Exercises for Algebra II, WS 12/13

Sheet 11 – Solutions

Exercise 47

The group ring has by detinition one generator whose n-th power is 0, thus isomorphic to R.

a) $(1-t)N = (1-t)(1+t+\cdots+t^{n-1}) = (1+t+\cdots+t^{n-1}) - (t+t^2+\cdots+t^n) = 1-t^n$.

b) $\mathbb{Z}$ is Noetherian, thus is $\mathbb{Z}[t]$. Since in $\mathbb{Z}[t]$ each prime element is irreducible (by Gauß' Theorem), $\mathbb{Z}[t]$ is in particular factorial, thus admits a unique prime decomposition (up to units).

c) If $f = h(1-t)$, then $fN = h(1-t)N = h(1-t^n)$. On the other hand, if $fN = g(1-t^n)$, then we may write (up to units)

$$f_1 \cdots f_m \cdot N_1 \cdots N_n = g_1 \cdots g_k \cdot L_1 \cdots L_l \tag{1}$$

where $L = (1-t^n)$. With $Q = (1-t)$ we have $Q_1 \cdots Q_q \cdot N_1 \cdots N_n = L_1 \cdots L_l$ and the the uniqueness of the prime factor decomposition there exists for each $1 \leq i \leq n$ some $1 \leq j \leq l$ with $N_i = L_j$ up to units and the product of the remaining factors of L give $(1-t)$ up to units. Thus (1) implies

$$f = c \cdot g \cdot (1-t).$$

d) From c) it follows that the sequence is exact. Since each R is free (thus projective) it remains to verify that ε is an R-module morphism. On monomials we have $\varepsilon(t^n) = \varepsilon(t)^n = 1$, whence $\varepsilon(t^m \cdot t^n) = \varepsilon(t)$ and the rest follows from the linearity of ε.

In order to show the claim we observe that $R \otimes_R \mathbb{Z} \cong \mathbb{Z}$, the isomorphism sending $x \in \mathbb{Z}$ to $1 \otimes x \in R \otimes_R \mathbb{Z}$ and

$$(1+t+\cdots+t^{n-1}) \otimes x = 1 \otimes x + t.(1 \otimes x) + \cdots + t^{n-1}(1 \otimes x) = n(1 \otimes x) = 1 \otimes nx$$

to nx. Thus the tensored sequence becomes

$$\cdots \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to \cdots$$

and the homology of this gives the desired result.

Exercise 48

If $A \to I$ is an injection into an injective (equivalently divisible) abelian group, then I/A is also divisible (hence injective) and $0 \to A \to I \to I/A \to 0 \to \cdots$ is an injective resolution of A. Thus $R^iF \equiv 0$ for each left exact functor F and $i > 1$ (similar to the case of R-modules for R a pid).

Exercise 49

This will be done step by step in the exercises, the complete proof can be fond in the mentioned lecture notes.

Exercise 50

a) It is clear that the horizontal and vertical differentials square to 0 and since $d_v d_h = 0 = -d_h d_v$ $X_{\bullet\bullet}$ forms indeed a double complex. Since $d_v(...,1,1) = (...,2,2) = d_h(...,1,1)$ it follows that $(d_v + d_h)(...,1,1) = 0$.

b) The 0-boundaries are certainly contained in $\prod_{\mathbb{N}_0} 2\cdot\mathbb{Z}_4$. To get a sequence $(x_n)_{n\in\mathbb{N}_0}$ in $\prod_{\mathbb{N}_0} 2\cdot\mathbb{Z}_4$ as image of an element of $(\operatorname{Tot} X_{\bullet\bullet})_1 = \prod_{\mathbb{N}_0}\mathbb{Z}_4$ one starts with a sequence of zeros from below until one hits the first $x_i \neq 0$. Then one continuous with 1's until one hits the next $x_i \neq 2$, then one switches back to zeros and so on...

c) $(d_v + d_h)((x_i)_{i\in\mathbb{N}_0})$ vanishes if and only if the difference of x_i and x_{i+1} is in $\{0,2\}$ (i.e. all x_i are even or all x_i are odd). Thus mod $\prod_{\mathbb{N}_0} 2\cdot\mathbb{Z}_4$ it is either 0 or $(...,1,1)$.

d) Since the rows are exact, $|X_{\bullet\bullet}|$ is exact by the Acyclic Assembly Lemma.