Exercises for Algebra II, WS 12/13

Sheet 7

Exercise 31

Show that $F \dashv U$ for the the forgetful functor $U\colon \mathbf{R\text{-}Mod} \to \mathbf{Set}$ and the functor $F\colon \mathbf{Set} \to \mathbf{R\text{-}Mod}$ that assigns to a set X the free R-module with basis X and to a map $f\colon X \to Y$ the scalar extension of the map given by f on basis elements.

Exercise 32

Let $\mathcal{C}_{\mathbb{Z}}$ be the category associated to the poset $\mathbb{Z}$ with the standard order.

a) Show that functors $F\colon \mathcal{C}_{\mathbb{Z}} \to \mathcal{C}_{\mathbb{Z}}$ are in one-to-one correspondence with monotonous, but not necessarily strictly monotonous maps $f\colon \mathbb{Z} \to \mathbb{Z}$.

b) Define two series of functors via the maps $f_n, g_m n\colon \mathbb{Z} \to \mathbb{Z}$ by

$$f_n(x) = \begin{cases} x & \text{if } x \leq n \\ x-1 & \text{else} \end{cases}, \qquad g_n(x) = \begin{cases} x & \text{if } x \leq n \\ x+1 & \text{else} \end{cases}.$$

Compute for f_n and g_n both left and right adjoint functors.

Exercise 33

Show the following **Proposition:** Let $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{C}$ be functors and

$$\varepsilon\colon \mathrm{id}_{\mathcal{C}} \to G \circ F, \qquad \delta\colon F \circ G \to \mathrm{id}_{\mathcal{D}}$$

be natural transformation such that

$$\delta(F(X)) \circ F(\varepsilon(X)) = \mathrm{id}_{F(X)} \quad \text{and} \quad G(\delta(Y)) \circ \varepsilon(G(Y)) = \mathrm{id}_{G(Y)},$$

i.e.

$$F \xRightarrow{F\circ\varepsilon} FGF \xRightarrow{\delta\circ F} F \;=\; \mathrm{id}_F \quad \text{and} \quad G \xRightarrow{\varepsilon\circ G} GFG \xRightarrow{G\circ\delta} G \;=\; \mathrm{id}_G\,.$$

Then $\eta\colon \mathcal{D}(FX, Y) \to \mathcal{C}(X, GY)$, $\eta(\varphi) = G(\varphi) \circ \varepsilon(X)$ is natural such that $F \dashv G$.

Exercise 34

Let R be a ring let $F, G\colon \mathbf{R\text{-}Mod} \to \mathbf{R\text{-}Mod}$ be additive and $F \dashv G$.

a) If $\varphi\colon M \to N$ is a morphism in $\mathbf{R\text{-}Mod}$, then show that $\ker(\varphi)$ is the unique submodule of M such that for each $f\colon A \to M$ with $\varphi \circ f = 0$ we have $f(M) \subseteq \ker(\varphi)$:

$$\begin{tikzcd}
& & A \arrow[dl, dashed, "\exists!"'] \arrow[d, "f"] \arrow[dr, "0"] & \\
0 \arrow[r] & \ker(\varphi) \arrow[r] & M \arrow[r, "\varphi"] & N
\end{tikzcd}$$

with the bottom row exact.

b) Show that G preserves kernels, i.e., that $G(\ker(\varphi)) = \ker(G(\varphi))$ for each $\varphi\colon M \to N$.
Hint: Use Ex. 33.