Exercises for Algebra II, WS 12/13

Sheet 11

Exercise 47

Let $R = \mathbb{Z}[t]/(t^n - 1)$. Show that R is the group ring of $\mathbb{Z}_n$ and that

$$\operatorname{Tor}_k^R(\mathbb{Z}, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}_n & \text{if } k \text{ is odd} \\ 0 & \text{else} \end{cases}$$

through the following steps. Set $N = (1 + t + \cdots + t^{t-1})$ and $\varepsilon(t) = 1$.

a) Show that $(1 - t)N = 1 - t^n$.

b) Show that $\mathbb{Z}[t]$ admits a unique prime decomposition (up to units).

c) Use the latter to show that a multiple of N in $\mathbb{Z}[t]$ is a multiple of $1 - t^n$ if and only if it is a multiple of $1 - t$.

d) Show that $\cdots \xrightarrow{N} R \xrightarrow{1-t} R \xrightarrow{N} R \xrightarrow{1-t} R \xrightarrow{\varepsilon} \mathbb{Z}$ is a projective resolution of the trivial R-module $\mathbb{Z}$.

Exercise 48

Show that $\operatorname{Ext}^n_{\mathbb{Z}}(A, B)$ vanishes for $n \geq 2$ and each two abelian groups A and B.

Exercise 49

Go through the proof of the Acyclic Assembly Lemma (Lemma 6.6.4 in the lecture notes of Prof. Schweigert). In particular, verify that $d_h(x_{n_0}) = 0$, that $d_h(x_i + d_v(y_{i-1})) = 0$ and that $d(y_{n_0}, ..., y_0) = (x_{n_0}, ..., x_0)$. Moreover, verify part b) and give the reason why part b) works only for $\operatorname{Tot} X_{\bullet\bullet}$ and part a) only works for $|X_{\bullet\bullet}|$.

Exercise 50

Let $X_{\bullet\bullet}$ be the double complex with $X_{i,j} = \mathbb{Z}_2$ if $j \geq 0$ and $X_{i,j} = 0$ if $j < 0$ and all differentials given by multiplication with 2.

a) Show that $X_{\bullet\bullet}$ is a double complex and that $(..., 1, 1, 1)$ is a 0-cycle in $\operatorname{Tot} X_{\bullet\bullet}$.

b) Show that the 0-boundaries of $\operatorname{Tot} X_{\bullet\bullet}$ are $\prod_{\mathbb{N}_0}(2 \cdot \mathbb{Z}_4)$.

c) Show that $H_0(\operatorname{Tot} X_{\bullet\bullet}) \cong \mathbb{Z}_2$.

d) Show that $|X_{\bullet\bullet}|$ is exact.