Exercises for Higher Structures in Differential Geometry SS 2013

Sheet 02

Exercise 06

Show that in **Top** limits and colimits always exist. In contrast to this, show that in the category $\mathbf{Top_{Haus}}$ of topological Hausdorff spaces, limits do always exist, but colimits do not (**Hint:** The colimit of the embedding $\mathbb{Q} \hookrightarrow \mathbb{R}$ would have to have the property that each continuous map $\mathbb{Q} \to X$ to an arbitrary Hausdroff space extends to a continuous map on $\mathbb{R}$).

Exercise 07

a) Show that

$$\mathbf{Set}(X \times Y, Z) \to \mathbf{Set}(X, \mathbf{Set}(Y, Z)), \quad f \mapsto \widehat{f} \quad \text{with} \quad \widehat{f}(x)(y) := f(x, y)$$

is in fact a natural bijection (natural in the sense that is gives an adjunction $(\cdot \times Y) \dashv \mathbf{Set}(Y, \cdot)$ for each fixed Y).

b) Suppose that $\mathcal{C}, \mathcal{D}, \mathcal{E}$ are small categories. Show that we have natural isomorphism of categories

$$\mathbf{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \cong \mathbf{Fun}(\mathcal{C}, \mathbf{Fun}(\mathcal{D}, \mathcal{E})). \tag{1}$$

Exercise 08

Let $\prod_{\mathbb{N}} S^1$ be the product of $\mathbb{N}$ copies of S^1 in **Top**, i.e., the cartesian product endowed with the product topology. Show that $(1, 1, ...)$ (or equivalently each point) does not have an open neighbourhood which is homeomorphic to an open subset of a lcs.

Exercise 09

Fill in the details of Example B.2 f). For this it could help to first show/realise the following fact: If d is a metric on X, then

$$d'(x, y) := \frac{d(x, y)}{1 + d(x, y)}$$

is an equivalent metric on X (i.e., id_X is continuous with respect to d and d'). For this it suffices in turn to show that $\mathbb{R}^{\geq 0} \to \mathbb{R}_{\geq 0}$ $x \mapsto \frac{x}{1+x}$ is subadditive.

Exercise 10

Let $X_1, ..., X_n, Y$ be lcs and $f\colon X_1 \times ... \times X_n \to Y$ be continuous and multi-linear. Show that f is smooth.