Exercises for Algebra II, WS 12/13

Sheet 9

Exercise 39

Show that in **Set**, **Ab** and **R-Mod** a pull-back of $f\colon X \to Z$ and $g\colon Y \to Z$ is given by

$$X \times_Z Y := \{(x, y) \in X \times Y \mid f(x) = g(y)\},$$

along with the canonical maps $X \times_Z Y \to X$ and $X \times_Z Y \to Y$.

Exercise 40

Show that in **R-Mod** the sequence $A \to B \to C$ is short exact if and only if the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 0 \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

is a pull-back and a push-out.

Exercise 41

Let G be a group and A, B be subgroups such that $G = \langle A \cup B \rangle$. Show that the inclusions $A \to G$ and $B \to G$ give a push-out of the inclusions $A \cap B \to A$ and $A \cap B \to B$.

Exercise 42

In this exercise we will see that the category **R-Mod** has enough injectives. This we will do as follows:

a) Show that the functor $\mathbf{Ab} \to \mathbf{Mod\text{-}R}$, $A \mapsto \mathrm{Hom}_{\mathbf{Ab}}(R, A)$ is right adjoint to the forgetful functor $\mathbf{Mod\text{-}R} \to \mathbf{Ab}$.

b) Show that the R^{op}-module $\mathrm{Hom}_{\mathbf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$ is injective.

c) Show that a product of injective R^{op}-modules is injective if each factor is so.

d) Set $I_0 := \mathrm{Hom}_{\mathbf{Ab}}(R, \mathbb{Q}/\mathbb{Z})$ and

$$I(M) := \prod_{\mathrm{Hom}_{\mathbf{Mod\text{-}R}}(M, I_0)} I_0.$$

Show that M injects into $I(M)$.

Conclude that **R-Mod** has enough injectives.