Exercises for Algebra II, WS 12/13

Sheet 2 – Solutions

Exercise 5

a) Yes: for each $u \in R$ we have $1 \cdot u = u \neq 0$.

b) Yes: $u \in R$ is a torsion element if and only if there exists some $r \in R$ with $r \neq 0$ and $r \cdot u = 0$.

c) Yes: This is also true for any ring R.

d) No: $\mathbb{Z}_4$ is a free $\mathbb{Z}_4$-module but has zero divisors.

Exercise 6

Direct Product: If $\varphi_i \colon A \to M_i$ is given for each $i \in I$, then define $\Phi(a) := (\varphi_i(a))_{i \in A}$.

Direct Sum: If $\psi_i \colon M_i \to B$ is given, define $\Psi((m_i)_{i\in I}) := \sum_{i\in I} \psi_i(m_i)$. Note that this sum is defined since only finitely many of its summands are not equal to 0!

Exercise 7

a) We consider the map $f_0 \colon A/I \times M \to M/IM$, $(a + I, x) \mapsto ax + IM$ and show that this is a tensor product of A/I and M over A (why does this suffice?). If $f \colon A/I \times M \to L$ is A-bilinear, then define $\varphi \colon M/IM \to L$ by $\varphi(x + IM) := f(1 + I, m)$ (note that $x + IM = f_0(1 + I, x)$). Since f is bilinear we have $f(1 + I, im) = f(i + I, m) = f(I, m) = 0$ and φ is well-defined. It obviously is A-linear and since M/IM is generated by the image of f_0 it is uniquely determined by requiring $\varphi \circ f_0 = f$. Thus $(M/I, f_0)$ is a tensor product of A/I and M over A.

b) Since I acts trivially on $M \otimes_A N$, this is also an A/I-module and $M \times N \to M \otimes_A N$ is A/I-bilinear. This induces an A/I-linear map $\varphi \colon M \otimes_{A/I} N \to M \otimes_A N$. On the other hand, $M \otimes_{A/I} N$ is also an A-module and $M \times N \to M \otimes_{A/I} N$ is A-linear, inducing an A-linear map $\psi \colon M \otimes_A N \to M \otimes_{A/I} N$. With the universal property one sees as usual that these maps are inverse to each other.

c) $Z_m \otimes_{\mathbb{Z}} \mathbb{Z}_n = \gcd(m, n)$ and $\mathbb{Z}_m \otimes_{\mathbb{Z}_m} \mathbb{Z}_n = \mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n = \mathbb{Z}_n$.

Exercise 8

The reason is that $\mathbb{Q}/\mathbb{Z}$ has only torsion elementes (as $\mathbb{Z}$-modules, since $q \cdot \frac{p}{q} \in \mathbb{Z}$), but tensoring with $\mathbb{Q}$ kills all the torion. Likewise, $\mathbb{Q}/\mathbb{Z}$ is divisible (i.e. for each $x \in \mathbb{Q}/\mathbb{Z}$ and $n \in \mathbb{Z}\backslash\{0\}$ there exists $y \in \mathbb{Q}/\mathbb{Z}$ such that $ny = x$), such that $x \otimes m = n \cdot y \otimes m = y \otimes mn = 0$.

Exercise 9

a) No, $15\mathbb{Z} \cap 6\mathbb{Z} = 30\mathbb{Z}$.

b) No, $p\mathbb{Z} \cap q\mathbb{Z} = \mathrm{lcm}(p, q)\mathbb{Z}$.

c) Yes, this is eventually true!

d) Yes (likewise).

e) Yes: If A is a field, then each A-module (aka vector space) has a basis and is thus a free module. Conversely, If A is not a field, then there exists some $x \in A$ which is not a unit. Consequently, the ideal xA of A is not A and A/xA is not the trivial A-module. The map $A \to A/xA$ is a morphism of A-modules mapping the generator 1 of A to a generator of A/xA. If A/xA were free, then $A \to A/xA$ would have to be bijective, which is not the case.