Exercises for Algebra II, WS 12/13

Sheet 4 – Solutions

Exercise 17

The map $M_i \to \operatorname{coker}(d_{i-1}) = M_i/\operatorname{im}(d_{i-1})$ is the projection and the map $\operatorname{coker}(d_{i+2}) \to M_i$ is induced from $d_{i+1}\colon M_{i-1} \to M_i$ (since d_{i+1} vanishes on $\operatorname{im}(d_{i+2})$). Now $M_i \to \operatorname{coker}(d_{i-1}) = M_i/\operatorname{im}(d_{i-1})$ is surjective by definition and $\operatorname{coker}(d_{i+2}) \to M_i$ is injective if and only if $d_{i+1}(x) = 0 \Leftrightarrow x \in \operatorname{im}(d_{i+2})$. This shows the claim.

Exercise 18

This is an important fact. If follows simply from applying the 5-Lemma to

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & M' \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\varphi"] & M'' \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
0 \arrow[r] & M' \arrow[r] & \widetilde{M} \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

Exercise 19

a) If

$$\begin{tikzcd}
 & M \arrow[d, "\varphi"] & \\
N_1 \arrow[r] & N_2 \arrow[r] & 0
\end{tikzcd}$$

is given, then we choose inverse images y_i of $\varphi(x_i)$ in N_1 and define $M \to N_1$ by

$$x = \sum_{i=1}^{n} f_i(x)x_i \mapsto \sum_{i=1}^{n} f_i(x)y_i.$$

This is a lift of φ and thus M is projective.

On the other hand, suppose M is projective and finitely generated. Then there exists Q such that $M \oplus Q \cong \oplus_{i\in I} R$ is free. If $M = \langle x_1, ..., x_n\rangle$, then (the image in $\oplus_{i\in I} R$ of) each x_k is contained in $\oplus_{i\in I_k} R$ with I_k finite. Thus the image of M is contained in $\oplus_{i\in F} R$ for some finite F. Thus $Q' := Q \cap \oplus_{i\in F} R$ is a module satisfying $M \oplus Q' \cong \oplus_{i\in F} R$ and then the projections of $\oplus_{i\in F} R$ to R furnish the required maps f_i.

b) Let Q be such that $M \oplus Q \cong \oplus_{i\in I} R$ is free. Then

$$M/IM \oplus Q/IQ \cong (M \oplus Q)/I(M \oplus Q) \cong \oplus_{i\in I} R/I.$$

Exercise 20

a) $\mathrm{id}_R \otimes_R \iota = \iota$ under the isomorphism $R \otimes_R M \cong M$, thus $\mathrm{id}_R \otimes \iota$ is injective if ι is so.

b) We first observe that $\frac{r}{s} \otimes x = 0$ if and only if $x = 0$. In fact, $R \cong R \cdot \frac{1}{s}$ and thus $(R \cdot \frac{1}{s}) \otimes N \cong N$.

Let $\iota\colon N' \to N$ be injective. We have that the kernel of $\mathrm{id}_{S^{-1}R} \otimes \iota$ is the submodule generated by $\{\frac{r}{s} \otimes x \mid \frac{r}{s} \otimes \iota(x) = 0\}$. Since $\frac{r}{s} \otimes \iota(x) = \frac{1}{s} \otimes \iota(rx)$, this equals the submodule generated by $\{\frac{1}{s} \otimes x \mid \frac{1}{s} \otimes \iota(x) = 0\}$. By the above argument and the injectivity of ι, this equals the submodule generated by $\{\frac{1}{s} \otimes 0\} = \{0\}$.

c) For any S-module N we have isomorphisms $(M \otimes_R S) \otimes_S N \cong M \otimes_R (S \otimes_S N) \cong M \otimes_R N$ that make the diagram

$$\begin{tikzcd}
(M \otimes_R S) \otimes_S N' \arrow[r] \arrow[d] & (M \otimes_R S) \otimes_S N \arrow[d] \\
M \otimes_R N' \arrow[r] & M \otimes_R N
\end{tikzcd}$$

commute. Thus the morphism on the top is injective if and only if the bottom one is.