# Exercises for Algebra II, WS 12/13

## Sheet 7 – Solutions

---

**Exercise 31**
This will be discussed in detail in the exercise class.

**Exercise 32**

a) Each functor yields in particular a map $f\colon \mathbb{Z} \to \mathbb{Z}$ on objects. If we assume that $f$ is not monotone, then we have $n < m$ with $f(n) > f(m)$. Then there is a (unique) morphism $n \to m$, but no morphism $f(n) \to f(m)$. This cannot be, thus $f$ is monotone. On the other hand, a monotone map preserves the order, thus yields a functor.

b) We claim
$$\mathrm{Hom}(f_n(x), y) \cong \mathrm{Hom}(x, g_{n-1}(y)).$$
Both sides are either empty or have exactly one element. If $x \le y$, the both sides have one element. If $x > y+1$, then both sides are empty. If $x = y+1$, then
$$\mathrm{Hom}(f_n(x), x-1) = \begin{cases} \emptyset & \text{if } x \le n \\ \text{has one element} & \text{if } x > n \end{cases}$$
and
$$\mathrm{Hom}(x, g_{n-1}(x-1)) = \begin{cases} \emptyset & \text{if } x-1 \le n-1 \\ \text{has one element} & \text{if } x-1 > n-1 \end{cases}$$
Thus both sides are equal and one sees $f_n \dashv g_{n-1}$. Similarly, one shows $g_n \dashv f_n$

**Exercise 33**
This is spelled out in detail in Proposition II.7.2 in Hilton, Stambach „A Course in Homological Algebra.“ It is directly checked that $\eta$ is natural and $\psi \mapsto \delta(Y) \circ F(\psi)$ is a natural inverse.

**Exercise 34**

a) This is clear.

b) We show that $G(\ker(\varphi))$ satisfies the above property. Recall the units and counits $\varepsilon\colon \mathrm{id} \to GF$ and $\delta\colon FG \to \mathrm{id}$ of the adjunction. If $f\colon A \to G(M)$ is such that $G(\varphi) \circ f = 0$, then apply $F$ and $\delta$ to obtain $\varphi \circ \delta(M) \circ F(f) = 0$. Thus $\delta(M) \circ F(f)\colon F(A) \to \ker(\varphi)$. If we apply $G$ to this and precompose with $\varepsilon(A)$, then we get
$$\begin{tikzcd} A \arrow[r, "\varepsilon(A)"] \arrow[d, "f"] & GF(A) \arrow[r, "GF(f)"] \arrow[d, "GF(f)"] & GFG(M) \arrow[d, "G(\delta(M))"] \\ G(M) \arrow[r, "\varepsilon(G(M))"] \arrow[rr, bend right, "\mathrm{id}_{G(M)}"'] & GFG(M) \arrow[r, "G(\delta(M))"] & G(M) \end{tikzcd}$$
from $G(\delta(M)) \circ \varepsilon(G(M)) = \mathrm{id}_{G(M)}$. Since $G(\delta(M) \circ F(f))$ takes values in $G(\ker(\varphi))$, so does $f$.