Introduction To Lie algebras

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1 Lie algebras

1.1 Definition and examples

Definition 1.1. A Lie algebra is a vector space $g$ over a field $F$ with an operation $\cdot : g \times g \rightarrow g$ which we call a Lie bracket, such that the following axioms are satisfied:

- It is bilinear.
- It is skew symmetric: $[x, x] = 0$ which implies $[x, y] = -[y, x]$ for all $x, y \in g$.
- It satisfies the Jacobi Identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Definition 1.2. A Lie algebra Homomorphism is a linear map $H \in \text{Hom}(g, h)$ between to Lie algebras $g$ and $h$ such that it is compatible with the Lie bracket:

$H : g \rightarrow h$ and $H([x, y]) = [H(x), H(y)]$.

Example 1.1. Any vector space can be made into a Lie algebra with the trivial bracket: $[v, w] = 0$ for all $v, w \in V$.

Example 1.2. Let $g$ be a Lie algebra over a field $F$. We take any nonzero element $x \in g$ and construct the space spanned by $x$, we denote it by $Fx$. This is an abelian one dimensional Lie algebra: Let $a, b \in Fx$. We compute the Lie bracket.

$[\alpha x, \beta x] = \alpha \beta [x, x] = 0$.

where $\alpha, \beta \in F$. Note: This shows in particular that all one dimensional Lie algebras have a trivial bracket.

Example 1.3. Any associative algebra $\mathfrak{A}$ can be made into a Lie algebra by taking commutator as the Lie bracket:

$[x, y] = xy - yx$

for all $x, y \in \mathfrak{A}$.

Example 1.4. Let $V$ be any vector space. The space of $\text{End}(V)$ forms an associative algebra under function composition. It is also a Lie algebra with the commutator as the Lie bracket. Whenever we think of it as a Lie algebra we denote it by $\mathfrak{gl}(V)$. This is the General Linear Lie algebra.

Example 1.5. Let $V$ be a finite dimensional vector space over a field $F$. Then we identify the Lie algebra $\mathfrak{gl}(V)$ with set of $n \times n$ matrices $\mathfrak{gl}_n(F)$, where $n$ is the dimension of $V$. The set of all matrices with the trace zero $\mathfrak{sl}_n(F)$ is a subalgebra of $\mathfrak{gl}_n(F)$.

Example 1.6. The set of anti symmetric matrices with the trace zero denoted by $\mathfrak{so}_n$ forms a Lie algebra under the commutator as the Lie bracket .

Example 1.7. Heisenberg algebra: We look at the vector space $\mathfrak{h}$ generated over $F$ by the matrices:

$$
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
$$

This is a linear subspace of $\mathfrak{gl}_3(F)$ and becomes a Lie algebra under the commutator bracket. The fact that $\mathfrak{h}$ is closed under the commutator bracket follows from the well known commutator identity on the standard basis of $n \times n$ matrices:

$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}$

where $\delta_{ij}$ is the Kronecker delta.
1.2 Some Basic Notions

**Definition 1.3.** Let \( g \) be a Lie algebra over \( F \). Then a linear subspace \( U \subseteq g \) is a **Lie subalgebra** if \( U \) is closed under the Lie bracket of \( g \):

\[
[x, y] \in U
\]

for all \( x, y \in U \).

**Definition 1.4.** Let \( I \) be a linear subspace of a Lie algebra \( g \). Then \( I \) is an **ideal** of \( g \) if

\[
[x, y] \in I
\]

whenever \( x \in I \) and \( y \in g \).

**Definition 1.5.** A Lie algebra \( g \) is called **abelian** if the Lie bracket vanishes for all elements in \( g \):

\[
[x, y] = 0
\]

for all \( x, y \in g \).

**Definition 1.6.** Let \( U \) be a non empty subset of \( g \), we call \( \langle U \rangle \) the **Lie subalgebra (ideal) generated by** \( U \), where:

\[
\langle U \rangle = \bigcap \{ I \subseteq g : I \text{ is Lie subalgebra (ideal) containing } U \}
\]

2 Free Lie Algebras

Let \( X \) be a set. We define \( W_X = \bigoplus_{x \in X} F \), where \( F \) is an arbitrary field. Then we denote the tensor algebra of \( W_X \) by \( TW_X \) which is as well a Lie algebra. **The Free Lie algebra** on \( X \) is the Lie subalgebra in \( TW_X \) generated by \( X \). Where \( X \) can be canonically embedded into \( W_X \) via the map:

\[
f : X \rightarrow W_X \quad x \mapsto e_x
\]

3 Representations

A representation of a Lie algebra \( g \) is a Lie algebra homomorphism from \( g \) to the Lie algebra \( \text{End}(V) \):

\[
\rho : g \rightarrow \text{gl}(V).
\]

**Definition 3.1.** For a Lie algebra \( g \) and any \( x \in g \) we define a map

\[
ad_x : g \rightarrow g, \quad y \mapsto [x, y]
\]

which is the **adjoint action**.

Every Lie algebra has a representation on itself, the **adjoint representation** defined via the map:

\[
ad : g \rightarrow \text{gl}(g) \quad x \mapsto ad_x
\]

**Definition 3.2.** For two representations of a Lie algebra \( g \), \( \phi : g \rightarrow \text{gl}(V) \) and \( \phi' : g \rightarrow \text{gl}(V') \) a **morphism** from \( \phi \) to \( \phi' \) is a linear map \( \psi : V \rightarrow V' \) such that it is compatible with the action of \( g \) on \( V \) and \( V' \):

\[
\phi'(x)\psi = \psi\phi(x)
\]

For all \( x \in g \). This constitutes the category \( g \text{-Mod} \).
4 The Universal Enveloping Algebra

For any associative algebra we construct a Lie algebra by taking the commutator as the Lie bracket. Now let us think in the reverse direction. We want to see if we can construct an associative algebra from a given Lie algebra and its consequences. With this construction, instead of non-associative structures;Lie algebras, we can work with nicer and better developed structures: Unital associative algebras that captures the important properties of our Lie algebra.

4.1 Constructing $U(\mathfrak{g})$

Let us construct the tensor algebra of the Lie algebra $\mathfrak{g}$:

$$T_0 \mathfrak{g} = \bigoplus_{k=0}^{\infty} T^k \mathfrak{g} = \bigoplus_{k=0}^{\infty} \mathfrak{g} \otimes \ldots \otimes \mathfrak{g}$$

We look at the two sided ideal $I$ generated by: $g \otimes h - h \otimes g - [g,h]$ For $g, h \in T_0 \mathfrak{g}$. Its elements look like:

$$\sum_{i=0}^{k} c_i (x_1^{(i)} \otimes \ldots \otimes x_n^{(i)}) \otimes (g_i \otimes h_i - h_i \otimes g_i - [g_i,h_i]) \otimes (y_1^{(i)} \otimes \ldots \otimes y_n^{(i)})$$

for $g_i, h_i \in T_0 \mathfrak{g}$.

Now the universal enveloping algebra is constructed by taking the quotient of our tensor algebra: $U(\mathfrak{g}) = T(\mathfrak{g})/I$.

**Definition 4.1.** For two ring homomorphisms $S : U(\mathfrak{g}) \rightarrow \text{End}(W)$ and $S' : U(\mathfrak{g}) \rightarrow \text{End}(W')$ a morphism from $S$ to $S'$ is a map $t : W \rightarrow W'$ such that it is compatible with the action of $U(\mathfrak{g})$ on $W$ and $W'$:

$$S'(x)t = tS(x).$$

This constitutes the category $U(\mathfrak{g}) - \text{Mod}$.

**Theorem 4.1.** To each representation $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ we can associate some $S_\phi : U(\mathfrak{g}) \rightarrow \text{End}(V)$ and to each ring homomorphism $S : U(\mathfrak{g}) \rightarrow \text{End}(V)$ we can associate some $\phi_S : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, such that a morphism $\psi$ from $\phi$ to $\phi'$ is also a morphism from $S_\phi$ to $S_{\phi'}$ and a morphism $t$ from $S$ to $S'$ is also a morphism from $\phi_s$ to $\phi'_s$, and $\phi_{S_s} = \phi$ and $S_{\phi_s} = S$.

**Remark.** This is equivalent as to say $\mathfrak{g}$-Mod is equivalent to $U(\mathfrak{g})$-Mod.

**Proof.** Suppose we are given a representation $\phi$, we want to construct $S_\phi$ such that it will satisfy the properties of a ring homomorphism, that is:

1. $S_\phi(1) = 1$
2. $S_\phi(r + r') = S(r) + S(r')$
3. $S_\phi(rr') = S(r)S(r')$

We define $S_\phi$ as follows:

$$S_\phi(1) = 1$$

So that (1) holds. We consider an element of $x_1 \otimes \ldots \otimes x_n \in U(\mathfrak{g})$, we define $S$ on $x_1 \otimes \ldots \otimes x_n$:

$$S_\phi(x_1 \otimes \ldots \otimes x_n) = \phi(x_1) \circ \phi(x_2) \circ \ldots \circ \phi(x_n).$$

Now we define $S$ on the rest of $U(\mathfrak{g})$ by linear extension:

$$S_\phi(x_1 \otimes \ldots x_n + y_1 \otimes \ldots y_m) = S_\phi(x_1 \otimes \ldots x_n) + S_\phi(y_1 \otimes \ldots y_m)$$
It is clear that (2) is also satisfied. We show that (3) is satisfied as well:

\[ S_\phi(x_1 \otimes \ldots x_n \otimes y_1 \otimes \ldots y_m) = \phi(x_1) \circ \ldots \circ \phi(x_n) \circ \phi(y_1) \ldots \circ \phi(y_m) = S_\phi(x_1 \otimes \ldots x_n) \circ S_\phi(y_1 \otimes \ldots y_n) \]

Now suppose we have \( S \) and we want to define \( \phi \):

\[ \phi_S(x) = S(x) \]

we want to show that its a Lie algebra representation:

\[ \phi_S([x, y]) = S(x \otimes y - y \otimes x) = S(x \otimes y) - S(y \otimes x) = S(x) \circ S(y) - S(y) \circ S(x) = \phi_S(x) \circ \phi_S(y) - \phi_S(y) \circ \phi_S(x) \]

Now we will show if \( \psi \) is a morphism from \( \phi \) to \( \phi' \) then it is also a morphism from \( S_\phi \) to \( S_{\phi'} \). By definition \( S_\phi = \phi \), Thus:

\[ S_{\phi'}(x) \psi = \psi \circ S(x). \]

Taking another element \( x \otimes y \in U(g) - Mod \), we will have:

\[ S_{\phi'}((x \otimes y) \psi = \psi \circ S(x \otimes y) \]

\[ S'(x) \circ S'(y) \circ \psi = S'(x) \circ \psi \circ S(y) = \psi \circ s(x) \circ S(y) = \psi \circ S(x \otimes y) \]

From the equation above it is obvious that this holds for any arbitrary element \( x_1 \otimes \ldots \otimes x_n \in U(g) \). That is we have:

\[ S_{\phi'}(x_1 \otimes \ldots \otimes x_n) \psi = \psi \circ S_{\phi}(x_1 \otimes \ldots \otimes x_n) \]

The other way around to show that \( t \) is also a morphism from \( \phi \) to \( \phi' \) is obvious by using the one to one correspondence between \( \phi \) and \( s \).

The only thing left is to show that \( \phi_{S_\phi}(x) = \phi(x) \) and \( S_{\phi_{S_\phi}}(X) = S(x) \). For the first case we mean that if we start from \( \phi \) go to \( \phi_S \) and then to \( \phi_{S_\phi} \) we get the same \( \phi \). Let’s look at the definition of \( \phi_{S_\phi} \). It is a Lie algebra representation that we get from the ring homomorphism \( S_\phi \). By defining \( \phi_{S_\phi}(x) := S_\phi(x) \), and we defined \( S_\phi(x) = \phi(x) \). Therefore: \( \phi_{S_\phi}(X) = \phi(x) \).

To show \( S_{\phi_{S_\phi}}(x) = S(x) \), we use the same trick. By definition:

\[ S_{\phi_{S_\phi}}(x_1 \otimes \ldots \otimes x_n) = \phi_S(x_1) \circ \ldots \circ \phi(x_n) \]

We insert the definition of \( \phi_S(x) \):

\[ \phi_S(x_1) \circ \ldots \circ \phi_S(x_n) = S(x_1) \circ \ldots \circ S(x_n) \]

but since \( S \) is a ring homomorphism, we have that it is multiplicative:

\[ S(x_1) \circ \ldots \circ S(x_n) = S(x_1 \otimes \ldots \otimes x_n). \]

Therefore \( S_{\phi_{S_\phi}}(x) = S(x) \).

\[ \square \]

5 Simple Lie algebras, Semisimple Lie algebras, Killing form, Cartan criterion for semisimplicity

**Definition 5.1.** A non abelian Lie algebra \( g \) is called **simple** if it has no non trivial ideals.

**Definition 5.2.** We define a Lie algebra \( g \) to be **semisimple** if it is the finite direct sum of simple Lie algebras \( g_i \):

\[ g = g_1 \oplus g_2 \ldots \oplus g_n. \]
Definition 5.3. Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $F$. The Killing form $\kappa$ is the bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to F$ defined by

$$\kappa(x, y) = \text{Tr}(\text{ad}_x \circ \text{ad}_y)$$

$\forall x, y, z \in \mathfrak{g}$. It has the following properties:

- It is bilinear.
- It is symmetric.
- It is ad invariant:

$$\kappa([y, x], z) + \kappa(x, [y, z]) = 0$$

Definition 5.4. The Killing form is said to be non degenerate if: $\forall y = 0 \; \kappa(x, y) = 0$ implies $x = 0$.

Theorem 5.1. Cartan criterion: A Lie algebra $\mathfrak{g}$ over a field $F$ of characteristic zero is semisimple if and only if the Killing form is non degenerate.