Exercises for Higher Structures in Differential Geometry SS 2013

Sheet 09

Exercise 45

Suppose M is a compact manifold, N is a locally metrisable manifold and $L \subseteq M$ be a subset and $n_0 \in N$. Show that

$$C_L^\infty(M,N) := \{f \in C^\infty(M,N) \mid f|_L = n_0\},$$

where n_0 is identified with the constant map with values n_0 is a closed submanifold of $C^\infty(M,N)$.

Exercise 46

Let $\mathbb{Z}_n$ act on $\mathbb{C}$ (from the right) by $z.[s] := \rho(z,[s]) := e^{\frac{2\pi i}{s}} \cdot z$.

a) Show that $\mathbb{C}/\mathbb{Z}_n$, together with the quotient topology and the quotient map $\mathbb{C} \to \mathbb{C}/\mathbb{Z}_n$ is the colimit of the diagram

$$\mathbb{C} \times \mathbb{Z}_n \xrightarrow{\mathrm{pr}_1} \mathbb{C} \xleftarrow{\rho} \mathbb{C} \times \mathbb{Z}_n \tag{1}$$

in **Top**.

b) Show that $\mathbb{C}/\mathbb{Z}_n$ does not possess a manifold structure such that $\mathbb{C} \to \mathbb{C}/\mathbb{Z}_n$ is smooth and the colimit of in **Man**. **Hint:** The corresponding statement is true if one considers the action of $\mathbb{Z}_n$ on $\mathbb{C}\backslash\{0\}$. Assuming that there also exists a chart around $0 \cdot \mathbb{Z}_n$ in $\mathbb{C}/\mathbb{Z}_n$, consider the smooth curve $\mathbb{R} \hookrightarrow \mathbb{C}$ and show that this cannot be mapped to a smooth curve in the quotient (for instance if $n = 2$).

c) Show that the statements of a) and b) are true if one replaces "colimit of (1)" by "surjective submersion".

Exercise 47

a) Let $\mathcal{C}$ be a category with finite products. Define the notion of a *group object* in $\mathcal{C}$ (your definition should yield that group objects in **Man** are Lie groups), along with morphisms of group objects. Assure yourself that the definition also works if $\mathcal{C}$ does not have arbitrary products but only the products occurring in the definition and generalise the notion to groupoid objects (and morphisms of them).

b) Show that group objects (respectively morphisms between those) in $\mathbf{PSh}_{\mathcal{C}}$ are those functors $\mathcal{C} \to \mathbf{Set}$ (respectively natural transformations) that have values in **Grp** (respectively in group homomorphisms). Conclude that the category of group objects in $\mathbf{PSh}_{\mathcal{C}}$ is $\mathbf{Grp}^{\mathcal{C}^{\mathrm{op}}}$.

c) Does the same also work in the same manner for algebra objects?

Exercise 48

Work out Example III.3 c) in the case of the open covering topology of **Top** explicitly. Even more explicitly, describe $\check{C}(R)$ for R the open covering of $\mathbb{S}^1$ by two (respectively three) open connected arcs such that each $e^{\frac{2\pi i}{n}}$ is for $n = 2$ (respectively $n = 3$) contained in only one arc.

Exercise 49

If $(\mathcal{C}, K)$ is a site and $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Grpd}$ is a weak presheaf in groupoids, show that $\mathbf{Match}(R, F)$ is for each $R = \{f_i\colon D_i \to C \mid i \in I\} \in K(C)$ a category with respect to $\mathrm{id}_{(X_i,\varphi_{ij})} = (\mathrm{id}_{X_i})$ and $(\alpha_i) \circ (\beta_i) = (\alpha_i \circ \beta_i)$. Moreover, show that

$$F(C) \to \mathbf{Match}(R, F), \quad (X \mapsto ((X|_{D_i}), (\varphi_{ij}(F, X)))), \; (\alpha \mapsto (\alpha|_{D_i})), \tag{2}$$

where $\varphi_{ij}(F, X) := F(f_i, \pi_{ij})(X)^{-1} \circ F(f_j, \rho_{ij})(X)$, is a functor.