Exercises for Higher Structures in Differential Geometry SS 2013

Sheet 11

Exercise 54

Show that each principal G-space over M is in fact a principal G-bundle over M and that the functor $\mathbf{G\text{-}Sp}_M^{\mathbf{pr}} \to \mathbf{Bun}(M,G)$ actually takes values in $\mathbf{Bun}_{\mathbf{pr}}(M,G)$ and yields an isomorphism of categories $\mathbf{G\text{-}Sp}_M^{\mathbf{pr}} \to \mathbf{Bun}_{\mathbf{pr}}(M,G)$

Exercise 55

Let G be a Lie group, acting on itself by right translation. Show that the action groupoid $G \rtimes G$ is actually equal to the pair groupoid $\mathrm{Pair}(G)$.

Exercise 56

Fill in the details of the construction of the gauge groupoid $\mathrm{Gauge}(P)$ of a principal G-bundle $\pi\colon P \to M$ over G. In particular, show that the identity map $M \to (P \times P)/G$, $m \mapsto \Phi^{-1}(m,e), \Phi^{-1}(M,e)$ (for Φ a local trivialisation at m) is well-defined, that the composition map $[(p,q)],[(v,w)] \mapsto [(p, w.\delta(q,v))]$ is well-defined and smooth and that the inversion map is given by $[(p,q)] \mapsto [(q,p)]$ and is smooth.

Exercise 57

Let G be a Lie group and M be a manifold. Show that a principal BG-bundle over $\underline{M}$ is the same thing as a principal G-bundle over M. Then show that the bundlisation $P(f)$ of a smooth functor $f\colon \underline{M} \to BG$ always gives the trivial principal G-bundle over M.

Exercise 58

Show that the bundlisation functor

$$P\colon \mathbf{Fun}^{\mathbf{sm}}(X,Y) \to \mathbf{Bun}_{\mathbf{pr}}(Y,X).$$

is fully faithful, but in general not essentially surjective.

Exercise 59 (The Hopf fibration)

Let S^3 be the three sphere, given by $S^3 = \{(z,w) \in \mathbb{C}^2 \mid z\overline{z} + w\overline{w} = 1\}$. Then $S^1 = \{x \in \mathbb{C} \mid x\overline{x} = 1\}$ acts on S^3 by $(z,w).x = (z \cdot x, w \cdot x)$. Show that $S^3/S^1 \cong S^2$ and that the quotient map turns S^3 into a principal S^1-bundle over S^2 (with respect to the above identification). **Hint:** S^2 is also diffeomorphic to the projective space $\mathbb{CP}^2$.