Exercises for Algebra II, WS 12/13

Sheet 10

Exercise 43

Let $C_\bullet$ be a chain complex of abelian groups such that all groups C_n are free abelian groups. Are then the following assertions true or false?

a) The group of cycles $Z_n(C_\bullet)$ is free.

b) The group of boundaries $B_n(C_\bullet)$ is free.

c) The homology group $H_n(C_\bullet)$ is free.

Exercise 44

Let

$$\cdots \to M_{i+1} \xrightarrow{f_{i+1}} M_i \xrightarrow{f_i} M_{i-1} \xrightarrow{f_{i-1}} M_{i-2} \to \cdots$$

be an exact sequence. Show that f_i is an isomorphism if and only if f_{i-1} and f_{i+1} vanish.

Exercise 45

Let $f_\bullet \colon C_\bullet \to D_\bullet$ a morphism of chain complexes of R-modules. We construct from this a new chain complex $E(f)_\bullet$ by setting

$$E(f)_n := C_{n-1} \oplus D_n \quad \text{and} \quad d_n(c,d) := (-d_{n-1}(c), f_{n-1}(c) + d_n(d)).$$

a) Show that $E(f)_\bullet$ as defined above is in fact a chain complex.

b) Show that $f_\bullet$ is chain homotopic to 0 if and only if $f_\bullet \colon C_\bullet \to D_\bullet$ extends to a morphism $E(\mathrm{id}_{C_\bullet}) \to D_\bullet$ of chain complexes (if we consider $C_\bullet$ as a sub complex of $E(\mathrm{id}_{C_\bullet})$).

c) Let $C[-1]_\bullet$ be the chain complex with $C[-1]_n := C_{n-1}$. Show that $D_\bullet \to E(f)_\bullet$, $d \mapsto (0,d)$ and $E(f)_\bullet \to C[-1]_\bullet$, $(c,d) \mapsto c$ are morphisms of chain complexes and that

$$D_\bullet \to E(f)_\bullet \to C[-1]_\bullet \tag{1}$$

is a short exact sequence of chain complexes.

d) Let

$$\cdots \to H_{n+1}(E(f)_\bullet) \to H_n(C[-1]_\bullet) \xrightarrow{\delta} H_{n-1}(D_\bullet) \to H_{n-1}(E(f)_\bullet) \to \cdots \tag{2}$$

be the long exact sequence induced by (1). Show that $H_n(C[-1]_\bullet) \cong H_{n-1}(C_\bullet)$ and that $\delta = H_{n-1}(f_\bullet)$ with respect to this isomorphism.

e) Show that $f_\bullet$ is a quasi-isomorphism if and only if $E(f)_\bullet$ is exact.

Exercise 46

Let $F \colon \mathcal{C} \to \mathcal{D}$ be a right exact additive functor between abelian categories. Suppose that $\mathcal{C}$ and $\mathcal{D}$ have enough projectives and that we are given an exact sequence

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

such that each P_i is projective.

a) Show that for each $i > n$ we have the identity $L_iF(M) \cong L_{i-n}F(K)$.

b) Show that $L_1F = 0$ implies $L_iF = 0$ for all i.