Exercises for Algebra II, WS 12/13

Sheet 6

Exercise 25

Go through the steps of the proof of the Elementary Divisor Theorem in order to find a basis v_1, v_2, v_3 of the free $\mathbb{Z}$-module $M = \mathbb{Z}^3$ and elements $a_1, a_2, a_3 \in \mathbb{Z}$ such that $a_1 \cdot v_2, a_2 \cdot v_2$ is a basis of the submodule

$$N = \{(2m, 4n, 5n) \mid m, n \in \mathbb{Z}\} \leq M$$

Exercise 26

Collect further examples of categories, functors and of natural transformations that you have already encountered so far. Identify also the isomorphisms (i.e. the invertible morphisms) in the respective categories.

Exercise 27

Let $\mathbf{Gp}$ be the category of groups and G be an object of $\mathbf{Gp}$.

a) Is it true that $\mathrm{Hom}_{\mathbf{Gp}}(\mathbb{Z}, G) \cong G$ as sets?

b) Is it true that $\mathrm{Hom}_{\mathbf{Gp}}(G, \mathbb{Z}) \cong G$ as sets?

Exercise 28

A *generator* of a category $\mathcal{C}$ is an object U such that for any pair of morphisms $f, g\colon X \to Y$ with $g \neq f$ there exists a morphism $u\colon U \to X$ such that $f \circ u \neq g \circ u$.

a) Show that the category of groups and the category of sets have generators.

b) Show that functors do in general not map generators to generators, nor that generators are unique (even not up to isomorphism) if they exist.

Exercise 29

Let $\mathcal{C}$ be a small category. Show that the functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ also form a category (it is your task to define the objects, morphisms, source, target and composition in this category).

Exercise 30

Let $\mathcal{C}$ be a category and X be an object of $\mathcal{C}$. We consider the functor

$$h^X\colon \mathcal{C} \to \mathbf{Set}, \quad Y \mapsto \mathrm{Hom}_{\mathcal{C}}(X, Y), \quad \left(Y \xrightarrow{\varphi} Z\right) \mapsto (\varphi_*\colon \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z))$$

with $\varphi_*(f) := \varphi \circ f$. A functor which is isomorphic to such a functor is called *representable.*

a) Show that h^X defines indeed a functor.

b) Let $F\colon \mathcal{C} \to \mathbf{Set}$ be a functor. Show that the assignment

$$\mathrm{Hom}(h^X, F) \to F(X), \quad (\alpha\colon h^X \Rightarrow F) \mapsto \alpha_X(\mathrm{id}_X)$$

is a bijection.

c) Show that each category $\mathcal{C}$ embeds into $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$.