$

is continuous in the CO-topology.

Proof. Exercise. □

Another property of the CO-topology is that it preserves the Hausdorff property.

Proposition 2.2.2 *Let X, Y be topological spaces. If Y is Hausdorff, then the CO-topology on $\mathcal{TOP}(X, Y)$ is Hausdorff.*

Proof. Take two functions $f, g \in \mathcal{TOP}(X, Y)$ with $f \neq g$. Then there is an $x \in X$ such that $f(x) = y \neq z = g(x)$. Since Y is Hausdorff, we find disjoint open sets $U, V \subseteq Y$ with $y \in U$, $z \in V$. Since points are compact, the open sets $\mathcal{U}(x, U)$ and $\mathcal{U}(x, V)$ are obviously disjoint with $f \in \mathcal{U}(x, U)$, $g \in \mathcal{U}(x, V)$. We conclude that $\mathcal{TOP}(X, Y)$ is a Hausdorff space. □

We recall the adjunction

$$\mathcal{SET}(X \times Y, Z) \cong \mathcal{SET}(X, \mathcal{SET}(Y, Z)),$$

which gave rise to the dual characterization of group actions. In the environment of topological spaces, it would be desirable to have a similar adjunction

$$\mathcal{TOP}(X \times Y, Z) \cong \mathcal{TOP}(X, \mathcal{TOP}(Y, Z)),$$

in which case one would be able to derive continuity of ρ from that of α and vice versa. Unfortunately, the functor

$$\mathcal{TOP} \to \mathcal{TOP},\ X \mapsto X \times Y,\ f \mapsto f \times \mathbb{1}_Y$$

is not necessarily a left adjoint of the functor $\mathcal{TOP}(Y, \bullet)$. This is true, as we have seen, if Y is locally compact Hausdorff. However, we can not just restrict everything to the category of locally compact Hausdorff spaces, because the space $\mathcal{TOP}(X, Y)$ for X, Y locally compact Hausdorff is not necessarily locally compact. The solution of this problem is the introduction of a new topological category, the category of compactly generated weak Hausdorff spaces, $CGWH$-spaces for short. We will not go into detail of the definition here. It is a category containing all metric spaces and all locally compact Hausdorff spaces. Furthermore, function spaces of $CGWH$-spaces are still $CGWH$ and the adjunction of above holds. This is the reason why the theory of topological group actions is often carried out in the category of $CGWH$-spaces.

2.3 Topological Groups

So far we have defined group actions in a general context and we have seen some examples of actions in the category of topological spaces. To develop this theory further, we have to make a transgression to structure theory of topological groups. Recall that a topological group, or a $\mathcal{TOP}$-group, is a group together with a topology making group multiplication and inversion continuous.

Example 2.3.1

1. All groups are topological groups with the discrete topology. This topology is mainly of interest if G is finite.
2. A normed real or complex vector space is a topological group. Continuity of addition and inversion are immediate consequences of the triangle inequality.
3. $\mathbb{S}^1$ is the group of elements of norm 1 in the Banach space $\mathbb{C}$ and is topologized as a subspace of $\mathbb{C}$. Multiplication and inversion are induced from the respective maps on $\mathbb{C}$ and therefore easily seen to be continuous.
4. $\mathbb{S}^3$ is the group of unit quaternions and is topologized as a subspace of $\mathbb{C}^2$. Again, continuity of the group operations follows because they are induced by continuous maps on $\mathbb{H}$.

5. Let V be an inner product space. Then $L(V,V)$ is a Banach space with the induced operator norm and thus a topological group. The set $O(V)$ of orthogonal transformations is a subgroup and hence a topological group as well. In particular, the classical orthogonal groups $O(n)$ and the classical unitary groups $U(n)$ are topological groups.

6. Products of topological groups are topological groups with the product topology and the obvious group structure. In particular, the torus $\mathbb{S}^1 \times \mathbb{S}^1 \cong \mathbb{T}^2 \subseteq \mathbb{R}^3$ is a topological group.

7. As we have seen in the exercises, for a compact Hausdorff space X, $Homeo(X)$ is a topological group with the CO-topology.

An important feature of a topological group is that it acts on itself in various ways.

Lemma 2.3.2 *Let G be a topological group. Then G acts on itself in the following ways.*

1. *By left translation, that is, $g.h = g \circ h$.*
2. *By right translation, that is, $g.h = h \circ g^{-1}$.*
3. *By conjugation, that is, $g.h = g \circ h \circ g^{-1}$.*

Proof. One easily calculates that the defining diagrams commute and it remains to check continuity of the action maps. In the first case, the action map is just the group multiplication, so continuity is clear by definition. In the other two cases, the action map can be expressed as $\mu \circ (\mathbb{1}, i) \circ \tau$ and $\mu \circ (\mathbb{1}, i) \circ \tau \circ (\pi_1 \times \mu)$, respectively, where τ is the continuous transposition map $(g,h) \mapsto (h,g)$ and i is the inversion. □

Topological groups "look similar" around each of their points. In detail, since left translation is a homeomorphism, if V is a neighbourhood of g, then $g^{-1}V$ is a neighbourhood of e and vice versa. Moreover, since inversion is continuous, if U is a neighbourhood of e, $U^{-1} = \{g^{-1} \mid g \in U\}$ is a neighbourhood of e as well, and so is $V = U \cap U^{-1}$. This set has the property that $g \in V$ implies $g^{-1} \in V$. Such sets are called symmetric and it is clear that the symmetric neighbourhoods of e form a neighbourhood basis for e. By the initial remark on left translation, the topology of G is completely described by the symmetric neighbourhoods of e. We derive the existence of some additional special neighbourhoods of the identity element.

Proposition 2.3.3 *Let $g \in G$, U be a neighbourhood of e and let n be a positive integer.*

1. *There exists a neighbourhood V of e such that $VgV \subseteq gU$.*
2. *There exists a neighbourhood W of e such that $W^n \subseteq U$.*

Proof. 1. The map $\varphi : G \times G \to G$, $(h,k) \mapsto (h \circ g \circ k)$ is continuous, and so the preimage of gU under φ is open and contains (e,e). Since the sets $A \times B$, $A, B \subseteq G$ open, form a neighbourhood basis of the topology of $G \times G$, we find a neighbourhood V of e such that $V \times V \subseteq \varphi^{-1}(gU)$, i.e. $VgV \subseteq gU$.

2. This follows immediately from continuity of the multiplication map $G^n \to G$ and the same argument as above. $\square$

We turn our attention to the most basic constructions of group theory and their compatibility with topology. For a group G and a subgroup $H \subseteq G$, one can build the coset space ${G}/{H}$. This can be realized as the quotient of G by an equivalence relation, and so ${G}/{H}$ inherits the quotient topology from G. If H is a normal subgroup, ${G}/{H}$ is a group. So it is natural to ask whether the space ${G}/{H}$ is Hausdorff, or even a topological group if H is normal. The answer is positive in both cases, provided the subgroup H is closed. We need a lemma concerning the Hausdorff property of topological spaces.

Lemma 2.3.4 *Let $f : X \to Y$ be a continuous surjective and open map. Then Y is a Hausdorff space if and only if*

$$R = \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$$

is closed in $X \times X$.

Proof. If Y is Hausdorff, the diagonal $\Delta(Y) = \{(y, y) \mid y \in Y\}$ is closed. R is the preimage of the diagonal under the map $(f, f) : X \times X \to Y \times Y$, hence it is closed. Now assume R to be closed. The complement of R is open and the image of this complement under (f, f) is the complement of the diagonal in Y. So the diagonal is closed, which implies that Y is Hausdorff. $\square$

Proposition 2.3.5 *Let H be a subgroup of the topological group G. Then the canonical projection is open. H is closed if and only if ${G}/{H}$ is a Hausdorff space. If H is closed and normal, ${G}/{H}$ is a topological group.*

Proof. Let $p : G \to {G}/{H}$ be the projection. Then for $U \subseteq G$ open we have to show that $p^{-1}(p(U))$ is open. But

$$p^{-1}(p(U)) = \{gh \mid g \in U,\ h \in H\} = \bigcup_{h \in H} Uh$$

is open as a union of open sets. For the Hausdorff property, assume first that ${G}/{H}$ is Hausdorff. Then the point $[e] \in {G}/{H}$ is closed, and its preimage under the projection p is just H, so H is closed. Conversely, since p is continuous, surjective and open we can apply the preceeding lemma to see that ${G}/{H}$ is Hausdorff if and only if the set

$$R = \{(g, h) \mid p(g) = p(h)\}$$

is closed in $G \times G$. We can write R alternatively as

$$R = \{(g, gh) \mid g \in G,\ h \in H\}.$$

Looking at the map $f : G \times G \to G,\ (g, h) \mapsto g^{-1}h$, we have that f is continuous, and the preimage of H under f is precisely R. Hence, R is indeed closed.

Finally, to see that in case H is normal we obtain a topological group, we have to check continuity of the group operations. For this purpose, we define $\eta : G \times G \to G,\ (g,h) \mapsto g^{-1}h$ and let ν be the map induced by η in the quotient. Then it suffices to show that ν is continuous. We have the diagram

$$\begin{tikzcd} G \times G \arrow[r, "\eta"] \arrow[d, "p\times p"'] & G \arrow[d, "p"] \\ G/_H \times G/_H \arrow[r, "\nu"] & G/_H \end{tikzcd} .$$

Hence, if $U \subseteq G/_H$ is open, then

$$\nu^{-1}(U) = (p \times p)\eta^{-1}p^{-1}(U).$$

η and p are continuous and p is open, implying that $p \times p$ is open as well, so this set is open, proving continuity of ν. □

We remark that, by the preceeding result, a topological group is Hausdorff if and only if $\{e\}$ is a closed subset. For example, T_1-topological groups are already Hausdorff.

The question remains what happens if the subgroup H is not closed. In this case, $G/_H$ will never be Hausdorff, however, one does not have to go far to rectify the situation.

Proposition 2.3.6 *Let $H \subseteq G$ be a subgroup. Then the closure $\overline{H}$ of H is a subgroup as well. If H is normal, so is $\overline{H}$.*

Proof. Define $\eta : G \times G \to G,\ (g,h) \mapsto gh^{-1}$. We have to show that $\eta(\overline{H} \times \overline{H}) \subseteq \overline{H}$. We have

$$\eta(\overline{H} \times \overline{H}) = \eta(\overline{H \times H}) \subseteq \overline{\eta(H \times H)} = \overline{H},$$

since H is a subgroup. For normality, the map $c_g : G \to G,\ h \mapsto ghg^{-1}$ is a homeomorphism, hence,

$$c_g(\overline{H}) \subseteq \overline{c_g(H)} = \overline{H}$$

because H is normal. □

2.4 Basic Properties of G-Spaces

In this section we will study the elementary properties of topological spaces with a continuous action by a topological group.

We write $\rho(g)(x) = \alpha(g,x) = g.x$ in this context, and a continuous map $f : X \to Y$ between G-spaces is G-equivariant, if it satisfies the condition $f(g.x) = g.f(x)$ for all $g \in G$, $x \in X$. Note the different meaning of the dot. On the left hand side it stands for the action of G on X, whereas on the right hand side it stands for the action of G on Y.

Sometimes it is useful to consider a G-space with an action map $\alpha : G \times X \to X$ as a left G-space, whereas a map $X \times G \to X$, making the analogous diagrams to the action diagrams commute, specifies the structure of a right G-space. Of course, from a more general point of view, this is only an artificial distinction. A left action always induces a right action by $x.g = g^{-1}.x$.

We continue to sketch some basic topological properties of G-spaces. First of all, it is important to note that a G-action induces a special kind of equivalence relation.

Definition 2.4.1 Let X be a G-space. Then there is an equivalence relation on X, specified by $x \sim y \iff \exists\, g \in G : g.x = y$. The quotient space of this relation is denoted by $X/_G$ and it is called the quotient space of X by G.

In relation to taking group quotients, the following objects are of importance, and they will turn out to be prevalent throughout the theory.

Definition 2.4.2 Let X be a topological space with an action of the topological group G and let x be an element of X.

1. The set of points
$$Gx = \{g.x \mid g \in G\}$$
of translates of the point x is called the orbit of x.

2. The set
$$G_x = \{g \in G \mid g.x = x\}$$
of elements of G fixing the point x is called the isotropy subgroup of x.

The quotient space of a group action therefore can be regarded as the space whose points are G-orbits in X. The isotropy groups can be interpreted as a measure of symmetry for a point. The larger the isotropy, the more symmetric the point. We have to clarify the word "subgroup" in the definition of the isotropy subgroup.

Lemma 2.4.3 *The isotropy subgroup G_x of a point is a subgroup of G. If X is a T_1-space, G_x is closed.*

Proof. For $g, h \in G_x$ we have to show that $gh^{-1} \in G_x$. We have
$$gh^{-1}.x = gh^{-1}.hx = g.x = x,$$
since h and g both fix x. So G_x is a subgroup. For closedness, we note that G_x is the preimage of the point $(x, x) \in X \times X$ under the map
$$G \to X \times X,\ g \mapsto (x, gx).$$
If X is T_1, points are closed, so G_x is closed. □

One of the fundamental concepts of the theory of transformation groups is that all involved objects shall become invariant under the group action. From a general point of view, since orbits are the smallest G-invariant subspaces of the space X, they take the role of points from general topology.

The following result identifies the structure of all possible orbits that may occur in a space. It is completely described by the acting group G.

Proposition 2.4.4 *Let X be a G-space. There is a canonical bijection*

$$\varphi : G/_{G_x} \to Gx,\ [g] \mapsto g.x.$$

If G is compact Hausdorff and X is Hausdorff, this map is a homeomorphism.

PROOF. The map is well defined since $[gh]$ for $h \in G_x$ maps to $gh.x = g.(h.x) = g.x$. Surjectivity is obvious and for injectivity, if $g.x = h.x$ for $g, h \in G$, then $h^{-1}g.x = x$, or equivalently, $h^{-1}g \in G_x$. This shows that $[g] = [h] \in G/_{G_x}$.

The map is continuous by definition of the quotient topology, since the map $G \to Gx,\ g \mapsto gx$ is continuous as the composition $G \xrightarrow{\mathrm{id}_G \times x} G \times X \xrightarrow{\mu} X$. As a continuous bijection between a compact space and a Hausdorff space, it is a homeomorphism. $\square$

Example 2.4.5

1. Consider the orthogonal group $O(n)$, acting on $\mathbb{R}^n$ in the canonical way. Let $v \in \mathbb{R}^n \setminus \{0\}$ be a non-zero vector. Then v is fixed by all rotations around the axis $\mathbb{R}v$ and the corresponding reflections. This group is the full orthogonal group of the subspace $\mathbb{R}v^{\perp}$. Hence, the isotropy subgroup $O(n)_v$ of v can be identified with $O(n-1)$. The $O(n)$-orbit through v is the sphere of radius $\|v\|$. So by the preceeding theorem, we obtain a homeomorphism

$$\mathbb{S}^{n-1} \cong O(n)/_{O(n-1)}.$$

2. The Stiefel manifold $V_{k,n}$ of k-frames in $\mathbb{R}^n$ is given by k-tuples of orthonormal vectors $v_1, \dots, v_k \in \mathbb{R}^n$. This can be seen as a subset of $\mathbb{R}^{k \cdot n}$ and inherits the subspace topology. Again, the group $O(n)$ acts on $V_{k,n}$ in the obvious way. An element $A \in O(n)$ fixes the element $(v_1, \dots, v_k) \in V_{k,n}$, if it is an element of the orthogonal group of the subspace

$$\mathbb{R}v_1^{\perp} \cap \dots \cap \mathbb{R}v_k^{\perp}.$$

This is just the orthogonal complement of $\langle v_1, \dots, v_k \rangle$ and so the isotropy subgroup is homeomorphic to $O(n-k)$. If $(v_1, \dots, v_k)$ and $(w_1, \dots, w_k)$ are two elements of $V_{k,n}$, we find an element A_i of $O(2)$, acting in the (v_i, w_i)-plane, sending v_i to w_i for $1 \le i \le k$. A_i specifies an element of $O(n)$ by letting it act trivially on the space orthogonal to the (v_i, w_i)-plane. In particular, the elements v_j for $i \neq j$ remain fixed under A_i. We see that the composition $A_1 \circ \dots \circ A_k$ is an element of $O(n)$ that sends $(v_1, \dots, v_k)$ to $(w_1, \dots, w_k)$, i.e. the action is transitive. Hence, we obtain a homeomorphism

$$O(n)/_{O(n-k)} \cong V_{k,n}.$$

3. Let $G_{k,n}$ be the set of k-dimensional subspaces of $\mathbb{R}^n$. We have an obvious map $\pi : V_{k,n} \to G_{k,n}$, sending a k-tupel to the space it spans. $G_{k,n}$ is topologized by the quotient topology with respect to π. Then $O(n)$ acts on $G_{k,n}$ in the obvious way. It is also obvious that this action is transitive (it is induced by the transitive action

on $V_{k,n}$). An element of $O(n)$ fixes a given subspace V if either it acts on the space orthogonal to V, or it fixes the space orthogonal to V, or it is a combination of both. Hence, the isotropy subgroup $O(n)_V$ is homeomorphic to $O(n-k) \times O(k)$. We conclude that

$$G_{k,n} \cong O(n)/_{O(n-k) \times O(k)}.$$

The spaces $G_{k,n}$ are called Grassmannian manifolds and are important spaces throughout topology, especially in the area of bundle theory. We note that $G_{1,n}$ is real projective $n-1$-space, and we have identified

$$\mathbb{R}P^{n-1} \cong O(n)/_{O(n-1) \times O(1)} \cong O(n)/_{O(n-1) \times \mathbb{Z}_2}.$$

One point of interest is, in what way the topological structure of a G-space is preserved by passing to a subset or to the quotient. The following proposition summarizes some elementary properties concerning these questions.

Proposition 2.4.6 *Let X be a G-space.*

(i) If $A \subseteq G$, $B \subseteq X$ are open, then $AB \subseteq X$ is open.

(ii) If $A \subseteq G$, $B \subseteq X$ are compact, then $AB \subseteq X$ is compact.

(iii) If $A \subseteq G$ is compact, $B \subseteq X$ is closed, then AB is closed.

(iv) The projection $p : X \to X/_G$ is an open map.

(v) If G is compact and X is Hausdorff, then $X/_G$ is Hausdorff.

PROOF. (i) We have

$$AB = \bigcup_{a \in A} \rho(a)(B).$$

Since $\rho(a)$ is a homeomorphism, this is a union of open sets, hence open.

(ii) The product $A \times B \subseteq G \times X$ is compact and AB is the image of $A \times B$ under the continuous action map α, hence compact.

(iii) Let $x \in X$ be an element not contained in AB. For any $a \in A$, $X \setminus \rho(a)(B)$ is open, so there are neighbourhoods V_a of $e \in G$, V_a symmetric, and W_a of x such that

$$V_a W_a \subseteq X \setminus \rho(a)(B)$$

and consequently, $W_a \cap V_a^{-1}\rho(a)(B) = W \cap V_a\rho(a)(B) = \emptyset$. By compactness of A, there are finitely many elements $a_1 \ldots, a_n \in A$ such that $A \subseteq \bigcup_{i=1}^n V_{a_i} a_i$. Let W be the intersection of the corresponding sets W_{a_i}. Then W is a neighbourhood of x and $W \cap AB = \emptyset$. Otherwise, we would have $ab \in W$ with $a \in V_{a_i} a_i$ for some $1 \leq i \leq n$. But

$$W \cap V_{a_i}\rho(a_i)(B) \subseteq W_{a_i} \cap V_{a_i}\rho(a_i)(B) = \emptyset.$$

(iv) Let $U \subseteq X$ be open. By definition of the quotient topology, $p(U) \subseteq X/_G$ is open if and only if $p^{-1}(p(U))$ is open in X. We have

$$p^{-1}(p(U)) = \bigcup_{g \in G} \rho(g)(U),$$

which is indeed open.

(v) A space X is a Hausdorff space if and only if the diagonal $\Delta(X) = \{(x, y) \in X \times X \mid x = y\}$ is closed in the product topology. Assume X to be Hausdorff. Then the preimage under p of the complement of the diagonal in $X/_G$ is given by

$$\begin{aligned} p^{-1}(\Delta(X/_G)^c) &= \{(x, y) \in X \times X \mid y \neq gx \,\forall\, g \in G\} \\ &= \{(x, gx) \in X \times X \mid x \in X, g \in G\}^c. \end{aligned}$$

By openness and surjectivity of p, the diagonal in $X/_G$ is closed if and only if $\{(x, gx) \in X \times X \mid x \in X, g \in G\}$ is closed. But this set is the image of the set $G \times \Delta(X)$ under the action map $G \times X \times X \to X \times X$, $(g, x, y) \mapsto (x, gy)$, and therefore it is closed by (iii). □

The condition that G is compact is indeed necessary and many highly pathological examples arise from actions of non-compact groups. Consider the torus $\mathbb{T}^2$, given by the unit square $[0, 1]^2$ with its boundary identified in the proper way. For an element $v \in \mathbb{R}^2$, define an action α_v of $\mathbb{R}$ on $\mathbb{R}^2$ by

$$\alpha_v : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2,\ (\lambda, x) \mapsto x + \lambda \cdot v.$$

This induces an action on the torus $\mathbb{T}^2$. Assume now that $v = (a, b)$ and $\frac{a}{b}$ is irrational. Then $\lambda.x \neq x$ for every $\lambda \neq 0$ and every $x \in \mathbb{T}^2$. Otherwise, we would have an element $(y, z) \in \mathbb{R}^2$ and integers $k, \ell \in \mathbb{Z}$ such that

$$(y, z) + \lambda \cdot (a, b) = (y + k, z + \ell)$$

and in conclusion, $\frac{a}{b} = \frac{k}{\ell}$ which is impossible. In this case, every orbit is dense. To see this, let $x \in \mathbb{R}^2$ be any element and let $\lambda_k \in \mathbb{R}$ be the element such that $x_1 + \lambda_k \cdot a = k$ for $k \in \mathbb{Z}$. We have

$$x_2 + \lambda_k \cdot b = x_2 + \frac{k - x_1}{a} \cdot b = x_2 + k \cdot \frac{a}{b} - \frac{x_1}{a} \cdot b.$$

The second component of the point $\lambda_{k+1}.x$ differs from the second component of $\lambda_k.x$ by $\frac{a}{b}$, i.e. we have an induced action of the group $\mathbb{Z}$ on $\mathbb{S}^1 \subseteq \mathbb{T}^2$ given by

$$\mathbb{Z} \times \mathbb{S}^1 \to \mathbb{S}^1,\ n.[0, x] = [0, x + n \cdot \frac{a}{b}].$$

We claim that every orbit of this action is dense. In that case, every orbit of our original action would meet the circle $\{[0, y] \in \mathbb{T}^2\}$ in a dense subset. The orbit of the circle clearly is all of $\mathbb{T}^2$, hence, every orbit would be dense.

So let $A \subseteq \mathbb{S}^1$ be the closure of an orbit. Then $\mathbb{S}^1 \setminus A$ is open and therefore is the union of disjoint intervals. Let I be the longest of these intervals. Then $n.I \cap I = \emptyset$ for every $n \in \mathbb{Z} \setminus \{0\}$. Otherwise, either their intersection would be an interval longer than I, or $n.I = I$. But in that case, an endpoint $[0, x_0]$ of the interval would map to itself, which we have seen is not possible. But then, the intervals $n.I$ are pairwise disjoint and of equal length, which is impossible unless $I = \emptyset$.

We conclude that every orbit of the $\mathbb{R}$-action on $\mathbb{T}^2$ is dense. So an open subset in the quotient $\mathbb{T}^2/_{\mathbb{R}}$ containes all orbits in a small neighbourhood of a given orbit, meaning that it contains all of $\mathbb{T}^2/_{\mathbb{R}}$. The topology on the quotient therefore is trivial, in particular it is not Hausdorff.

Next, we will make use of Proposition 2.4.6 to prove some useful mapping properties of the action map itself.

Proposition 2.4.7 *Let X be a G-space and $\alpha : G \times X \to X$ be the action map. Then α is open. If G is compact, α is closed.*

PROOF. The first part is considerably easier than the second. Take $U \subseteq G \times X$ open and let $U_g = \{x \in X \mid (g, x) \in U\}$. Then U_g is the preimage of U under the inclusion $i_g : X \to G \times X,\ x \mapsto (g, x)$, so U_g is open. Let $V_g = \{g\} \times U_g$. We have

$$U = \bigcup_{g \in G} V_g,$$

hence,

$$\alpha(U) = \alpha\left(\bigcup_{g \in G} V_g\right) = \bigcup_{g \in G} \rho(g)(U_g).$$

Since $\rho(g)$ is a homeomorphism, $\rho(g)(U_g)$ is open and so the image of U under α is as well.

For closedness of α, let $C \subseteq G \times X$ be closed. Define an action of G on $G \times X$ by

$$\beta : G \times (G \times X) \to G \times X,\ (g, h, x) \mapsto (hg^{-1}, g.x).$$

This is the canonical diagonal action on the product $G \times X$, where G is considered as a left G-space via right translation. We claim that

$$\beta(G \times C) = \alpha^{-1}(\alpha(C)).$$

Indeed, take $g \in G$ and $(h, x) \in C$. Then

$$\alpha \circ \beta(g, h, x) = \alpha(hg^{-1}, g.x) = h.x = \alpha(h, x) \in \alpha(C).$$

Conversely, if $(h, x) \in \alpha^{-1}(\alpha(C))$, then $\alpha(h, x) \in \alpha(C)$ and we find $(g, y) \in C$ such that $h.x = \alpha(h, x) = \alpha(g, y) = g.y$. Thus, $h^{-1}g.y = x$ or $\alpha(h^{-1}g, y) = x$. This yields

$$\beta(h^{-1}g, g, y) = (gg^{-1}h, h^{-1}g.y) = (h, x).$$

$\beta(G \times C)$ is closed by Proposition 2.4.6 (iii), and so $\alpha^{-1}(\alpha(C))$ is closed, implying that $\alpha^{-1}(\alpha(C))^c$ is open. By elementary set theory,

$$\alpha^{-1}(\alpha(C))^c = \alpha^{-1}(\alpha(C)^c),$$

so this last set is seen to be open. By surjectivity of α, $\alpha(\alpha^{-1}(A)) = A$ for every subset $A \subseteq X$. By what we have already proven, α is an open map and so

$$\alpha(\alpha^{-1}(\alpha(C)^c)) = \alpha(C)^c$$

is open, showing that $\alpha(C)$ is closed. □

As we already pointed out, the consideration of G as a G-space, with the action being either translation or conjugation, is of particular importance. These actions induce actions on the quotients $G/_H$, for $H \subseteq G$ a closed subgroup, and the behaviour of these spaces is to be investigated in the following. If $f : X \to Y$ is an equivariant map, it maps orbits into orbits by definition. So, f restricted to an orbit exhibits the behaviour that is described in the next proposition.

Proposition 2.4.8 *Let G be a group, H, K subgroups of G and let G act (as a $\mathcal{SET}$-group) on $G/_H$, $G/_K$ via left translation.*

(i) If $\varphi : G/_H \to G/_K$ is a G-map, then it has the form $\varphi([g]) = [ga]$ for some $a \in G$ such that $a^{-1}Ha \subseteq K$ and every element a satisfying this condition specifies a G-map $G/_H \to G/_K$.

(ii) Two G-maps $\varphi, \psi : G/_H \to G/_K$, defined as in (i) by elements $a, b \in G$, respectively, are equal if and only if $ab^{-1} \in K$.

Proof. Let $\varphi : G/_H \to G/_K$ be a G-map. Then $\varphi([g]) = g.\varphi([e]) = g.[a] = [ga]$ for some $a \in G$. Substituting gh for g with any $h \in H$ yields

$$[ga] = \varphi([g]) = \varphi([gh]) = gh.\varphi([e]) = [gha],$$

so we must have $a^{-1}Ha \subseteq K$. On the other hand it is obvious that any such element a defines a G-map. For the second statement, in particular we have $[a] = [b]$, hence $ab^{-1} \in K$. □

Since G-maps between orbits have the simple structure we have just determined, every set G-map between orbits is automatically continuous. Even more is true. An equivariant self map of an orbit is automatically a G-homeomorphism.

Proposition 2.4.9 *If G is a compact Hausdorff topological group and $H \subseteq G$ a closed subgroup, then for any $g \in G$, $gHg^{-1} \subseteq H$ if and only if $g^{-1}Hg \subseteq H$. In particular, every set G-map $G/_H \to G/_H$ is a G-homeomorphism.*

Proof. Suppose $gHg^{-1} \subseteq H$ and consider the map $c : G \times G \to G,\ (h,k) \mapsto hkh^{-1}$. Denote by A the set of powers of g, $A = \{g^n \mid n \in \mathbb{N}\}$. By assumption, $c(A \times H) \subseteq H$, hence by continuity of c, we have $c(\overline{A \times H}) = c(\overline{A} \times H) \subseteq H$ as well. We proceed to show that $g^{-1} \in \overline{A}$, which would prove our claim. Let $B = \{g^n \mid g \in \mathbb{Z}\}$. Then B is a subgroup of G, so $\overline{B}$ is a subgroup as well. By compactness of G, $\overline{B}$ is a compact subgroup. We consider two cases.

- The identity element e is isolated in $\overline{B}$. This implies that every element of $\overline{B}$ is isolated by continuity of left translation. In conclusion, $\overline{B}$ is compact and discrete, so it is a finite group, implying that $g^n = g^{-1} \in B = A$ for some $n \in \mathbb{N}$.

- If e is not isolated, for any symmetric neighbourhood U of e there is an $n > 0$ such that $g^n \in U$. This implies that $g^{n-1} \in g^{-1}U \cap A$. The sets $g^{-1}U$ form a neighbourhood basis for g^{-1}, so in every neighbourhood of g^{-1} we find an element of A, proving that $g^{-1} \in \overline{A}$. □

The self-homeomorphisms of the orbit $G/_H$ are therefore completely determined by the elements $a \in G$ such that $aHa^{-1} = H$, and two of these elements determine the same homeomorphism if they are equal modulo H. We recall from algebra that the set of $a \in G$ such that $aHa^{-1} \subseteq H$ is called the normalizer of H and is a subgroup of G. If H is closed, $N(H)$ is a closed subgroup. Altogether, we have almost proven the following result.

Proposition 2.4.10 *Let G be a compact Hausdorff topological group, H a closed subgroup. Then there is a homeomorphism*

$$\Phi : N(H)/_H \to Homeo_G(G/_H),\ [a] \mapsto ([g] \mapsto [ga]).$$

Proof. It remains to show that Φ indeed is a homeomorphism. The right translation $G/_H \times N(H) \to G/_H$ is continuous, so by the exponential law, the adjoint map $N(H) \to \mathcal{TOP}(G/_H, G/_H)$ is continuous. It has image in $Homeo_G(G/_H)$, which is topologized as a subspace, so by definition of the quotient topology, Φ is continuous. It is a bijection by the preceeding comments. Since $N(H)/_H$ is compact and $Homeo_G(G/_H)$ is a Hausdorff space, the claim follows. □

The group $N(H)/_H$ is called the Weyl group of H, denoted with $W(H)$. If G is abelian, $W(H) = G/_H$, so the Weyl group is the best possible substitute for quotient groups in the non-abelian case.

2.5 Constructing New Actions

When dealing with a new class of objects, one is often interested to obtain new examples from already existing ones. We have already considered such basic constructions as taking subspaces, passing to quotients or taking products. We will now sketch several other constructions which will be of importance later. We will also fit these constructions into a more general categorical context.

Twisted Products

Let G, H be topological groups. Let X be a left G-space and a right H-space such that $g.(x.h) = (g.x).h$. Let Y be any H-space. The product $X \times Y$ carries an H-action, namely the diagonal action

$$H \times (X \times Y) \to X \times Y,\ h.(x, y) = (x.h^{-1}, h.y).$$

We denote the orbit space $X \times Y/_{H}$ by $X \times_H Y$. This space becomes a G-space by letting G act on the first component by left translation, that is,

$$G \times (X \times_H Y) \to X \times_H Y,\ g.[x, y] = [g.x, y].$$

A special case is given when X is the group G itself, H is a closed subgroup and the actions on X are given by left and right translation. An important application of these spaces is described in the next paragraph on the induction functor. Another interesting feature is that, under some assumptions, such spaces are the prototypes of G-spaces that admit a map into an orbit.

Proposition 2.5.1 *Let X be a G-space such that there is a G-map $f : X \to G/_{H}$ for some closed subgroup H of G. Let $A = f^{-1}([e])$. A is an H-space with the induced action from X and there is a canonical map*

$$F : G \times_H A \to X,\ [g, a] \mapsto g.a.$$

If G is compact Hausdorff, then F is a homeomorphism of G-spaces.

Proof. F is well-defined. If $g.a = h.b$ for $g, h \in G$, $a, b \in A$, we have $a = g^{-1}h.b$. Applying f yields $g^{-1}h = k \in H$ or equivalently, $h = gk$. Hence, $(h, b) = (gk, k^{-1}a)$ and we conclude that $[g, a] = [h, b] \in G \times_H A$ and F is injective. For surjectivity, take $x \in X$ arbitrary and $f(x) = [g] \in G/_{H}$ for some $g \in G$. Then $g^{-1}.x \in A$ and thus, $F([g, g^{-1}.x]) = x$. Continuity of F is clear from the definition of the quotient topology.

We claim that F is a closed map. For this it suffices to show that the map $G \times A \to X,\ (g, a) \mapsto g.a$ is closed. This is the restriction of the action map to a closed subset. By Proposition 2.4.7, since G is compact Hausdorff, the action map is closed, and so its restriction is closed as well. □

Restriction and Induction

Let G be a topological group and $H \subseteq G$ a subgroup. If X is a G-space, there is the obvious structure of an H-space on X. Furthermore, any G-equivariant map $f : X \to Y$ between G-spaces is also H-equivariant when regarded as a map of H-spaces. Thus, we obtain a functor

$$res^G_H : \mathcal{TOP}_G \to \mathcal{TOP}_H.$$

This functor has a left adjoint named induction. The induction functor on spaces is defined to be the twisted product, that is,

$$ind^G_H : \mathcal{TOP}_H \to \mathcal{TOP}_G,\ \mathrm{ind}^G_H(X) = G \times_H X.$$

For an H-equivariant map $f : X \to Y$ of H-spaces, $ind_H^G(f) : G \times_H X \to G \times_H Y,\ [g, x] \mapsto [g, f(x)]$. We have the adjointness property with the restriction functor.

Proposition 2.5.2 *For H-spaces X and G-spaces Y, there is a natural bijection*

$$\mathcal{TOP}_G(ind_H^G(X), Y) \to \mathcal{TOP}_H(X, \mathrm{res}_H^G(Y)),\ f \mapsto (x \mapsto f([e, x])).$$

Proof. Denote the map from the proposition by Φ. If $F : X \to Y$ is an H-map, the map $G \times X \to Y,\ (g, x) \mapsto g.F(x)$ induces a map on the quotient space $G \times_H X \to Y$. Denote this map by $\Psi(F)$. Then we have

$$\Phi \circ \Psi(F) : X \to \mathrm{res}_H^G(Y),\ x \mapsto \Psi(F)([e, x]) = e.F(x) = F(x)$$

and

$$\Psi \circ \Phi(f) : G \times_H X \to Y,\ [g, x] \mapsto g.\Phi(f)(x) = g.f([e, x]) = f([g, x]),$$

since f is G-equivariant. $\square$

Push-Outs

Let A, X, Y be G-spaces and $f : A \to X$, $h : A \to Y$ G-maps. The push-out of f and h is defined as a G-space P together with G-maps $F : X \to P$, $H : Y \to P$ such that $F \circ f = H \circ h$ and with the following property. Whenever Z is a G-space with G-maps $\varphi : X \to Z$, $\psi : Y \to Z$ such that $\varphi \circ f = \psi \circ h$, then there is a G-map $P \to Z$, making the following diagram commutative.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & X \arrow[d, "F"'] \arrow[ddr, "\varphi"] & \\
Y \arrow[r, "H"] \arrow[drr, "\psi"'] & P \arrow[dr] & \\
& & Z
\end{tikzcd}$$

Proposition 2.5.3 *Let $f : A \to X$ and $h : A \to Y$ be G-maps. The push-out of f and h exists. It can be defined as the space $P = X \cup Y/\sim$, where two points $x, y \in X \cup Y$, the disjoint union, are equivalent, if either $x = y$, or else $x \in X$, $y \in Y$ and there exists $a \in A$ such that $f(a) = x$, $h(a) = y$.*

Proof. Define P as stated. P obviously carries a G-action and this action is continuous by definition of the quotient topology. Define the maps $F : X \to P$, $H : Y \to P$ as the obvious inclusions. It remains to show the universal property. So let Z be a G-space together with maps φ, ψ as stated in the definition. Define a map $h : P \to Z$ by sending $x \in X$ to $\varphi(x)$ and $y \in Y$ to $\psi(y)$. Again by definition of the quotient topology, this map is continuous and it makes the required diagram commutative. So indeed P is a push-out of f and h. $\square$

Pull-Backs

Let X, Y, Z be G-spaces and $f : X \to Z$, $h : Y \to Z$ be G-maps. A pull-back of f and h is a G-space B together with G-maps $F : B \to X$, $H : B \to Y$ such that $f \circ F = h \circ H$ and the following universal property holds. Whenever A is a G-space with G-maps $\varphi : A \to X$, $\psi : A \to Y$ such that $f \circ \varphi = h \circ \psi$, then there is a G-map $A \to B$ making the diagram

$$\begin{tikzcd}
A \arrow[dr] \arrow[drr, "\varphi"] \arrow[ddr, "\psi"'] & & \\
& B \arrow[r, "F"'] \arrow[d, "H"] & X \arrow[d, "f"] \\
& Y \arrow[r, "h"'] & Z
\end{tikzcd}$$

commutative.

Proposition 2.5.4 *Let $f : X \to Z$, $h : Y \to Z$ be G-maps. A pull-back of f and h exists. It can be defined as the space $B = \{(x, y) \in X \times Y \mid f(x) = h(y)\}$ with the diagonal G-action.*

Proof. Define B as stated. We let $F : B \to X$, $H : B \to Y$ be the projections to the first and second factor, respectively. Then $f \circ F = h \circ H$ by definition of B. It remains to show the universal property. Let A be a G-space and $\varphi : A \to X$, $\psi : A \to Y$ be G-maps with $f \circ \varphi = h \circ \psi$. Define a G-map $k : A \to B$ by $k(a) = (\varphi(a), \psi(a))$. This map is continuous and well defined by the conditions on φ and ψ. The necessary diagram commutes, so B is indeed a pull-back. $\square$

Examples

1. Let $H \subseteq G$ be a closed subgroup of the topological group G and let V be an H-representation. That is, V is a topological vector space with an action given by a map $G \times V \to V$ which is linear in the second variable. The twisted product $G \times_H V$ is a G-space with a canonical map $G \times_H V \to G/_H$, $[g, v] \mapsto [g]$. We will see in chapter two that every G-vector bundle over an orbit arises in this fashion. Following the equivariant philosophy, G-vector bundles over orbits replace vector bundles over a point. So, from the bundle point of view, the equivariant equivalent of a vector space is a bundle $G \times_H V$.

2. Let $Z = X \cup Y$ be the union of two G-subsets and take $A = X \cap Y$. Let $\varphi : A \to A$ be an equivariant homeomorphism. Then the push-out of the inclusion $A \to X$ and the map $A \xrightarrow{\varphi} A \to Y$ is denoted by $X \cup_\varphi Y$. In case φ extends to a self-homeomorphism $X \to X$, we have an equivariant homeomorphism $Z \cong X \cup_\varphi Y$. However, many interesting examples of G-spaces are obtained from this setting if φ can not be extended to a self-homeomorphism of X.

3. Let X be a G-space and consider the projection map $p : X \to X/_G$. $X/_G$ is an ordinary topological space. Let $h : Y \to X/_G$ be any continuous map. The pull-back of p and h is denoted by $h^*(X)$. The structure map $p' : h^*(X) \to Y$ induces a map $\sigma : h^*(X)/_G \to Y$, since Y is trivial. We claim that p' is an open map. To see this, it suffices to check openness of $p'(U)$ for a basic neighbourhood $U \subseteq h^*(X)$, so we assume $U = (U_1 \times U_2) \cap h^*(X)$ with $U_1 \subseteq X$ open, $U_2 \subseteq Y$ open. By openness of p, the set $h^{-1}(p(U_1)) \cap U_2$ is open in Y. But this is precisely the image of U under p', so p' is open. It is obviously surjective, and if $p'(x, y) = p'(x', y')$, we have $y = y'$ and $p(x) = h(y) = h(y') = p(x')$. Thus, x and x' are in the same G-orbit, showing that the induced map σ is injective. Altogether, we have proven that σ is a homeomorphism. We thus have constructed a G-space with quotient space Y together with a map to X covering a given map of the quotients.

2.6 Orbits and Fixed Points

Definition 2.6.1 Let G be a topological group. Let *Sub* G be the set of closed subgroups of G. G acts on the set *Sub* G by conjugation. The equivalence class of a closed subgroup H of G in *Sub* $G/_G$ is denoted by (H). It is called the orbit type of H.

There is a partial order defined on the set of orbit types by letting $(H) \leq (K)$ if and only if there exists $g \in G$ such that $gHg^{-1} \subseteq K$.

Note that two subgroups H, K define the same orbit type if and only if there is a $g \in G$ such that $gHg^{-1} = K$. It does not suffice for the two subgroups to be isomorphic.

Example 2.6.2 Consider the dihedral group D_4. The subgroup H_τ generated by the reflection τ is isomorphic to $\mathbb{Z}_2$. The subgroup $H_{\tau\sigma}$ generated by $\tau\sigma$ is isomorphic to $\mathbb{Z}_2$ as well. The orbit of H_τ under the conjugation action is $\{H_\tau,\ H_{\tau\sigma^2}\}$, whereas the orbit of $H_{\tau\sigma}$ is $\{H_{\tau\sigma},\ H_{\tau\sigma^3}\}$. Hence, H_τ and $H_{\tau\sigma}$ do not define the same orbit type.

Let X be a G-space and take $x \in X$. If G_x is the isotropy subgroup of x, for any $g \in G$ and $h \in G_x$ we have $g.x = gh.x = ghg^{-1}.gx$, thus, $ghg^{-1} \in G_{gx}$. It follows easily that $G_{gx} = gG_xg^{-1}$. Hence, points on the same orbit do not necessarily have the same isotropy group, but their isotropy groups define the same orbit type. We therefore also speak of (G_x) as the orbit type of the point $x \in X$.

Definition 2.6.3 Let X be a G-space, $H \subseteq G$ a closed subgroup. We define the following subspaces of X.

1.
$$X^H = \{x \in X \mid h.x = x \,\forall\, h \in H\},$$
the fixed space of H (in X).

2.
$$X_{(H)} = \{x \in X \mid (G_x) = (H)\}.$$

3.

$$X_{\geq(H)} = \{x \in X \mid (G_x) \geq (H)\}.$$

4. $X_H = X^H \cap X_{(H)}$.

Note that in general, only $X_{(H)}$ and $X_{\geq(H)}$ are G-subspaces of X. However, also the other spaces are naturally equipped with group actions.

Proposition 2.6.4 *Let X be a G-space, H be a closed subgroup of G and $W(H)$ the Weyl group of H, $N(H)$ the normalizer of H. The G-action on X induces a $W(H)$-action on X^H and X_H. More precisely, the assignment*

$$\mathcal{TOP}_G \to \mathcal{TOP}_{W(H)},\ X \mapsto X^H,$$

is a functor, acting as restriction on maps. The action of $W(H)$ on X_H is free.

Proof. We have to check that the $W(H)$-actions and restriction of G-maps is well defined. Clearly, $N(H)$ acts on X^H and it is obvious that H acts trivially on X^H. Hence, $W(H)$ acts on X^H. If $x \in X_H$ and $n \in N(H)$, then $gn.x = n.x$ implies $n^{-1}gn \in G_x$ or equivalently, $g \in nG_xn^{-1}$. By definition of X_H, $G_x = H$ and $nG_xn^{-1} = H$, since $n \in N(H)$. So $G_{n.x} = H$ and the action is well defined.

If X, Y are G-spaces and $f : X \to Y$ is a G-map, we have $h.f(x) = f(h.x) = f(x)$ for $x \in X^H$ and $h \in H$. So f induces a map $f^H : X^H \to Y^H$ by restriction and this map is clearly $W(H)$-equivariant.

Finally, if $x \in X_H$ and $n \in N(H)$ are given such that $n.x = x$, then we have $n \in H$ by definition of X_H, showing that $[n] = [e] \in W(H)$. We conclude that the action is free.□

The assignment $X \mapsto X_{(H)}$ can not be turned into a functor, since the image of a point $x \in X$ with orbit type (H) under an equivariant map f need not have orbit type (H). With respect to the partial order on orbit types, $f(x)$ will always have an orbit type lesser or equal than (H). It is sometimes convenient to look only at those equivariant maps that respect the orbit type, that is, $(G_{f(x)}) = (G_x)$. $X \mapsto X_{(H)}$ will be a functor into the category of $W(H)$-spaces with orbit type preserving morphisms.

3 Fibre Bundles

The theory of G-spaces is closely connected to the theory of fibre bundles. The reason is that every free G-space arises as a fibre bundle and more general G-spaces can be decomposed into almost free parts. Therefore we will develop some material from the general material of fibre bundles and use this to obtain structural results for G-spaces, assuming some slight regularity of the space.

3.1 Principal G-Bundles

Let $p : E \to B$ be any map of topological spaces. A local (bundle)-trivialization for p at a point $b \in B$ is a homeomorphism $\varphi : F \times U \to p^{-1}(U)$ for some space F and an open neighbourhood U of x such that the diagram

$$\begin{tikzcd} F \times U \arrow[r, "\varphi"] \arrow[dr, "\pi_2"'] & p^{-1}(U) \arrow[d, "p"] \\ & U \end{tikzcd}$$

commutes. The homeomorphism φ is called a chart for p. Assume that F is a right K-space. Then two charts $\varphi : F \times U \to p^{-1}(U)$, $\psi : F \times V \to p^{-1}(V)$ are said to be compatible, if there is a function $\vartheta : U \cap V \to K$ such that

$$\varphi^{-1} \circ \psi(f, u) = (f.\vartheta(u), u).$$

An atlas for a map $p : E \to B$ is given by a cover $\{U_i\}_{i \in I}$ of B together with local trivializations φ_i defined on U_i such that any two trivializations are compatible. The notion of a subatlas is obvious and an atlas is called maximal, if it is not the subatlas of a strictly larger atlas.

Definition 3.1.1 A fibre bundle $p : E \to B$ with typical fibre F and structure group K is a map $p : E \to B$ together with a right K-space F and a maximal atlas of local trivializations with fixed fibre F. The group K is called the structure group of the bundle.

We note that it follows directly from the existence of local trivializations that p is an open map.

Example 3.1.2 1. Every covering is a fibre bundle, where the fibre F is a discrete topological space.

2. The Moebius strip M is defined as the space $[-1, 1] \times [-1, 1]$, where the points $(s, -1)$ and $(-s, 1)$ are identified. The space $[-1, 1]$ carries an obvious $\mathbb{Z}_2$-action and the projection $p : M \to \mathbb{S}^1$, $(s, t) \mapsto t$ turns M into a fibre bundle over $\mathbb{S}^1$. Here, we regard $\mathbb{S}^1$ as the interval $[-1, 1]$ with -1 identified with 1. We define two open sets in $\mathbb{S}^1$ by

$$U = \left[-1, -\frac{1}{4}\right) \cup \left(\frac{1}{4}, 1\right], \; V = \left(\frac{1}{2}, \frac{1}{2}\right).$$

We define charts for p on these sets. For U, define

$$\varphi^{-1} : p^{-1}(U) \to [-1,1] \times U,\ (s,t) \mapsto \begin{cases} (s,t) & t < 0 \\ (-s,t) & t > 0 \end{cases}.$$

φ^{-1} is well defined, since $\varphi^{-1}(s,-1) = (s,-1) = (s,1) = \varphi^{-1}(-s,1)$. It is easy to see that φ^{-1} is a homeomorphism. On V, we define

$$\psi^{-1} : p^{-1}(V) \to [-1,1] \times V,\ (s,t) \mapsto (s,t).$$

The intersection $U \cap V$ is the disjoint union of two intervals and the composition $\varphi^{-1} \circ \psi$ is the identity on one of these intervals, and is twisting in the fibre on the second. Hence, φ and ψ are compatible and the maximal atlas containing these two defines the structure of a fibre bundle on M with structure group $\mathbb{Z}_2$.

3. Consider the n-sphere $\mathbb{S}^n$ as a subset of $\mathbb{R}^{n+1}$ and the set

$$T\mathbb{S}^n = \{(x,v) \in \mathbb{S}^n \times \mathbb{R}^{n+1} \mid \langle x,v \rangle = 0\}.$$

There is the obvious projection $p : T\mathbb{S}^n \to \mathbb{S}^n$ onto the first component. We claim that p is a fibre bundle with fibre $\mathbb{R}^n$ and structure group $GL(\mathbb{R}^n)$.

To see this, let $U_i = \{x \in \mathbb{S}^n \mid x_i \neq 0\}$ and

$$e_i : \mathbb{R}^n \to \mathbb{R}^{n+1},\ e_i(v) = (v_1, \dots, v_{i-1}, 0, v_i, \dots, v_n),$$

$$c_i : \mathbb{R}^{n+1} \to \mathbb{R}^n,\ c_i(w) = (w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_{n+1}).$$

Then we can define

$$\varphi : U_i \times \mathbb{R}^n \to p^{-1}(U_i),\ (x,v) \mapsto (x, e_i(v) - \langle e_i(v), x \rangle \cdot x)$$

and

$$\psi : p^{-1}(U_i) \to U_i \times \mathbb{R}^n,\ (x,w) \mapsto \left(x, c_i \left(w - \frac{w_i}{x_i} \cdot x \right) \right).$$

It is easy to see that φ and ψ are inverse to each other and that coordinate changes induces linear isomorphisms, continuously dependent on x. Therefore, these charts define compatible bundle charts for $T\mathbb{S}^2$.

If the fibre F of a bundle $p : E \to B$ is not only a right K-space but also a left G-space, the space E carries a G-action in a natural way.

Theorem 3.1.3 *Let $p : E \to B$ be a fibre bundle with typical fibre F and structure group K. Let G act on F such that $g.(f.k) = (g.f).k$ for all $g \in G$, $f \in F$, $k \in K$. Then there is a unique G-action on E such that p is invariant and each bundle chart is equivariant.*

Proof. Take any $x \in E$. It lies in the domain of a chart φ, that is, there is a neighbourhood U of $p(x)$ and a homeomorphism

$$\varphi : F \times U \to p^{-1}(U).$$

In particular, $x = \varphi(f, p(x))$ for some $f \in F$. Clearly, $g.x$ has to be defined by

$$g.x = g.\varphi(f, p(x)) = \varphi(g.f, p(x)),$$

if charts are going to be equivariant. Therefore we have to check independence from particular charts. Let $\psi : F \times U \to p^{-1}(U)$ be a second bundle chart and $x = \psi(f', p(x))$. We have to show that $\varphi(g.f, p(x)) = \psi(g.f', p(x))$. For this, it suffices to show equivariance of the map $\varphi^{-1} \circ \psi : F \times U \to F \times U$. We have

$$\begin{aligned} g.\varphi^{-1} \circ \psi(f, u) &= g.(f.\vartheta(u), u) \\ &= (g.(f.\vartheta(u)), u) \\ &= ((g.f).\vartheta(u), u) \\ &= \varphi^{-1} \circ \psi(g.f, u). \end{aligned}$$

This shows equivariance and the result is proven. $\square$

A class of important examples of fibre bundles with G-action on the fibre are bundles where the fibre itself is G and the G-action is left translation. If the structure group happens to be G as well, we make the following definition.

Definition 3.1.4 A principal G-bundle is a fibre bundle $p : E \to B$ with typical fibre G and structure group G, the action given by right translation.

Obviously, left and right translation in a group commute, hence the preceeding theorem is applicable in the situation of fibre bundles.

Corollary 3.1.5 *Let $p : E \to B$ be a principal G-bundle. Then there is a unique G-action on E turning E into a free G-space. Moreover, p induces a homeomorphism $E/_G \to B$.*

Proof. The action is free since it comes from left translation on G, which is a free action. $p(x) = p(y)$ implies that, in a chart φ, $x = \varphi(g, u)$, $y = \varphi(g', u)$ for $g, g' \in G$. Thus, $g'g^{-1}.x = y$. Since p is invariant, open and surjective, it induces a homeomorphism $E/_G \to B$. $\square$

Let $p : E \to B$ be a G-principal bundle and F be any right G-space. We can form the twisted product $F \times_G E$ and obtain a map $\pi : F \times_G E \to B$, $[f, x] \mapsto p(x)$.

Proposition 3.1.6 *The map $\pi : F \times_G E \to B$ constitutes a fibre bundle with structure group G and typical fibre F. It is called the associated F-bundle of $p : E \to B$.*

Proof. Let $\varphi : G \times U \to p^{-1}(U)$ be a chart for p. Define a map

$$\psi : F \times U \to \pi^{-1}(U),\ (f,u) \mapsto [f, \varphi(e,u)].$$

We have $\pi^{-1}(U) = \{[f,x] \in F \times_G X \mid p(x) \in U\} = F \times_G p^{-1}(U)$. There are obvious G-homeomorphisms $F \cong F \times_G G$ and $F \times_G (G \times U) \cong (F \times_G G) \times U$. Then we can write ψ as the composite

$$F \times U \xrightarrow{\cong} (F \times_G G) \times U \xrightarrow{\cong} F \times_G (G \times U) \xrightarrow{\mathrm{id}_F \times_G \varphi} F \times_G p^{-1}(U) = \pi^{-1}(U).$$

Hence, ψ is a homeomorphism. If ψ' is a second chart over U, constructed from the chart φ' of p, then we have

$$\psi^{-1} \circ \psi'(f,u) = \psi^{-1}([f, \varphi'(e,u)]).$$

Since φ and φ' are compatible, $\varphi'(e,u) = \varphi(e.\vartheta(u),u) = \varphi(\vartheta(u),u)$ and so we see that

$$\psi^{-1} \circ \psi'(f,u) = \psi^{-1}([f, \varphi(\vartheta(u),u]) = \psi^{-1}([f.\vartheta(u), \varphi(e,u)]) = (f.\vartheta(u),u).$$

Hence, ψ and ψ' are not only compatible, but they even have the same transition function as φ and φ'. □

Before passing on to vector bundles, we will prove some structure theorems concerning actions with a single orbit type. So we are dealing with G-spaces X such that $X = X_{(H)}$ for some closed subgroup H of G. Such spaces are called monotypic. The results apply in particular to the G-subspaces $X_{(H)}$ of arbitrary G-spaces X.

Theorem 3.1.7 *Let G be compact and X be a monotypic G-space with orbit type (H). Then the map*

$$\varphi : G \times_{N(H)} X^H \to X,\ [g,x] \mapsto g.x$$

is a G-homeomorphism. In particular, if the projection $G \to G/_{N(H)}$ is an $N(H)$-principal bundle, X is bundle equivalent to the associated X^H-bundle of the bundle $G \to G/_{N(H)}$.

Proof. It is immediate that φ is well-defined and equivariant. Since φ comes from the restriction of the action map to $G \times X^H$, it is continuous. It is surjective, since if $x \in X$, then $G_x = gHg^{-1}$ for some $g \in G$ and thus, $G_{g^{-1}.x} = H$, implying that $g^{-1}.x \in X^H$. Clearly, $[g, g^{-1}.x]$ maps to x. φ is closed. To see this, we recall that, since G is assumed compact, the action map is closed and restrictions of closed maps to closed subsets are closed. Hence, the map $G \times X^H \to X$ is closed and for $C \subseteq G \times_{N(H)} X^H$ closed, $\varphi(C) = \alpha(p^{-1}(C))$ is closed, where α is the action map. It remains to show that φ is injective. Hence, suppose that $g.x = h.y$. We have $h^{-1}g.x = y \in X^H$ and consequently, $G_{h^{-1}g.x} = h^{-1}gHg^{-1}h = H$. We conclude that $h^{-1}g \in N(H)$. This shows that $[g,x] = [gg^{-1}h, h^{-1}g.x] = [h,y] \in G \times_{N(H)} X^H$. □

Corollary 3.1.8 *If G is compact and X a monotypic G-space, the map*

$$G/_H \times_{W(H)} X^H \to X,\ [[g],x] \mapsto g.x$$

is a G-homeomorphism.

Proof. By definition,

$$G \times_{N(H)} X^H = G \times X^H/_{N(H)}.$$

It is easy to see (compare exercises) that, if $H \subseteq G$ is normal, the space $X/_G$ is homeomorphic to $X/_H/_{G/_H}$. In our case, this implies that

$$G \times_{N(H)} X^H \cong G \times_H X^H/_{W(H)}.$$

But H acts trivially on X^H, so the twisted product $G \times_H X^H$ is just $G/_H \times X^H$, which yields the result. □

3.2 G-Vector Bundles

A special but also very important case of fibre bundles are bundles such that each fibre carries the structure of a vector space. We already have seen the example of the tangential bundle of the n-sphere. Of course, every tangential bundle of a differentiable manifold is a vector bundle. The precise definition is the following.

Definition 3.2.1 Let k be a field. A (finite dimensional) vector bundle over k is a fibre bundle $p : E \to B$ with typical fibre k^n for some $n \in \mathbb{N}$ and structure group $GL(n,k)$.

We should specify a topology on $GL(n,k)$ for arbitrary k to make real sense of the definition, however, we will only consider cases where k is $\mathbb{R}$ or $\mathbb{C}$. One could also look at vector bundles over skew fields and consider the quaternions, which we will not pursue.

In the literature, one often finds a different definition of vector bundles and our first result will be to establish that our definition agrees with the second one.

Proposition 3.2.2 *Vector bundles over k are precisely given by maps $p : E \to B$ such that the following two conditions are satisfied.*

1. *For each $b \in B$, $p^{-1}(b)$ has the structure of a vector space.*
2. *For each $b_0 \in B$ there is a neighbourhood $U \subseteq B$ of b_0 and a homeomorphism $\varphi : k^n \times U \to p^{-1}(U)$ such that $\pi_2 = p \circ \varphi$ and $\varphi\big|_{k^n \times \{b\}}$ is a linear isomorphism between k^n and $p^{-1}(b)$ for every $b \in U$.*

Proof. Let $p : E \to B$ be a vector bundle over k. Take $b_0 \in B$ and an open neighbourhood U of b_0 such that $\varphi : k^n \times U \to p^{-1}(U)$ is a bundle chart. Then φ defines the structure of a vector space on $p^{-1}(b)$ for every $b \in U$ such that φ is a linear isomorphism on fibres. Any other bundle chart over U differs from φ only by a linear isomorphism and hence defines the same linear structure, so it is independent of the chosen chart and conditions 1) and 2) are satisfied.

Conversely, assume the two conditions to hold and let $\varphi : k^n \times U \to p^{-1}(U)$, $\psi : k^n \times U \to p^{-1}(U)$ two maps as in 2). For $b \in U$, we define $\varphi_b : k^n \to p^{-1}(b)$ by $v \mapsto \varphi(v, b)$, similarly for ψ. The map $\psi_b^{-1} \circ \varphi_b : k^n \to k^n$ is a linear isomorphism, hence, we can define a map $\vartheta : U \to GL(n, k)$, $\vartheta(b) = \varphi_b^{-1} \circ \psi_b$ (note the reversal in order due to the requirement of a right action), satisfying the condition from the definition of a fibre bundle. □

From a categorical viewpoint, again we have to take care of the morphisms between vector bundles. Clearly, the category of vector bundles exists as a subcategory of fibre bundles with structure group $GL(n, k)$. However, morphisms in this category would not take the linear structure in the fibre into account. Therefore we make the following definition.

Definition 3.2.3 Let $p : E \to B$, $q : Y \to C$ be vector bundles over k. A morphism of vector bundles is pair of maps $F : X \to Y$, $f : B \to C$ such that

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ B \arrow[r, "f"] & C \end{tikzcd}$$

commutes and $F\big|_{p^{-1}(b)} : p^{-1}(b) \to q^{-1}(f(b))$ is a linear map for every $b \in B$. If $B = C$, we say that a bundle map is a bundle map over B if the lower map is the identity on B.

With this definition, it is apparent what a G-vector bundle should be. It is just a symmetric object in the category of vector bundles over the category of topological spaces. In other words, we have a simultaneous action of a topological group G on X and on B such that an element $g \in G$ constitutes a self-map of the vector bundle $p : E \to B$.

From Proposition 2.5.1 we know that every G-vector bundle over an orbit $G/_H$ is G-homeomorphic to a bundle $G \times_H V \to G/_H$ for some H-representation V, provided G is compact Hausdorff or $\pi : G \to G/_H$ has a local cross section. In the special situation of vector bundles, this result holds in general.

Theorem 3.2.4 *Let $p : E \to G/_H$ be a G-vector bundle over the orbit $G/_H$ with $H \subseteq G$ a closed subgroup. Then the canonical map*

$$f : G \times_H V \to E, \ [g, v] \mapsto g.v$$

is a G-homeomorphism of bundles over $G/_H$. Here, $V = p^{-1}([e])$ is an H-representation.

Proof. It is obvious that f is well-defined, continuous and equivariant. We will construct a continuous inverse map for f. Take $z \in E$ arbitrarily. Then there is a $g \in G$ such that $p(z) = [g] \in G/_H$. Let $v = g^{-1}.z$. Then $p(v) = [e]$, so $v \in V$ and we can define

$$f'(z) = [g, v].$$

For $h \in H$, we have $h^{-1}.v = (gh)^{-1}.z$, so working with gh instead of g in the definition gives $f'(z) = [gh, h^{-1}.v] = [g, v]$. We conclude that f' is well defined. For continuity of f', consider the pull-back of the maps $p : E \to G/_H$ and $\pi : G \to G/_H$. Denoting this space by P, we recall that

$$P = \{(g, z) \in G \times E \mid \pi(g) = p(z)\}.$$

We claim that the induced map $\pi' : P \to E$ is surjective and open. Surjectivity is clear, since the map π is surjective. For openness, consider an open subset $U \subseteq P$ and $(g, z) \in U$. We find open neighbourhoods $V \subseteq G$ and $W \subseteq E$ of g, z, respectively, such that $V \times W \cap P \subseteq U$. Since $\pi(V)$ is open, $p^{-1}(\pi(V))$ is open as well and $z \in p^{-1}(\pi(V))$. Let $W' = W \cap p^{-1}(\pi(V))$.

By definition of π', we have $\pi'(V \times W' \cap P) \subseteq W'$. On the other hand, for $w' \in W'$, there is a $v \in V$ such that $\pi(v) = p(w')$. This shows that $(v, w') \in V \times W' \cap P$ and $\pi'(v, w') = w'$. In conclusion, we have shown that $\pi'(V \times W' \cap P) = W'$, so

$$W' = \pi'(V \times W' \cap P) \subseteq \pi'(V \times W \cap P) \subseteq \pi'(U),$$

showing that $\pi'(U)$ is open.

Back in the main proof, we define a map $\beta : G \times E \to G \times E,\ (g, z) \mapsto (g, g^{-1}.z)$. β is obviously continuous and maps P onto $G \times V$. The map β fits into a diagram

$$\begin{tikzcd} P \arrow[r, "\beta|_P"] \arrow[d, "\pi'"'] & G \times V \arrow[d] \\ E \arrow[r, "f'"] & G \times_H V \end{tikzcd}$$

which is readily seen to commute. By openness and surjectivity of π' and continuity of the other two involved maps, f' is continuous as well. f' is inverse to f since, for $z \in E$ and g, v as in the definition of f', we have

$$f \circ f'(z) = f([g, v]) = g.v = gg^{-1}.z = z$$

and for arbitrary $[g, v] \in G \times_H V$, we have

$$f' \circ f([g, v]) = f'(g.v) = [g, g^{-1}g.v] = [g, v]. \qquad \square$$

The next concept is an important notion in the context of fibre bundles.

Definition 3.2.5 Let $p : E \to B$ be a map. A cross section for p is a continuous map $s : B \to E$ such that $p \circ s = \mathrm{id}_B$. We say that p has local cross sections, if for every $b \in B$ there is a neighbourhood U of b such that $p\big|_{p^{-1}(U)} : p^{-1}(U) \to U$ has a cross section.

In general, fibre bundles need not have a global cross section. In vector bundles, there is obviously the zero cross section, mapping a point $b \in B$ to $0 \in p^{-1}(b)$. The existence of local cross sections follows immediately from the existence of trivializations. If $p : E \to G/_H$ is a bundle over an orbit, then it is clear that the existence of a cross section at $[e]$ implies that p has local cross sections.

Proposition 3.2.6 *Let H be a closed subgroup of the topological group G. The projection $p : G \to G/_H$ has a local cross section if and only if it is an H-principal bundle.*

PROOF. Assume that $p : G \to G/_H$ is an H-principal bundle. For $g \in G$ there is a neighbourhood $U \subseteq G/_H$ of $[e]$ and a bundle chart

$$\varphi : G \times U \to p^{-1}(U).$$

Define a map $s : U \to p^{-1}(U),\ s([g]) = \varphi(e, [g])$. Then s is a local cross section. Conversely, assume that p has local cross sections and let $U \subseteq G/_H$ be an open neighbourhood of $[e]$ with a cross section $s : U \to G$. Define maps

$$\varphi : p^{-1}(U) \to H \times U,\ \varphi(g) = ((s \circ p(g))^{-1}g, p(g))$$

and

$$\psi : H \times U \to p^{-1}(U),\ (h, u) \mapsto s(u)h.$$

Both maps are clearly continuous. We compute

$$\begin{aligned} \varphi \circ \psi(h, u) &= \varphi(s(u)h) \\ &= (s \circ p(s(u)h))^{-1}s(u)h, p(s(u)h)) \\ &= (s(u)^{-1}s(u)h, u) \\ &= (h, u) \end{aligned}$$

and

$$\begin{aligned} \psi \circ \varphi(g) &= \psi((s \circ p(g))^{-1}g, p(g)) \\ &= s \circ p(g)(s \circ p(g))^{-1}g \\ &= g. \end{aligned}$$

Hence, these two maps are inverse homeomorphisms. We obtain a similar result for the translated sets gU as neighbourhoods of $[g]$, by defining $s_g(u) = gs(g^{-1}.u)$ and defining the homeomorphisms as above, with s replaced by s_g. It remains to show that two bundle charts obtained in this way differ by a transition function into H. Take $u \in g_0U \cap g_1U$. Then

$$\begin{aligned} \varphi_{g_0} \circ \psi_{g_1}(h, u) &= \varphi_{g_0}(s_{g_1}(u)h) \\ &= (s_{g_0} \circ p(s_{g_1}(u)h))^{-1}s_{g_1}(u)h, p(s_{g_1}(u)h)) \\ &= (s_{g_0}(u)^{-1}s_{g_1}(u)h, u). \end{aligned}$$

The transition function in prospect therefore is the function $U \to G,\ u \mapsto s_{g_0}(u)^{-1}s_{g_1}(u)$ and we have to show that its image is contained in H. But this is obvious since both s_{g_0} and s_{g_1} are sections and so we have $p \circ s_{g_0}(u) = u = p \circ s_{g_1}(u)$. □

Theorem 3.2.7 *If $p : G \to G/_H$ has a local cross section, then for any H-representation V, $q : G \times_H V \to G/_H$ is a G-vector bundle with $q^{-1}([e])$ H-isomorphic to V.*

Proof. We already have identified $V \times_H G$ as a fibre bundle with typical fibre V over $G/_H$, the associated bundle to the principal bundle $p : G \to G/_H$. If $\vartheta : U \to H$ is a transition function for two charts over U, we extend ϑ to a map into $GL(n,k)$ via the action of H on V. Clearly, this turns $V \times_H G$ into a vector bundle. Twisting the coordinates is a homeomorphism and we obtain a vector bundle $G \times_H V$ over $G/_H$. The G-action on $G \times_H V$ and on $G/_H$ is induced by translation on G. For $g, g_0 \in G$, g acts on the fibre $q^{-1}([g_0])$ via

$$q^{-1}([g_0]) \to q^{-1}([gg_0]),\ [g_0, v] \mapsto [gg_0, v],$$

which is linear. Hence, $G \times_H V$ is a G-vector bundle over $G/_H$ and the canonical map $q^{-1}([e]) \to V$ is an H-isomorphism. □

4 G-Manifolds

4.1 Manifolds

Definition 4.1.1 1. Let M be a topological space. A chart for M around $x \in M$ is a pair (U, φ), where φ is a homeomorphism $\varphi : U \to \mathbb{R}^n$ with U a neighbourhood of x. We will also speak of φ as the chart and of U as a chart neighbourhood.

2. An atlas for M is a cover of M by chart neighbourhoods.

3. An n-dimensional topological manifold is a second countable Hausdorff space together with an atlas such that the image of all charts is $\mathbb{R}^n$.

4. An atlas for a space M is called a $\mathcal{C}^k$-atlas, if for all charts (U, φ), (V, ψ), the map

$$\varphi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$$

is a $\mathcal{C}^k$-diffeomorphism. There is the obvious notion of a subatlas and a $\mathcal{C}^k$-atlas is called maximal, if it is not the subatlas of a properly larger atlas.

5. An n-dimensional $\mathcal{C}^k$-manifold is a second countable Hausdorff space together with a maximal $\mathcal{C}^k$-atlas.

6. For a $\mathcal{C}^k$-manifold with $k \geq 1$ and $x \in M$, define the tangential space T_xM at x as the space

$$\{[\gamma] \mid \gamma : (-\varepsilon, \varepsilon) \to U,\ \varphi \circ \gamma \text{ differentiable at } 0,\ \gamma(0) = x\},$$

where ε varies in the positive real numbers and (U, φ) is a chart around x. The equivalence relation determining the class $[\gamma]$ is given by $\gamma \sim \delta$ if and only if

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \varphi \circ \gamma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \psi \circ \delta(t).$$

It is easily seen that the operations on $\mathbb{R}^n$ determine the structure of a vector space on T_xM.

7. The collection
$$TM = \bigcup_{x\in M} T_xM$$
is called the tangential bundle of M. The bundle projection is given by $p : TM \to M,\ v \in T_xM \mapsto x$.

Proposition 4.1.2 *T_xM is canonically, up to choice of charts, isomorphic to $\mathbb{R}^n$. The tangential bundle of a manifold M is a vector-bundle.*

Proof. Let (U, φ) be a chart for M. Then we have the canonical map
$$\Phi_x : T_xM \to \mathbb{R}^n,\ [\gamma] \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \varphi \circ \gamma(t).$$
It is obviously well-defined and injective by definition of $[\gamma]$. Surjectivity is also clear: The path $t \mapsto \varphi(x) + t \cdot v$ defines, after application of φ^{-1}, a path in M through x, and its image under the canonical map is v.

To see that the tangential bundle is a vector bundle, define a bundle chart for TM as follows.
$$\tilde{\varphi} : \mathbb{R}^n \times U \to p^{-1}(U) = TU,\ (v, x) \mapsto \Phi_x^{-1}(v) \in T_xM.$$
This is an isomorphism in the fibres covering the identity map, therefore by exercise 4 (iii), it is a homeomorphism. If we have two charts, defined on the same domain U from the chart ψ, then
$$\tilde{\varphi}^{-1} \circ \tilde{\psi}(v, x) = \tilde{\varphi}^{-1}(\Phi_x^{-1} \circ \Phi_x \circ \Psi_x^{-1}(v)) = (\Phi_x \circ \Psi_x^{-1}(v), x).$$
$\Phi_x \circ \Psi_x^{-1}$ is an element of $GL(n, \mathbb{R})$ and we have to show that it depends continously on x. For this, we can assume that $M = \mathbb{R}^n$, i.e. we can work in a chart. Then we can estimate
$$\begin{aligned}
\left\|\Phi_x \circ \Psi_x^{-1} - \Phi_y \circ \Psi_y\right\|_{L(\mathbb{R}^n,\mathbb{R}^n)} &= \sup_{\|v\|=1} \left\|\Phi_x \circ \Psi_x^{-1}(v) - \Phi_y \circ \Psi_y(v)\right\| \\
&= \sup_{\|v\|=1} \left\| \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \left(\varphi \circ \psi^{-1}(\psi(x) + tv) - \varphi \circ \psi^{-1}(\psi(y) + tv)\right)\right\| \\
&= \sup_{\|v\|=1} \left\|T_{\psi(x)}(\varphi \circ \psi^{-1})(v) - T_{\psi(y)}(\varphi \circ \psi^{-1})(v)\right\| \\
&= \left\|T_{\psi(x)}(\varphi \circ \psi^{-1}) - T_{\psi(y)}(\varphi \circ \psi^{-1})\right\|_{L(\mathbb{R}^n,\mathbb{R}^n)}.
\end{aligned}$$
Since we assumed that M is a $\mathcal{C}^k$-manifold with $k \geq 1$, this last expression will be small if x is close to y. Hence, the assignment $x \mapsto \Phi_x \circ \Psi_x^{-1}$ is continuous. The claim follows.□

We call a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ differentiable at 0, if for some chart φ at $\gamma(0)$, the map $\varphi \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is differentiable at 0.

Definition 4.1.3 Let $f : M \to N$ be a map of $\mathcal{C}^k$-manifolds. f is said to be differentiable at $x \in M$, if for any smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = x$, the curve $f \circ \gamma$ is differentiable at x. In that case, the map
$$T_xf : T_xM \to T_{f(x)}N,\ [\gamma] \mapsto [f \circ \gamma]$$
is linear and is called the differential of f at x.

For a curve $\gamma : \mathbb{R} \to M$, we make the following convention. The expression $\dot{\gamma}(t) \in T_{\gamma(t)}M$ is defined to be the value $T_t\gamma(1)$, where $1 \in T_t\mathbb{R}$ is the element corresponding to $1 \in \mathbb{R}$ under the canonical isomorphism defined by the global chart $\mathrm{id}_{\mathbb{R}}$ for $\mathbb{R}$.

4.2 The Exponential Map

Let G be a Lie group. Hence, G is a manifold together with smooth multiplication and inversion maps. We collect some basic facts about these objects, stemming from the interplay of group structure and analysis.

Proposition 4.2.1 *The map*

$$G \times T_eG \to TG,\ (g, v) \mapsto T_eg(v)$$

is an isomorphism of vector bundles.

Proof. The proposed map is a bundle map covering the identity. It is a homeomorphism on the fibres, because T_eg is a linear isomorphism. Hence, the map is a homeomorphism. □

Definition 4.2.2 A smooth equivariant vector field $\xi : G \to TG$ is called a left invariant vector field.

Any vector $v \in T_eG$ determines a left invariant vector field by the assignment

$$\xi_v(g) = T_eg(v).$$

Proposition 4.2.3 *Let $\mathfrak{X}_G$ be the space of left invariant vector fields. The map*

$$\Phi : T_eG \to \mathfrak{X}_G,\ v \mapsto \xi_v$$

is a linear isomorphism.

Proof. Linearity is obvious. We define an inverse map by

$$\Psi : \mathfrak{X}_G \to T_eG,\ \xi \mapsto \xi(e).$$

We calculate

$$\Phi \circ \Psi(\xi)(g) = \Phi(\xi(e))(g) = \xi_{\xi(e)}(g) = T_eg(\xi(e)) = \xi(g),$$

the last equality by equivariance of ξ. On the other hand,

$$\Psi \circ \Phi(v) = \Psi(\xi_v) = \xi_v(e) = T_ee(v) = v,$$

because e acts as the identity. □

Lemma 4.2.4 *Let ξ be a left invariant vector field on a Lie group G. Then ξ induces a global flow $\varphi_\xi : G \times \mathbb{R} \to G$ on G. The map $\varphi_\xi(e, \cdot) : \mathbb{R} \to G$ is a homomorphism of Lie groups.*

Proof. We first show that the integral curves are homomorphisms of Lie groups locally. Take $s, t > 0$ such that $t + s < \varepsilon$ for some $\varepsilon > 0$ such that the integral curve φ through e is defined on $(-\varepsilon, \varepsilon)$. Let $\varphi_s : (-\varepsilon, \varepsilon) \to M$ be the curve $\varphi_s(t) = \varphi(s+t)$. By definition,

$$\dot{\varphi}(t+s) = T_t\varphi_s(1) = \xi(\varphi(t+s)).$$

On the other hand,

$$\dot{\overline{\varphi(s) \circ \varphi(t)}} = T_t(\varphi(s) \circ \varphi)(1).$$

The map to be differentiated is just the integral curve φ, composed with left translation by the fixed element $\varphi(s)$. Hence by the chain rule,

$$T_t(\varphi(s) \circ \varphi)(1) = T_{\varphi(t)}(\varphi(s)) \circ T_t\varphi(1) = T_{\varphi(t)}(\varphi(s))(\xi(\varphi(t))) = \xi(\varphi(s) \circ \varphi(t)),$$

by equivariance of ξ. The curves $t \mapsto \varphi(s+t)$ and $t \mapsto \varphi(s) \circ \varphi(t)$ both satisfy the same differential equation with the same initial condition, hence, they are equal.

Next, we show that the locally defined curve φ can be extended to a global integral curve. Let $t \in \mathbb{R}$ be arbitrary and let k be an integer such that $\frac{t}{k} \in (-\varepsilon, \varepsilon)$. Then we define

$$\varphi(t) = \varphi\left(\frac{t}{k}\right)^k.$$

If $t \in (-\varepsilon, \varepsilon)$, we have

$$\varphi\left(\frac{t}{k}\right)^k = \varphi(t)$$

by the homomorphism property. Therefore, for any other integer ℓ such that $\frac{t}{\ell} \in (-\varepsilon, \varepsilon)$, we have

$$\varphi\left(\frac{t}{k}\right)^k = \varphi\left(\frac{t}{k \cdot \ell}\right)^{k \cdot \ell} = \varphi\left(\frac{t}{\ell}\right)^\ell,$$

so the definition does not depend on k. It is clear that φ constitutes a homomorphism of Lie groups. Now we have

$$T_t\varphi_s(1) = T_t(\varphi(s) \circ \varphi)(1) = T_{\varphi(t)}(\varphi(s)) \circ T_t\varphi(1),$$

so in particular,

$$\dot{\varphi}(s) = T_0\varphi_s(1) = T_e(\varphi(s)) \circ T_0\varphi(1) = \xi_{T_0\varphi(1)}(\varphi(t)).$$

φ is therefore an integral curve of the vector field $\xi_{T_0\varphi(1)}$. But clearly,

$$\dot{\varphi}(0) = T_0\varphi(1) = \xi(\varphi(0)) = \xi(e),$$

hence, $\xi = \xi_{T_0\varphi(1)}$. We conclude that the integral curve of ξ through e is defined globally. Since

$$g.\dot{\varphi} = T_{\varphi(t)}g \circ T_t\varphi(1) = T_{\varphi(t)}g\xi(\varphi(t)) = \xi(g.\varphi(t)),$$

the curve $t \mapsto g.\varphi(t)$ is the integral curve of ξ through g and we see that all integral curves are defined globally, i.e. ξ induces a global flow. $\square$

Definition 4.2.5 Let G be a Lie group. The exponential map of G is defined as

$$\exp : T_eG \to G,\ v \mapsto \varphi_{\xi_v}(e, 1).$$

Proposition 4.2.6 *The exponential map* exp *is differentiable. Its derivative at* 0 *is the identity.*

Proof. We show differentiability first. Consider the map

$$\mathbb{R} \times G \times T_eG \to G \times T_eG,\ (t, g, \xi) \mapsto (g.\varphi_\xi(e, t), \xi).$$

This is the flow of the vector field on $G \times T_eG$, defined as $(g, \xi) \mapsto (\xi(g), 0)$. As a flow of a smooth field, it is differentiable. Thus the restriction to $\{1\} \times \{e\} \times T_eG$ is differentiable as well, and its first component is the exponential map.

To calculate the differential, let ξ be a left-invariant vector field. Then $t \cdot \xi$ is as well for any $t \in \mathbb{R}$. If $\varphi : (-\varepsilon, \varepsilon) \to G$ is an integral curve for ξ with $\varphi(0) = e$, let $\psi(s) = \varphi(t \cdot s)$. Then we have

$$\dot{\psi}(s) = t \cdot \dot{\varphi}(t \cdot s) = t \cdot \xi(\varphi(t \cdot s)) = t \cdot \xi(\psi(s)).$$

Hence, ψ is the integral curve for $t \cdot \xi$ with $\psi(0) = e$. In particular,

$$\exp(t\xi) = \psi(1) = \varphi(t)$$

and we obtain $\xi = \frac{\partial}{\partial t}\big|_{t=0} \exp(t \cdot \xi) = T_0 \exp(\xi)$. $\square$

By the implicit function theorem, we see that exp is a diffeomorphism locally around $e \in G$.

Example 4.2.7 Consider the Lie group $Aut(V)$ for some finite-dimensional vector space V. It is well known from basic calculus that $Aut(V)$ is an open subset of the set of linear endomorphisms of V, hence $T_eG = End(V)$. An element $A \in End(V)$ determines a left-invariant vector field $\xi_A : Aut(V) \to End(V),\ B \mapsto T_eB(A) = B \circ A$. Finding the flow of this field amounts to solving the differential equation

$$\dot{\varphi}(t) = \varphi(t) \cdot A,\ \varphi(0) = \mathrm{id}_V$$

which is readily seen to be given by

$$\varphi(t) = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}.$$

Hence, the exponential map is given as

$$\exp : End(V) \to Aut(V),\ A \mapsto \exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

4.3 Subgroups of Lie Groups

Lemma 4.3.1 *For $k \in \mathbb{Z}$, we have* $\exp(k \cdot \xi) = \exp(\xi)^k$.

Proof. This follows immediately from the homomorphism property of the integral curve through e, since

$$\exp(k \cdot \xi) = \varphi(k) = \varphi(1)^k = \exp(\xi)^k. \qquad \square$$

Theorem 4.3.2 *Let H be a subgroup of a Lie group G. Then H is a submanifold of G if and only if H is closed.*

Proof. Suppose that H is a submanifold of G. Then H is locally closed and therefore we find a neighbourhood U of $e \in G$ such that $U \cap H$ is closed in U. Let $h \in \overline{H}$ be an element of the closure of H. Since U is open, we find an element k in H such that $h \in kU$. But $k^{-1}\overline{H} = \overline{H}$, since $\overline{H}$ is a subgroup, hence we have that $k^{-1}h \in \overline{H} \cap U = H \cap U$, since this set is closed in U. We conclude that $h \in H$, so H is closed.

Now let H be a closed subgroup of G. It suffices to show that $H \cap U$ is a submanifold of U for U a neighbourhood of e in G. We use the exponential map to define a subspace L of T_eG which will be seen to correspond to the tangential space T_eH. Then using that the exponential map is a local diffeomorphism at e, we will see that $\exp(T_eG) = U$ and $\exp(L) = H \cap U$. It follows immediately that $H \cap U$ is a submanifold of U.

We turn to the construction of L. Choose an inner product on T_eG and an open neighbourhood $U' \subseteq T_eG$ of 0 such that the exponential map, restricted to U', is a diffeomorphism onto $U \subseteq G$ and let $\log : U \to U'$ be its inverse. Let $H' = \log(H \cap U)$. We show the following three properties.

1. Let $\{h_n\}_{n \in \mathbb{N}}$ be a sequence in H' converging to 0 such that the sequence $h_n / \|h_n\|$ converges to an element ξ of T_eG. Then $\exp(t\xi) \in H$ for $t \in \mathbb{R}$.

2. The set
$$W = \{s\xi \mid \xi = \lim(h_n / \|h_n\|),\ h_n \in H',\ s \in \mathbb{R}\}$$
is a linear subspace of T_eG.

3. $\exp(W)$ is a neighbourhood of e in H.

Provided (i)-(iii) holds, we see that $H \cap U \cap \exp(W)$ is diffeomorphic to $U' \cap W$ via exp and thus a submanifold of U. So we proceed to prove (i)-(iii).

1. We have that $(t/\|h_n\|) \cdot h_n \to t\xi$ for $n \to \infty$. We choose elements $m_n \in \mathbb{Z}$ such that $m_n \cdot \|h_n\| \to t$, which is possible since h_n converges to 0. With this choice, $\exp(m_n \cdot h_n) = \exp(m_n \cdot \|h_n\| \cdot (h_n/\|h_n\|))$ converges to $\exp(t\xi)$. On the other hand, $\exp(m_n \cdot h_n) = \exp(h_n)^{m_n} \in H$, and since H is closed, $\exp(t\xi) \in H$.

2. Let ξ, η be two elements of W and $h(t) = \log(\exp(t\xi) \cdot \exp(t\eta))$. We note that the differential of the multiplication map $\mu : G \times G \to G$ at (e, e) is given by $T_eG \times T_eG \to T_eG,\ (\xi, \eta) \mapsto \xi + \eta$. This follows immediately from the relation

$\mu(g,e) = g$ and $\mu(e,g) = e$, because $T_{(e,e)}\mu$ must be a linear map restricting to the identity in the two components. The differential of log at e is the identity, because it is the inverse of exp. Therefore,

$$\lim_{t\to 0} \frac{h(t)}{t} = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} h(t) = \xi + \eta.$$

Hence,

$$\lim \frac{h(t)}{\|h(t)\|} = \frac{\xi+\eta}{\|\xi+\eta\|},$$

And we conclude that $\xi + \eta \in W$.

3. We can write $T_eG = W \oplus V$ for some subspace V, where the sum is orthogonal. As before, we see that the map

$$E : W \oplus V \to G,\ (\xi,\eta) \mapsto \exp(\xi) \circ \exp(\eta)$$

is locally invertible at the origin. Assume that the statement is false. Then we find elements $h_n \in H$ with $h_n \to e$ and $h_n \notin \exp(W)$. Since the exponential map is locally surjective, we therefore find elements $\xi_n \in W$ and $\eta_n \in V$, $\eta_n \neq 0$, such that $E(\xi_n, \eta_n) = h_n$. By invertibility, we must have $(\xi_n, \eta_n) \to (0,0)$. The space V is closed, therefore we can assume that

$$\frac{\eta_n}{\|\eta_n\|} \to \eta \in V$$

with $\|\eta\| = 1$. But $\exp(\xi_n) \in H$ by (i), and so $\exp(\eta_n) \in H$ as well, i.e. $\eta_n \in H'$. By definition of W, $\eta \in W$, which is impossible. $\square$

Theorem 4.3.3 *Let $f : G \to H$ be a continuous group homomorphism. Then f is smooth. In particular, if a topological group has the structure of a Lie group, this structure is unique up to diffeomorphism.*

Proof. We prove that f is smooth. Consider the set

$$\Gamma_f = \{(g, f(g)) \mid g \in G\} \subseteq G \times H.$$

Then Γ_f is a closed subgroup of $G \times H$ and therefore it is a submanifold. The projection $p_1 : \Gamma_f \to G$ is a differentiable homeomorphism and $T_e p_1$ is bijective. Hence, p_1 is a diffeomorphism locally around e and by translation, it is a diffeomorphism. Then we can write $f = p_2 \circ p_1^{-1}$, so f is smooth. The second statement follows immediately by considering the homeomorphism id_G. $\square$

The property of being a Lie group is not as strong as one might think, at least in addition to the requirement to be a topological group. It has been shown, by the way as a solution to Hilbert's fifth problem, that a connected, locally compact topological group has a unique differentiable structure turning it into a Lie group if and only if there is a neighbourhood of e containing no subgroups other than $\{e\}$. This purely topological characterization of a smooth property has the impact that Lie group actions really stand out among actions of topological groups not only in the smooth category, but also in the topological category. We therefore will consider in the following smooth actions of Lie groups on manifolds, as well as actions of Lie groups on topological spaces.

Since taking quotients is such a fundamental concept in the theory of group actions, it is obviously desirable to have criteria that determine when a quotient space of a manifold by a Lie group action is again a manifold. The most easy examples show that this is not the case in general. We need an auxiliary result from differential topology.

Lemma 4.3.4 *Let $R \subseteq M \times M$ be an equivalence relation on the smooth manifold M and $p : M \to M/_R$ the quotient map. If R is a closed submanifold of $M \times M$ and the projection $\pi_1 : R \to M$ is a submersion, then there is a smooth structure on $M/_R$ turning p into a submersion.*

Proof. Let r be the dimension of R. Since $\pi_1 : R \to M$ is a submersion, for any $x \in M$ there is a neighbourhood $U \subseteq M$ of x such that $U \times U \cong \mathbb{R}^m \times \mathbb{R}^m$ and $U \times U \cap R$ corresponds to an r-dimensional subspace V of $\mathbb{R}^m \times \mathbb{R}^m$ which is an equivalence relation on $\mathbb{R}^m$. So V splits into a direct sum $\Delta(\mathbb{R}^m) \oplus W$, where $\Delta(\mathbb{R}^m)$ is the diagonal in $\mathbb{R}^m$ and W is an $(r-m)$-dimensional subspace of the antidiagonal $\Delta^-(\mathbb{R}^m) = \{(v, -v) \mid v \in \mathbb{R}^m\}$. We have a projection $\pi : V \to \mathbb{R}^m$, making the diagram

$$\begin{tikzcd}
U \times U \arrow[r, "(\varphi,\varphi)"] & \mathbb{R}^m \oplus \mathbb{R}^m & \\
U \times U \cap R \arrow[u] \arrow[r, "\psi"] \arrow[d, "\pi_1"'] & \mathbb{R}^m \oplus \mathbb{R}^{r-m} \arrow[u] \arrow[r, "\cong"] \arrow[d, "\pi_1"] & \Delta(\mathbb{R}^m) \oplus W = V \arrow[ul] \arrow[dl] \\
U \arrow[r, "\varphi"] & \mathbb{R}^m &
\end{tikzcd}$$

commutative, where φ, ψ are charts for M, R, respectively. The induced equivalence relation on $\mathbb{R}^m$ is given by

$$v \sim w \Longleftrightarrow (v, w) \in V.$$

Thus, $v \sim w$ if and only if the projection of (v, w) to the antidiagonal $\Delta^-(\mathbb{R}^m)$ has image in W, i.e.

$$v - w \in W.$$

Therefore the quotient space $\mathbb{R}^m/_\sim$ is equal to the quotient space $\mathbb{R}^m/_W \cong \mathbb{R}^{2m-r}$ and this induces a homeomorphism $p(U) \cong \mathbb{R}^{2m-r}$ which determines a differentiable structure on $M/_R$. It is immediate that, with this differentiable structure, p becomes a submersion of manifolds. $\square$

Before we continue, we recall briefly the constant rank theorem from analysis, which will be important in several cases to come.

Proposition 4.3.5 *Let M, N be smooth manifolds and $f : M \to N$ a smooth map. Let $x \in M$ and $U' \subseteq M$ an open neighbourhood of x such that $\operatorname{rank} T_y f = \operatorname{rank} T_x f = r$ for all $y \in U'$. Then there are chart neighbourhoods $U \subseteq M$ of x and $V \subseteq N$ of $f(x)$ together with charts $\varphi : U \to \mathbb{R}^m$, $\psi : V \to \mathbb{R}^n$, such that $\psi \circ f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ has the form $x \mapsto (x_1, \ldots, x_r, 0, \ldots, 0)$.*

The proof is an easy application of the inverse function theorem.

Proposition 4.3.6 *Let G be a compact Lie group acting freely on the manifold M. Then $M/_G$ has a unique differentiable structure such that $p : M \to M/_G$ is a smooth submersion.*

Proof. The action of G defines the relation $R = \{(x, g.x) \mid x \in M,\ g \in G\}$. According to the preceeding lemma, we have to show that this is a submanifold and projection to the first component is a submersion. Let

$$\sigma : G \times M \to M \times M,\ (g, x) \mapsto (x, g.x).$$

Clearly, the image of σ is R and σ is smooth. Furthermore, it is injective and closed, the latter since the action map is closed. Therefore, σ is a homeomorphism onto its image. To show that σ is an immersion, we look at the differential of $\pi_1 \circ \sigma$. This is just the map $(g, x) \mapsto x$, hence its differential at (g, x) has kernel $T_g G \times \{0\}$. If we show the differential of $\pi_2 \circ \sigma$ to be injective on this subspace, it follows that the differential of σ itself is injective. Calculating the differential at (g, x) of $\pi_2 \circ \sigma$ on $T_g G$ amounts to calculating the differential of $f : G \to M,\ g \mapsto gx$. By freeness of the action, f is an injective map. Hence, we are done if we can show that $T_g f$ has constant rank. Since f is equivariant with respect to the left translation action on G, the rank of $T_g f$ equals the rank of $T_e f$, so indeed it has constant rank. In conclusion, we have shown that σ is an immersion, so R is a submanifold. Finally, since $\pi_1 \circ \sigma$ is projection to the second factor, π_1 itself must be a submersion. □

This result has several important applications.

Corollary 4.3.7 *Let $H \subseteq G$ be a closed subgroup of the compact Lie group G. Then $G/_H$ carries the structure of a smooth manifold such that the projection is a submersion.*

Proof. This follows at once, since the action of H on G by left translation is free. □

Corollary 4.3.8 *Let G be a compact Lie group, H a closed subgroup and $p : G \to G/_H$ the projection. Then p has local cross sections.*

Proof. Since p is a submersion, for $g \in G$ we find a neighbourhood U of $[g] \in G/_H$ and V of g as well as charts $\varphi : U \to \mathbb{R}^m$, $\psi : V \to \mathbb{R}^{m+n}$ such that

$$\begin{tikzcd}
V \arrow[r, "\psi"] \arrow[d, "p"'] & \mathbb{R}^{n+m} \arrow[d] \\
U \arrow[r, "\varphi"] & \mathbb{R}^m
\end{tikzcd}$$

commutes, where the map on the right is projection to the first m entries. The canonical cross section of this projection thus induces a cross section $U \to V$ of p. $\square$

In particular it follows that $p : G \to G/_H$ is an H-principal bundle if G is compact Lie and H a closed subgroup.

Proposition 4.3.9 *Let M be a smooth G-manifold, G a compact Lie group. For $x \in M$, the orbit Gx is a smooth submanifold of M.*

Proof. Since G is compact, Gx is a topological submanifold of M. The map $G \to M$, $g \mapsto g.x$ has constant rank, as we saw in the proof of Proposition 4.2.4. It induces an injective differentiable map $G/_{G_x} \to M$ of constant rank, which therefore is an immersion. $\square$

4.4 Invariant Integration on Topological Groups

Before we delve deeper into the structure theory of G-manifolds, where G is a compact Lie group, we must introduce a very important tool that already exists on compact topological groups. It often serves to make things equivariant. This tool is an invariant measure on the group G. Unfortunately, the details to construct such a measure are quite exhausting, and so we stick to the common technique of citing [Po62] for the existence proof. We will state the theorem nevertheless.

Theorem 4.4.1 *Let G be a compact topological Hausdorff group. Let $r_h, \ell_h : \mathcal{C}^0(G, \mathbb{R}) \to \mathcal{C}^0(G, \mathbb{R})$ be the functions sending f to the function*

$$r_h f : G \to \mathbb{R},\ r_h f(g) = f(gh), \quad \ell_h f : G \to \mathbb{R},\ \ell_h f(g) = f(h^{-1}g).$$

Then there is a unique real valued function $I : \mathcal{C}^0(G, \mathbb{R}) \to \mathbb{R}$ such that

1. *I is a linear homomorphism.*
2. *I is order preserving, that is, if $f(g) \geq 0$ for all $g \in G$, then $I(f) \geq 0$.*
3. *I is normalized, that is, $I(1) = 1$.*
4. *I is left and right invariant, meaning that $I(f) = I(r_h f) = I(\ell_h f)$ for all $h \in G$, $f \in \mathcal{C}^0(G, \mathbb{R})$.*

The function I is obtained from a σ-additive measure on the Borel σ-algebra of G. We will therefore write

$$I(f) = \int_G f(g)\, dg.$$

This function is often called the Haar Integral of f. The next result is a standard application for the Haar Integral. We call two representations ρ, σ of G on $\mathbb{R}^n$ equivalent, if there is a matrix $A \in GL(n)$ such that $\rho(g) = A^{-1}\sigma(g)A$. This means that, after a change of basis, the matrix representation of $\rho(g)$ and $\sigma(g)$ is identical.

Proposition 4.4.2 *Let G be a compact group acting linearly on a finite dimensional vector space V. Then V is equivalent to an orthogonal action of G.*

PROOF. Since V is finite dimensional, we can assume $V = \mathbb{R}^n$, equipped with the standard scalar product, and G acts by matrix multiplication. We can now define a new scalar product on V by

$$\langle v, w \rangle_G = \int_G \langle g.v, g.w \rangle \; dg.$$

This is obviously a symmetric bilinear form. For definiteness, we have

$$\langle v, v \rangle_G = \int_G \|g.v\|^2 \; dg.$$

The right hand side is strictly positive, so the form is positive definite. Finally, it is invariant under translations, since

$$\langle h.v, h.w \rangle_G = \int_G \langle gh.v, gh.w \rangle \; dg = \int_G \langle g.v, g.w \rangle \; dg$$

by right invariance. The scalar product is represented by a positive definite symmetric matrix $B \in GL(n)$, i.e.

$$\langle v, w \rangle_G = v^T B w,$$

and we have shown that

$$(gv)^T B(gw) = v^T g^T B g w = v^T B w$$

for every $g \in G$ and $v, w \in \mathbb{R}^n$. We define a new action of G by

$$g_\bullet v = \sqrt{B} g \sqrt{B}^{-1} v.$$

Note that $\sqrt{B}$ is symmetric and commutes with B. By definition, this new action is equivalent to the old one. In addition, we have

$$\begin{aligned}
\langle g_\bullet v, g_\bullet w \rangle &= (\sqrt{B} g \sqrt{B}^{-1} v)^T (\sqrt{B} g \sqrt{B}^{-1} w) \\
&= (\sqrt{B}^{-1} v)^T g^T \sqrt{B} \sqrt{B} g (\sqrt{B}^{-1} w) \\
&= (\sqrt{B}^{-1} v)^T g^T B g (\sqrt{B}^{-1} w) \\
&= v^T \sqrt{B}^{-1} B \sqrt{B}^{-1} w \\
&= v^T w \\
&= \langle v, w \rangle .
\end{aligned}$$

We see that the new action is orthogonal. The proposition is proven. □

A second important result is concerned with the existence of equivariant maps. We recall from point set topology that if $A \subseteq X$ is a closed subspace of a normal space X and $f : A \to \mathbb{R}^n$ is a continuous map into some euclidean space, then there is an extension $F : X \to \mathbb{R}^n$ of f. The equivariant analog is quite similar und easy to prove, given existence of the Haar integral.

Theorem 4.4.3 *Let X be a normal G-space, G a compact group, and let $A \subseteq X$ be a closed invariant subspace. Then every equivariant map $f : A \to V$ into a representation of G has an equivariant extension $F : X \to \mathbb{R}^n$.*

Proof. Let $F' : X \to \mathbb{R}^n$ be any extension of f. F' need not be equivariant, so we define

$$F(x) = \int_G g^{-1} F(g.x)\, dg.$$

F is equivariant, since

$$\begin{aligned} F(h.x) &= \int_G g^{-1} F'(gh.x)\, dg \\ &= \int_G h(gh)^{-1} F'(gh.x)\, dg \\ &= h \int_G (gh)^{-1} F'(gh.x)\, dg \\ &= h \int_G g^{-1} F'(g.x)\, dg \\ &= hF(x) \end{aligned}$$

by right invariance of the Haar integral. F extends f, because on A, by equivariance of f, we obtain

$$\begin{aligned} F(a) &= \int_G g^{-1} F'(g.a)\, dg \\ &= \int_G g^{-1} f(g.a)\, dg \\ &= \int_G g^{-1} g f(a)\, dg \\ &= \int_G f(a)\, dg \\ &= f(a). \end{aligned}$$

The theorem is proven. □

4.5 The Tubular Neighbourhood Theorem

In this section we are going to prove one of the most important theorems concerning the local structure of G-manifolds. If M is an ordinary manifold, by definition every point has a neighbourhood which is homeomorphic to a euclidean space. The general equivariant philosophy would like to find invariant charts for G-manifolds, which certainly is an impossible task. So instead, we take the different point of view to think of charts as neighbourhoods of points that are homeomorphic to a vector bundle over the point. The equivariant generalization now is to look for invariant neighbourhoods of orbits that are homeomorphic to a G-vector bundle over the orbit. The next lemmata are aiming at the proof of the existence of such neighbourhoods.

Lemma 4.5.1 *Let G be a compact Lie group which acts orthogonally on $\mathbb{R}^n$. Let $v_0 \in \mathbb{R}^n$ and $G_{v_0} = H$. Let V be the normal space of the orbit Gv_0 at v_0. Then there is a neighbourhood U of $[e]$ in $G/_H$, a local cross section $\sigma : U \to G$ and a number $\varepsilon > 0$ such that the restriction of the action map to $\sigma(U) \times \mathbb{B}_\varepsilon$ is a homeomorphism onto an open neighbourhood of v_0.*

Proof. Let σ be any smooth cross section at $[e]$ for $p : G \to G/_H$ with $\sigma([e]) = e$. Then σ is inverse to the restriction of the action map $\alpha : \sigma(U) \times \{v_0\} \to Gv_0 \cong G/_H$ and therefore, this restriction is a diffeomorphism. It follows that the differential of α at (e, v_0) is an isomorphism onto the tangent space of Gv_0. Clearly, the differential of the embedding $\{e\} \times V \to \mathbb{R}^n$ is an isomorphism onto V, and therefore, the differential at (e, v_0) of the action map $\sigma(U) \times V \to \mathbb{R}^n$ is an isomorphism. By the implicit function theorem, this map is a diffeomorphism of some open neighbourhood of (e, v_0) onto a neighbourhood of v_0. □

Lemma 4.5.2 *Under the same assumptions as above, the map*

$$G \times_H V \to \mathbb{R}^n,\ [g, v] \mapsto g.v$$

induces a homeomorphism of $G \times_H \mathbb{B}_\varepsilon$ onto the open neighbourhood $G(\mathbb{B}_\varepsilon)$ of Gv_0 in $\mathbb{R}^n$.

Proof. Choose $\sigma : U \to G$ as above and denote $G \setminus p^{-1}(U)$ by K. Then K is compact and $K.v_0 \subseteq \mathbb{R}^n \setminus \{v_0\}$. Any neighbourhood of $K.v_0$ contains a set of the form $K.C$, where $C \subseteq \mathbb{R}^n$ is a compact neighbourhood of v_0. Hence, for C sufficiently small, $K.C \cap C$ must be empty. So we must have $K(\mathbb{B}_\varepsilon) \cap \mathbb{B}_\varepsilon = \emptyset$ for $\varepsilon > 0$ sufficiently small. We claim that the map of the statement of the lemma is injective for such a $\mathbb{B}_\varepsilon$. Assume $g.v = g'.w$. Then $g^{-1}g'.w = v$, implying that $g^{-1}g' \notin K$. Hence, this is an element of $p^{-1}(U)$ and we find $u \in U$ such that $g' = g\sigma(u)h$ for some $h \in H$. Consequently, we have $\sigma(u)h.w = v$. $h.w$ is an element of $\mathbb{B}_\varepsilon$, since H acts orthogonally on V. Taking ε so small that the conclusion of Lemma 4.5.1 holds, we must have $\sigma(u) = e$ and $h.w = v$. In conclusion we have shown that

$$[g, v] = [g\sigma(u), h.w] = [g\sigma(u)h, w] = [g', w],$$

so the map is indeed injective. It is induced by the action map, which is closed by compactness of G, hence, our map is closed as well. Surjectivity is obvious, and the claim is proven. □

Definition 4.5.3 Let X be a G-space and $x \in X$ and V be a G_x-space. A tube around the orbit Gx is a G-homeomorphism $\varphi : G \times_{G_x} V \to X$ onto its image U, such that U is an open neighbourhood of Gx. A linear tube is a tube where V is a G_x representation, and the homeomorphism $\varphi : G \times_{G_x} V \to U$ satisfies $\varphi([g, 0]) = g.x$. The set U is also called a tubular neighbourhood of Gx.

For the existence result, we will need the following fact. It is not hard to prove but needs some preliminaries from the structure theory of compact Lie groups. Therefore, we just cite [Br72] as a reference for its proof.

Theorem 4.5.4 *Let G be a compact Lie group and $H \subseteq G$ a closed subgroup. Then there exists an orthogonal representation $\rho : G \to O(n)$ of G and a point $v \in \mathbb{R}^n$ such that $G_v = H$.*

Theorem 4.5.5 *Let X be an invariant subset of a G-manifold with G a compact Lie group. Then every orbit $Gx \subseteq X$ has a tubular neighbourhood.*

Proof. Take $x \in X$ and denote G_x by H. We find an orthogonal representation V of G and a point $v \in V$ with $G_v = H$. The map $\varphi : Gx \to Gv,\ g.x \mapsto g.v$ extends to a G-map $\Phi : X \to V$. Choose an $\varepsilon > 0$ such that $G \times_H \mathbb{B}_\varepsilon(v) \to V$ is a homeomorphism onto an open neighbourhood of v and let $W = \Phi^{-1}(G.\mathbb{B}_\varepsilon(v))$. W is invariant and open, since $G.\mathbb{B}_\varepsilon(v)$ is open. Furthermore, the composition

$$r : W \xrightarrow{\Phi} G(\mathbb{B}_\varepsilon(v)) \longrightarrow Gv \xrightarrow{\varphi^{-1}} Gx$$

is an equivariant retraction. In particular, there is a G-map $W \to Gx$ and so W is G-homeomorphic to the space $G \times_H r^{-1}(x)$. $r^{-1}(x)$, in turn, clearly is G-homeomorphic to $\mathbb{B}_\varepsilon(v)$, which is G-homeomorphic to V itself. $\square$

Remark 4.5.6 It is a bit more difficult to show that if $X = M$ is a manifold, then linear tubes exist around each orbit. The proof can be found in chapter VI of [Br72]. It uses the exponential map of the geodesic flow on a G-manifold M, where it can be assumed that M is a smooth Riemannian manifold and G acts via isometries. We will use existence of linear tubes in some of the following results.

Example 4.5.7 Consider the 2-sphere $\mathbb{S}^2 = M$ with the following action of $\mathbb{S}^1 \times \mathbb{Z}_2$. $\mathbb{S}^1$ acts via rotation around the z-axis. $\mathbb{Z}_2$ acts as eflection at the (x, y)-plane. The points in M are divided into three orbit types: The north and south pole have type $(\mathbb{S}^1)$, the points on the equator in the (x, y)-plane have type $(\mathbb{Z}_2)$, all other points have type (e). A tubular neighbourhood of the orbit of the north pole has the form $(\mathbb{S}^1 \times \mathbb{Z}_2) \times_{\mathbb{S}^1} \mathbb{R}^2$, where $\mathbb{R}^2$ carries the canonical action of $\mathbb{S}^1$. It is easily seen that this space is $\mathbb{S}^1 \times \mathbb{Z}_2$-difeomorphic to $\mathbb{Z}_2 \times \mathbb{R}^2$, where $\mathbb{S}^1$ acts on the second component, $\mathbb{Z}_2$ on the first. Hence, a tubular neighbourhood of the orbit through the pole is just the union of two symmetric polar caps.

For a point in the equator, a tubular neighbourhood has the form $(\mathbb{S}^1 \times \mathbb{Z}_2) \times_{\mathbb{Z}_2} \mathbb{R}$, where $\mathbb{R}$ has the canonical $\mathbb{Z}_2$-action. As above, this is equivariantly diffeomorphic to the space $\mathbb{S}^1 \times \mathbb{R}$. So a tubular neighbourhood of the equator is a $\mathbb{S}^1$-symmetric ring around the equator, fibred by lines meeting the equator transversally with the flipping action of $\mathbb{Z}_2$ on such a line.

Finally, a point of orbit type (e) has as a tubular neighbourhood the space $\mathbb{S}^1 \times \mathbb{Z}_2 \times \mathbb{R}$, trivial action on $\mathbb{R}$. Hence, this is the disjoint union of two $\mathbb{S}^1$-symmetric rings, flipped by the $\mathbb{Z}_2$-action.

The tubular neighbourhood theorem has an almost arbitrary amount of applications. We will sketch at least some of them.

Corollary 4.5.8 *Let X be as in the theorem and take $x \in X$. Then there is an invariant neighbourhood U of Gx such that, for every $y \in U$, $(G_y) \leq (G_x)$.*

Proof. Let U be a tubular neighbourhood of Gx. Then there is a G-homeomorphism $\varphi : U \to G \times_{G_x} V$. For $y \in U$, we have $\varphi(y) = [g, v]$ for some $g \in G$, $v \in V$. Define a map

$$G.y \to G.x,\ g'.y = \varphi^{-1}([g'g, v]) \mapsto \varphi^{-1}([g'g, 0]).$$

This is obviously a G-map and we conclude that $(G_y) \leq (G_x)$. □

Proposition 4.5.9 *Let M be a G-manifold and $H \subseteq G$ a closed subgroup. The sets $M_{(H)}$ and M^H are smooth submanifolds of M. Furthermore, $M_{(H)}$ is a smooth $G/_H$-bundle over its orbit space.*

Proof. Take any $x \in M_{(H)}$. Then Gx has a tubular neighbourhood U G-diffeomorphic to $G \times_H V$ for some H-representation V. The set $U_{(H)}$ corresponds to the set

$$(G \times_H V)_{(H)} = G \times_H V^H.$$

Since H acts trivially on V^H, this space is G-diffeomorphic to $G/_H \times V^H$. $G/_H$ is a smooth manifold, so the claim of the proposition for $M_{(H)}$ follows (the manifold part as well as the bundle part). For the sets M^H, these can be identified with the manifolds $res^G_H(M)_{(H)}$, so the claim follows from what we have already proven. □

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