Exercises for Algebra II, WS 12/13

Sheet 12 – Solutions

Exercise 51

If $0 \to \mathbb{Z}_p \to A \to \mathbb{Z}_p \to 0$ is exact, then we know from the classification of the finite abelian groups that either $A = \mathbb{Z}_p \oplus \mathbb{Z}_p$ or $A = \mathbb{Z}_{p^2}$. In the first case the extension is trivial, since then we have a homomorphism $\mathbb{Z}_p \to A$ lifting $\mathrm{id}_{\mathbb{Z}_p}$. Thus all extensions with $A = \mathbb{Z}_p \oplus \mathbb{Z}_p$ are equivalent. If $A = \mathbb{Z}_{p^2}$, then there exist exactly $(p-1)$ injective homomorphisms $\mathbb{Z}_p \to \mathbb{Z}_{p^2}$, given respectively by multiplication with $\{p, 2p, 3p, ..., (p-1)p\}$. Then $\mathbb{Z}_{p^2}/np\mathbb{Z}_p \cong \mathbb{Z}_p$ canonically (since $p^2\mathbb{Z} + np\mathbb{Z} = p\mathbb{Z}$), turning

$$0 \to \mathbb{Z}_p \xrightarrow{\cdot np} \mathbb{Z}_{p^2} \to \mathbb{Z}_p \to 0 \tag{1}$$

into an extension.

Exercise 52

a) $\mathcal{C}$ has enough projectives and injectives since **R-Mod** does so for any ring. In the explicit construction of projectives and injectives in **R-Mod** the module structure becomes trivial if it was so initially, so $\mathcal{D}$ also has enough projectives and injectives.

b) This is the case by definition.

c) If $0 \to A \to B \xrightarrow{\pi} C \to 0$ is exact, then $A^G \to B^G$ is injective and if $b \in B^G$ and $\pi(b) = 0$, then $b \in B^G \cap A = A^G$. Moreover, if A, B, C are in $\mathbb{Z}[\mathbf{G}]\text{-}\mathbf{Mod}^{triv}$, then $A^G = A$, $B^G = b$ and $C^G = C$ and thus $B^G \to C^G$ is also surjective. Since the right derived functors of $A \mapsto A^G$ do not vanish in general, the functor is in general not exact.

Exercise 53

a) If multiplication with n is an isomorphism on a filed, then it also is an isomorphism on each vector space over this field. Since multiplication with n is an isomorphism on the prime field $\mathbb{Z}_{\mathrm{char}(k)}$, the claim follows.

b) It is

$$h.(\frac{1}{n}\sum_{g\in G} g.x) = \frac{1}{n}\sum_{h^{-1}g\in G} g.x = \frac{1}{n}\sum_{g\in G} g.x.$$

c) If

$$0 \to U \to V \xrightarrow{\pi} W \to 0$$

is an exact sequence of $k[G]$-modules, then

$$0 \to U^G \to V^G \to W^G$$

is exact. Moreover, if $x \in W^G$, then there exists $v \in V$ with $\pi(v) = x$. Then $v_0 := \frac{1}{n}\sum_{g\in G} g.y$ is in V^G and satsifies

$$\pi(v_0) = \frac{1}{n}\sum_{g\in G} g.\pi(v) = \frac{1}{n}(n \cdot x)$$

(since $g.x = x$). Thus $V^G \to W^G \to 0$ is also exact.

Exercise 54

a) If $\sum n_g g$ satisfies $\sum n_g = 0$, then

$$\sum n_g g = \sum n_g g - \sum n_g 1 = \sum n_g (g-1),$$

Thus IG is generated by $(g-1)_{g\in G\setminus\{1\}}$. It is clear that $(g-1)_{g\in G\setminus\{1\}}$ is also linearly independent.

b) The sequence $0 \to IG \to \mathbb{Z}[G] \to \mathbb{Z} \to 0$ is exact, thus induces a long exact sequence

$$0 \to \operatorname{Ext}^0_{\mathbb{Z}[G]}(\mathbb{Z}, A) \to \operatorname{Ext}^0_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \to \\ \operatorname{Ext}^0_{\mathbb{Z}[G]}(IG, A) \to \operatorname{Ext}^1_{\mathbb{Z}[G]}(\mathbb{Z}, A) \to \operatorname{Ext}^1_{\mathbb{Z}[G]}(\mathbb{Z}[G], A).$$

Since $\mathbb{Z}[G]$ is free we have that $\operatorname{Ext}^1_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) = 0$ and we thus obtain

$$0 \to H^0(G, A) \to \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \to \operatorname{Hom}_{\mathbb{Z}[G]}(IG, A) \to H^1(G, A) \to 0$$

c) We first check that $\eta(\delta)$ is actually a $\mathbb{Z}[G]$-module morphism, i.e. that

$$\eta(\delta)(h.(g-1)) = \eta(\delta)(hg - 1 - (h-1)) = \\ \eta(\delta)(hg-1) - \eta(\delta)(h-1) = \delta(hg) - \delta(h) = h\delta(g) = h.(\eta(\delta)(g-1)).$$

Conversely, for any $\mathbb{Z}[G]$-module homomorphism $\varphi\colon IG \to A$ we define $\mu(\varphi)\colon G \to A$, $\mu(\varphi)(g) = \varphi(g-1)$. It is clear that μ is inverse to η.

d) A morphism $\varphi\colon IG \to A$ comes from $\operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \cong A$ is and only if $\varphi(g-1) = ga - a$ for some $a \in A$. This is by definition the case if and only if $\mu(\varphi)(g)$ is an inner dirivation.

Exercise 55

A group of order 42 has exactly one Sylow 7-subgroup, thus a normal subgroup of order 7. We thus always have an abelian extension

$$\mathbb{Z}_7 \to G \to H, \tag{2}$$

where H is a group of order 6, so is either $\mathbb{Z}_6$ or $D_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$. From the last exercise we deduce that $H^2(H, \mathbb{Z}_7) = 0$, no matter what the actual action of H on $\mathbb{Z}_7$ actually is. Thus all extensions (2) split and G is always a semi-direct product $\mathbb{Z}_7 \rtimes H$. It thus remains to classify the different ways in which H may act on $\mathbb{Z}_7$ (and check whether we have duplicates). For this note that $\operatorname{Aut}(\mathbb{Z}_7) \cong \mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$.

$H = D_3$**:** There is only one non-trivial homomorphism $D_3 \to \mathbb{Z}_6$. Since D_3 has no normal subgroups of order 2 the kernel of $D_3 \to \mathbb{Z}_6$ has to be $\mathbb{Z}_3$ (provided it is not D_3). This can only be the canonical morphism $D_3/\mathbb{Z}_3 \to \mathbb{Z}_2$. Thus in this case we get either $G = \mathbb{Z}_7 \times D_3$ or $G \cong \mathbb{Z}_7 \rtimes D_3 \cong D_7 \times \mathbb{Z}_3$.

$H = \mathbb{Z}_6$ **:** In the semi-direct product $\mathbb{Z}_7 \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_3)$ we first consider the case where one of the factors of $\mathbb{Z}_3 \times \mathbb{Z}_2$ acts trivially on Z_7 (in this case this factor is a normal subgroup):

- $\mathbb{Z}_3$ and $\mathbb{Z}_2$ act trivially $\Rightarrow G \cong \mathbb{Z}_7 \times \mathbb{Z}_3 \times \mathbb{Z}_2 \cong \mathbb{Z}_{42}$.
- $\mathbb{Z}_3$ acts trivially (but $\mathbb{Z}_2$ not) $\Rightarrow G \cong (\mathbb{Z}_7 \rtimes \mathbb{Z}_2) \times \mathbb{Z}_3 = D_7 \times \mathbb{Z}_3$ (which we already had above).
- $\mathbb{Z}_2$ acts trivially (but $\mathbb{Z}_3$ not) $\Rightarrow G \cong (\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \times \mathbb{Z}_2$ (the two possible homomorphism $\mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z}_7)$ give isomorphic groups since the non-trivial automorphism of $\mathbb{Z}_3$ intertwines the two actions on $\mathbb{Z}_7$ ans thus induces an isomorphism).

If $\mathbb{Z}_3 \times \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_7)$ has no kernel, then we get two different groups, one is D_{21} and the other is $(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$. We thus obtain the list of groups

G	$Z(G)$
$\mathbb{Z}_7 \times D_3$	$\mathbb{Z}_7$
$D_7 \times \mathbb{Z}_3$	$\mathbb{Z}_3$
$\mathbb{Z}_{42}$	$\mathbb{Z}_{42}$
$(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \times \mathbb{Z}_2$	$\mathbb{Z}_2$
D_{21}	0
$(\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	0

since $Z(A \times B) = Z(A) \times Z(B)$ and $Z(D_{2n+1}) = 0 = Z(\mathbb{Z}_7 \rtimes \mathbb{Z}_3)$. By counting the elements of order 2 one sees that $D_{21} \ncong (\mathbb{Z}_7 \rtimes \mathbb{Z}_3) \rtimes \mathbb{Z}_2$ and from comparing the centers one sees that also the others are pair-wise non isomorphic.