Exercises for Algebra II, WS 12/13

Sheet 12

Exercise 51

Let p be a prime. Show by hand (without the use of homological algebra) that there are (up to equivalence) exactly p extensions of length 1

$$0 \to \mathbb{Z}_p \to A \to \mathbb{Z}_p \to 0.$$

Exercise 52

Let G be a group and consider the categories $\mathcal{C} = \mathbb{Z}[G] - Mod$ of $\mathbb{Z}[G]$-modules and $\mathcal{D} = \mathbb{Z}[G] - Mod^{triv}$ of $\mathbb{Z}[G]$-modules with trivial module structure (which is the same as the category of abelian groups).

a) Assure yourself, that $\mathcal{C}$ and $\mathcal{D}$ are abelian categories with enough injectives and projectives.

b) Assure yourself, that $\mathcal{D}$ is a full subcategory of $\mathcal{C}$.

c) Show that the invariants functor $M \mapsto M^G := \{m \in M \mid g.m = m \ \forall g \in G\}$ is left exact on $\mathcal{C}$ and $\mathcal{D}$, is even exact on $\mathcal{D}$, but is in general not exact on $\mathcal{C}$.

Exercise 53

Let G be a finite group and k be a field such that $\operatorname{char}(k)$ does not divide $n := \operatorname{ord}(G)$.

a) Show that multiplication with n is an isomorphism on each $k[G]$-module M (the inverse is denoted by $\frac{1}{n}$).

b) Show that $\frac{1}{n}\sum_{g\in G} g.x \in M^G$ for each $x \in M$ and that $x = \frac{1}{n}\sum_{g\in G} g.x$ if $x \in M^G$.

c) Use part b) to show that $k[G] - Mod \to Ab$, $M \mapsto M^G$ is an exact functor.

d) Conclude that the derived functors $H^n_k(G, V)$ vanish for each $k[G]$-module V.

Exercise 54

If G is a group and A is a G-module, then a map $\delta\colon G \to A$ is called *derivation* if $\delta(gh) = \delta(g) + g.\delta(h)$. It is called *inner derivation* if $\delta(g) = a - g.a$ for some $a \in A$. The set $\operatorname{Der}(G, A)$ of all derivations is a group w.r.t. point-wise addition and the inner derivations form a subgroup. We denote the quotient of derivations by inner derivations by $\operatorname{PDer}(G, A)$ (for *principal* derivations).

We will now show that $\operatorname{PDer}(G, A) \cong H^1(G, A)$ through the following steps.

a) Let $\varepsilon\colon \mathbb{Z}[G] \to \mathbb{Z}$ be the morphism of $\mathbb{Z}[G]$-modules determined by $\varepsilon(g) = 1$ for all $g \in G$ and let IG denote the kernel of ε. Show that IG is the free abelian group on $(g-1)_{g\in G\setminus\{1\}}$.

b) Show that there is an exact sequence

$$0 \to H^0(G, A) \to \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \to \operatorname{Hom}_{\mathbb{Z}[G]}(IG, A) \to H^1(G, A) \to 0$$

c) Show that $\eta\colon \operatorname{Der}(G, A) \to \operatorname{Hom}_{\mathbb{Z}[G]}(IG, A)$, $\eta(\delta)(g-1) = \delta(g)$ is well-defined and an isomorphism (the latter be explicitly constructing an inverse morphism).

d) Show that under this isomorphism the inner derivations correspond exactly to elements in the image of $\operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \to \operatorname{Hom}_{\mathbb{Z}[G]}(IG, A)$.

e) Conclude that $\operatorname{PDer}(G, A) \cong H^1(G, A)$.

Exercise 55

Classify all groups of order 42!