Exercises for Algebra II, WS 12/13

Sheet 1 – Solutions

Exercise 1

No, for each invertible matrix A we have an isomorphism $S \to S^{\mathrm{op}}$, $X \mapsto A^{-1} \cdot X \cdot A$.

Exercise 2

a) Each polynomial x^n over $\mathbb{F}_2$ has the identity as associated function.

b) If (A, X) and (B, Y) are two polynomial rings over R, then there exist unique algebra morphisms $\varphi\colon A \to B$ and $\psi\colon B \to A$ with $\varphi(X) = Y$ and $\psi(Y) = X$. Thus the composition $\varphi \circ \psi$ is the unique algebra morphism mapping Y to Y, and so equals the identity on Y. The same argument also shows that $\psi \circ \varphi$ is the identity on X.

c) By definition we have that $x^n = (0, ..., 1, ...)$, where the 1 is in the n-th entry. Thus by definition we have the unique decomposition

$$(r_0, r_1, ...) = r_0 \cdot (1, 0, ...) + r_1(0, 1, ...) + ... = r_0x^0 + r_1x^1 + ...$$

d)
- $\mathbb{C}[x]/(x^2)$ is not free, for we have the equality $x^2 = 0$ (thus uniqueness will fail).
- $\mathbb{C}[x, y]$ is not a polynomial algebra over $\mathbb{C}$ (although it is one over $\mathbb{C}[x]$), since no polynomial f in x and y will allow to write both monomials x and y as a linear combination of powers of f.
- $\mathbb{C}[x, y]/(x) \cong \mathbb{C}[y]$ is a polynomial algebra
- $\mathbb{C}[X, y]/(x-y)$ is a polynomial algebra: an inductive argument shows that each $f(X, Y)$ may we written modulo sums of powers of $(x - y)$ as a polynomial in only one variable. Thus $\mathbb{C}[x, y]/(x - y)$ is isomorphic to $I/I \cap (x - y)$, where $I = \{f(x, y) \mid f = \sum c_i x^i\}$. Since $I \cap (x - y) = \{0\}$, we have that $\mathbb{C}[x, y]/(x - y) \cong I \cong \mathbb{C}[x]$.

Exercise 3

a) We trivially have that $\rho^\varphi = \rho \circ \varphi$ is a morphism of groups.

b) The linear map $\rho(x)\colon V \to V$ is a morphism of representations since we have

$$\rho(x)(\rho(g)(v)) = \rho(x \cdot g)(v) = \rho(x) \circ \rho(x^{-1} \cdot g \cdot x)(v) = \rho(x)(\rho^\varphi(g)(v))$$

c) No, take for instance $G = \mathbb{Z}_4$, acting on $\mathbb{C}$ (considered as a *complex* vector space) by identigying $\mathbb{Z}_4$ with $\{1, i, -1, -i\} \subseteq \mathbb{C}^\times$. If we take the automorphism φ that exchanges i and $-i$, then any isomorphism ψ would have to satisfy $\psi(i \cdot v) = -i \cdot \psi(v)$ for all $v \in \mathbb{C}$, and thus cannot be $\mathbb{C}$-linear.

Exercise 4

a) If $x, y \in Z(R[G])$, then we also have $(xy)z = x(yz) = x(zy) = z(yx) = z(xy)$ and $(x+y)z = xz + yz = zx + zy = z(x+y)$ for all $z \in R[G]$. This argument works of course for an arbitrary R-algebra. In general, $Z(R[G])$ is not an ideal, since it is not all of $R[G]$ (see below) but always contains the identity δ_e.

b) Restriction yields a $Z(R[G])$-module structure for each $R[G]$-module. Since $Z(R[G])$ is commutative, this is the same thing as a bimodule structure.

c) The equality $R(Z[G]) \subseteq Z(R[G])$ follows directly from the definition. In general, equality does not hold as one sees as follows: $X \in Z(R[G])$ is equivalent to $\delta_g \star X = X \star \delta_g$ for all $g \in G$ (why?). If we write $X = \sum X_h \cdot \delta_h$, then the previous equation is equivalent to $X_{g \cdot h} = X_{h \cdot g}$, i.e. $X_h = X_{g \cdot h \cdot g^{-1}}$ for all $g \in G$. Thus X has to be constant on conjugate elements. E.g. for $G = S_3$ the functions that are constant on

$$\{(12), (23), (13)\} \text{ or } \{(123), (132)\}$$

are in $Z(R[G])$, although $Z(S_3) = \{e\}$.