Exercises for Algebra II, WS 12/13

Sheet 5

Exercise 21

Let M be an R-module of finite length.

a) Show the following **Corollary** from the lecture: If N is a submodule, then N and M/N are of finite length and $l(M) = l(N) + l(M/N)$.

b) Show that for submodules P and Q of M we have $l(P)+l(Q) = l(P+Q)+l(P\cap Q)$.

Exercise 22

Classify all abelian groups of finite length. Calculate the length of each. Classify all abelian groups that have a *unique* composition series.

Exercise 23

a) Show that $\mathbb{Z}_n$ is semi-simple if and only if n is not divided by any square of a natural number $\neq 1$.

b) Find a ring R, an R-module M and a submodule $N \leq M$ such that N and M/N are semi-simple but not M.

Exercise 24

Let k be a field, $n \in \mathbb{N}^+$ and R be the ring of all upper triangular matrices in $M_n(k)$.

a) Show that $R/\operatorname{rad}(R)$ is semi-simple.

b) An element s of an arbitrary Ring S is called *nilpotent* if $s^n = 0$ for some $n \in \mathbb{N}^+$. Show that each nilpotent element in contained in $\operatorname{Rad}(S)$.

c) Determine $\operatorname{rad}(R)$ explicitly.

d) Show that $R/\operatorname{rad}(R) \cong k^n$ as vector spaces. Is is also true that $R/\operatorname{rad}(R) \cong k^n$ as R-modules (where the action of R on k^n is the natural one)?

e) Find a non-Artinian module $M \neq 0$ over $\mathbb{Z}$ such that $M/\operatorname{rad}(M)$ is not semi-simple.