Exercises for Algebra II, WS 12/13

Sheet 6 – Solutions

Exercise 25

φ_1 can be taken to be $(x,y,z) \mapsto x+z$, since then $\varphi_1(N) = \gcd(2,5)\mathbb{Z} = \mathbb{Z}$ is maximal. Thus $a_1 = 1$, $x_1 = (-4,4,5)$ and $v_1 = (-4,4,5)$ is a valid choice. Then $\ker(\varphi_1) = \{(n,m,-n) \mid n,m \in \mathbb{Z}\}$, $\mathbb{Z}^3 = \ker(\varphi_1) \oplus (-4,4,5)\mathbb{Z}$ and

$$\ker(\varphi_1) \cap N = \{(10 \cdot n, -8 \cdot n, -10 \cdot n)\}.$$

By induction we now have to apply the procedure to $\ker(\varphi_1)$ and $\ker(\varphi_1) \cap N$. Since $\gcd(10,8,-10) = 2$ we have that $\psi(\ker(\varphi_1) \cap N) \subseteq 2\mathbb{Z}$ for all $\psi \in \mathrm{Hom}(\ker(\varphi_1), \mathbb{Z})$ and thus $a_2 = 2$ and we can take φ_2 to be $(x,y,z) \mapsto x+y$ and $x_2 = (10,-8,-10)$ and $v_2 = (5,-4,-5)$. Then $\ker(\varphi_2) = \{(n,-n,m) \mid n,m \in \mathbb{Z}\}$, thus $\ker(\varphi_1) \cap \ker(\varphi_2) = (1,-1,-1)\mathbb{Z}$ and $N \cap \ker(\varphi_1) \cap \ker(\varphi_2) = 0$. We thus may extend $\{v_1, v_2\}$ to a basis of $\mathbb{Z}^3$, e.g., by setting $v_3 = (1,-1,-1)$ and chose $a_3 = 0$. Eventually, we get

$$N = (-4,4,5)\mathbb{N} \oplus 2(5,-4,-5)\mathbb{Z} \oplus 0(1,-1,-1)\mathbb{Z},$$

so that $\{(-4,4,5), 2(5,-4,-5)\}$ is a basis of N.

Exercise 26

- Vector spaces and linear maps. Isomorphisms are bijective linear maps (since inverse maps of bijective linear maps are again linear).
- Metric spaces and continuous maps. Isomorphisms are continuous bijective maps that are also open (i.e., map open sets to open sets).
- Ordered sets and order preserving maps (i.e., $x \leq y$ implies $f(x) \leq f(y)$). Isomorphisms are *not* the bijective order preserving maps (i.e. the set with four elements has the total order and one with one maximal element, one minimal and two incomparable elements between these two; these two ordered sets are not isomorphic (why?) but there is a bijective order preserving map between them).

Exercise 27

a) Yes, since $\mathbb{Z}$ is cyclic each homomorphism $\mathbb{Z} \to G$ is uniquely determined by its image at one and each $g \in G$ occurs as such.

b) No, for instance $\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z})$ is empty.

Exercise 28

a) In **Gp** the group $\mathbb{Z}$ is a generator. Indeed, if $f(x) \neq g(x)$, then x determines the homomorphism $u_x \colon \mathbb{Z} \to G$ $n \mapsto g^n$ and since $u_x(1) = x$ we have $f(u_x(1)) \neq g(u_x(1))$. By definition, in **Set** each point is a generator (this is of course unique up to isomorphisms). However, in **Set** each object is a generator.

b) We have already seen, that in **Set** generators are not isomorphic. If **1** is the category with one object and one (identity) morphism, then this object is a generator since there do not exists non-equal morphisms. But **1** embeds into **Gp** by sending the object to $\mathbb{Z}_2$ and $\mathbb{Z}_2$ is not a generator of **Gp** (why?).

Exercise 29

Let $F, G, H\colon \mathcal{C} \to \mathcal{D}$ be functors and $\alpha\colon F \Rightarrow G$, $\beta\colon G \Rightarrow H$ be natural transformations. Then we define the source of α to be F and the target of α to be G. The identity natural transformation $\mathrm{id}_F\colon F \Rightarrow F$ is simply given by $x \mapsto \mathrm{id}_{F(x)}$. The composition $F \circ G$ is defined by $x \mapsto \beta(x) \circ \alpha(x)$. That these assignments satisfy the necessary condition follows from the commutativity of

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\alpha(x)"] & F(y) \arrow[d, "\alpha(y)"] \\ G(x) \arrow[r, "G(f)"] \arrow[d, "\beta(x)"] & G(y) \arrow[d, "\beta(y)"] \\ H(x) \arrow[r, "H(f)"] & H(y) \end{tikzcd}$$

for each $f \in Hom_{\mathcal{C}}(x, y)$.

Exercise 30

a) Since the composition in **Set** is associative, one verifies $h^X(g \circ f) = h^X(g) \circ h^X(f)$ for composable morphisms $Y \xrightarrow{f} Z \xrightarrow{g} U$ in $\mathcal{C}$. The rest is obvious.

b) By definition a natural transformation $\alpha\colon h^X \Rightarrow F$ is the same thing as an assignment of a map $\alpha(Y)\colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \to F(Y)$. For each $\varphi\colon X \to Y$, this has to make the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(X, X) \arrow[r, "\varphi_*"] \arrow[d, "\alpha(X)"'] & \mathrm{Hom}(X, Y) \arrow[d, "\alpha(Y)"] \\ F(X) \arrow[r, "F(\varphi)"] & F(Y) \end{tikzcd}$$

commute, thus

$$\alpha(Y)(\varphi) = \alpha(Y)(\varphi_*(\mathrm{id}_X)) = F(\varphi)(\alpha(X)(\mathrm{id}_X)).$$

c) The functor $X \mapsto h^X$,

$$\left(X \xleftarrow{\varphi} Y\right) \mapsto \left(f^*\colon h^X \to h^Y\right)$$

with $f^*(Z)\colon \mathrm{Hom}(X, Z) \to \mathrm{Hom}(Y, Z)$, $\varphi \mapsto \varphi \circ f$ is an embedding, since by part b) we have $\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(h^X, h^Y)$.