# Exercises for Algebra II, WS 12/13

## Sheet 9 – Solutions

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### Exercise 39

If $U \to X$ and $U \to Y$ are given, then they determine a unique morphism into $X \times Y$ that gives the original morphism after composition with the respective projection morphisms. Since the compositions of $U \to X$ and of $U \to Y$ to a $Z$-valued morphisms coincide, it follows that the morphism $U \to X \times Y$ actually takes values in $\{(x, y) \mid f(x) = g(x)\}$.

### Exercise 40

Spelling out the definitions one sees that

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 0 \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

is a pull-back if and only if $A \to B$ is a kernel of $B \to C$. Similarly, the diagram is a push-out if and only if $B \to C$ is a cokernel of $A \to B$. Thus $A \to B$ is mono (kernels are always monomorphisms), $\ker(B \to C) \cong \operatorname{coker}(A \to B)$, and $B \to C$ is epi (likewise).

### Exercise 41

Let $f\colon A \to U$ and $g\colon B \to U$ be group homomorphisms that agree on $A \cap B$. Then we may define $\widetilde{f}\colon \langle A \cup B\rangle \to U$ by setting it to $f(a)$ if $a \in A$, $g(b)$ if $b \in B$ and extend it as a group homomorphism to $\langle A \cup B\rangle$. This is well-defined, e.g.

$$\begin{aligned} a_1 \cdot b_1 = a_2 \cdot b_2 \Rightarrow a_2^{-1} \cdot a_1 = b_2 \cdot b_1^{-1} \in A \cap B \\ \Rightarrow f(a_2)^{-1} \cdot f(a_1) = g(b_2) \cdot g(b_1)^{-1} \\ \Rightarrow \widetilde{f}(a_1 \cdot b_1) = f(a_1) \cdot g(b_1) = f(a_2) \cdot g(b_2) = \widetilde{f}(a_2 \cdot b_2). \end{aligned}$$

Since $\widetilde{f}$ is uniquely determined by its restriction to $A$ and $B$ this shows the claim since $G = \langle A \cup B\rangle$. This also shows that in general a push-out of two subgroups $A$ and $B$ is given by the subgroup generated by $A$ and $B$.

### Exercise 42

a) An inverse to the morphism

$$\mathrm{Hom}_{\mathbf{Ab}}(N, A) \to \mathrm{Hom}_{\textbf{Mod-R}}(N, \mathrm{Hom}_{\mathbf{Ab}}(R, A)), \quad \varphi \mapsto (n \mapsto \varphi(n \cdot r))$$

is given by

$$\mathrm{Hom}_{\textbf{Mod-R}}(N, \mathrm{Hom}_{\mathbf{Ab}}(R, A)) \to \mathrm{Hom}_{\mathbf{Ab}}(N, A), \quad \varphi \mapsto (a \mapsto \varphi(a, 1))\,.$$

b) Since the forgetful functor **Mod-R** $\to$ **Ab** is exact (check this), it follows from Proposition ... that the right adjoint preserves injectives. Since $\mathbb{Q}/\mathbb{Z}$ is divisible and thus is injective, $\mathrm{Hom}_{\mathbf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$ is injective.

c) This also follows from Proposition ..., since the product functor is right adjoint to the diagonal functor (which is of course exact).

d) We define

$$\Phi\colon M \to \prod_{\mathrm{Hom}_{\mathbf{Mod\text{-}R}}(M, I_0)} \mathrm{Hom}_{\mathbf{Ab}}(R, \mathbb{Q}/\mathbb{Z}), \quad (\Phi(m)_\psi)(r) := \psi(m)(r).$$

This is injective, since for each $m \in M$ we find a homomorphism $\psi_m\colon M \to \mathbb{Q}/\mathbb{Z}$ (of abelian groups) making the diagram

$$\begin{tikzcd} \langle M \rangle_{\mathbb{Z}} \arrow[r] \arrow[d] & M \arrow[dl, dashed, "\psi_m"] \\ \mathbb{Q}/\mathbb{Z} & \end{tikzcd}$$

commute. If we identify $\psi_m$ with the associated homomorphism $M \to \mathrm{Hom}_{\mathbf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$, then $(\Phi(m)_{\psi_m})(1) \neq 0$ and thus $\Phi$ is injective.

By b) we know that $I_0$ is injective, by c) we know that $I(M)$ is injective. Since $M$ injects into $I(M)$ this shows that **R-Mod** $\cong$ **Mod-R** has enough injectives.