Exercises for Algebra II, WS 12/13

Sheet 4

Exercise 17
Let

$$M_\bullet := \left(\cdots \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} \cdots \right)$$

be a chain complex of R-modules. Show that the following are equivalent:

a) $M_\bullet$ is an exact sequence.

b) In the diagram

$$\begin{tikzcd}
 & & \mathrm{coker}(d_{i+2}) \arrow[dr] & & & & \mathrm{coker}(d_i) & \\
\cdots \arrow[r] & M_{i+1} \arrow[ur] \arrow[rr, "d_{i+1}"] & & M_i \arrow[rr] \arrow[dr] & & M_{i-1} \arrow[ur] \arrow[rr] & & \cdots \\
\mathrm{coker}(d_{i+3}) \arrow[ur] & & & & \mathrm{coker}(d_{i+1}) \arrow[ur] & & &
\end{tikzcd}$$

the diagonal sequences are short exact. It is part of the exercise to determine all the maps in the previous diagram (recall that $\mathrm{coker}(f) := N/\mathrm{im}(f)$ for $f\colon M \to N$).

Exercise 18
Let $M' \to M \to M''$ and $M' \to \widetilde{M} \to M''$ be short exact sequences of R-modules and $\varphi\colon M \to \widetilde{M}$ be a morphism of R-modules such that

$$\begin{tikzcd}
M' \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\varphi"] & M'' \arrow[d, equal] \\
M' \arrow[r] & \widetilde{M} \arrow[r] & M''
\end{tikzcd}$$

commutes. Show that φ is then automatically an isomorphism.

Exercise 19

a) Let M be an R-module. Show that M is projective and finitely generated if and only if there exist $x_1, ..., x_n \in M$ and $f \in \mathrm{Hom}_R(M, R)$ such that $x = \sum_{i=1}^n f_i(x) \cdot x_i$.

b) Show that if I is a two-sided Ideal in R and M is a projective R-module, then M/IM is a projective R/I-module.

Exercise 20
Let R, S be commutative rings. Show that

a) R is flat as a module over itself.

b) If $S \subseteq R$ is multiplicative, then the localization $S^{-1}R$ is flat over R.

c) Show that if M is flat over R and that $R \to S$ is a morphism of rings, then $M \otimes_R S$ is flat over S.