Exercises for Higher Structures in Differential Geometry SS 2013

Sheet 03

Exercise 11

Fill in the details of Example B.2 f) on the completeness of $C(X,\mathbb{R})$ with respect to the metric
$$d(f,g) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{p_n(f-g)}{1+p_n(f-g)},$$
where $p_n(f) := \sup_{x\in K_n} |f(x)|$ and $K_1 \subseteq K_2 \subseteq ... \subseteq X$ are compact with $\cup_{n\in\mathbb{N}} K_n = X$.

Exercise 12

If X is a topological space and $U, V \subseteq X$ are open subsets, equipped with the subspace topology, then show that the pullback of the inclusions $U \hookrightarrow X$ and $V \hookrightarrow X$ is given by $U \cap V \hookrightarrow U$ and $U \cap V \hookrightarrow V$.

Exercise 13

Let A be a Banach algebra and denote by $A^\times$ the group of units in A. Show that the inversion $\iota\colon A^\times \to A^\times$, $a \mapsto a^{-1}$ is smooth by the following steps

1. Verify that $b^{-1} - a^{-1} = a^{-1}(a-b)b^{-1}$ for each $a, b \in A^\times$.
2. Calculate $d\iota(a)(v)$ and conclude that ι is of class C^1.
3. Show inductively that if ι is of class C^k, then $d\iota$ is of class C^k and conclude that ι is of class C^k for each $k \in \mathbb{N}_0$.

Exercise 14

If A is a Banach algebra and X is a compact topological space, show that $C(X,A)$ is a Banach algebra with respect to $(f\cdot g)(x) := f(x)\cdot g(x)$ and the supremum norm
$$\|f\|_\infty := \sup\{\|f(x)\| : x \in X\}.$$

Exercise 15

Let X, Y be lcs, $f\colon U \Subset X \to Y$ be a map and set $U^{[1]} := \{(x,v,s) \in U\times X\times\mathbb{R} \mid x+sv \in U\}$. Show that f is a C^1-map if and only there exists a continuous map $f^{[1]}\colon U^{[1]} \to Y$ such that
$$f^{[1]}(x,v,s) = \frac{1}{s}\left(f(x+sv) - f(x)\right) \qquad \text{if } s \neq 0.$$

Hint: Try
$$f^{[1]}(x,v,s) := \begin{cases} \int_0^1 df(x+stv)(v)dt & \text{if } x+[0,1]sv \subseteq U \\ \frac{1}{s}\left(f(x+sv)-f(x)\right) & \text{else} \end{cases}.$$

Use this to show the **Chain Rule:** If $f\colon U \Subset X \to Y$ and $g\colon V \Subset Z \to U$ are C^1-maps, then so is $f\circ g$ and
$$d(f\circ g)(z)(w) = df(g(z))(dg(z)(w)).$$