Conformal field theories and a new geometry

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To see a world in a grain of sand,
And a heaven in a wild flower,
Hold infinity in the palm of your hand,
And eternity in an hour.

---- William Blake, “Auguries of Innocence”

Contents

- Geometry and CFT
- A definition of CFT
- A classification of rational CFT
- Boundary-bulk duality and defects

Part 1. Geometry and CFT

String theory has been very successful in inspiring new mathematics especially in the field of geometry. To name a few: mirror symmetry, quantum cohomology, Gromov-Witten theory, topological string, elliptic genus, ..., etc.

Question: why can string theory provide so many new insights which is not available otherwise ?

A quick answer:

String theory emphasis the study of loop space instead of the original manifold. The loop space contain many new structures which can not be seen from the original manifold.

Question: what are the structures on loop space?

There are perhaps many different answers to this question.

One answer suggested by string theory is that the loop space has a natural algebraic structure which is called *(super-)conformal field theory* (CFT).

We list a few constructions which are inspired by this suggestion.

1. Malikov-Schetchman-Vaintrob's Chiral de Rham complex
= a sheaf of vertex operator algebras on smooth manifold
~ a shadow of certain structure on formal loop space.

2.Kapranov-Vasserot:
free loop space (Algebra-geometric version)
= factorization monoid
= a non-linear version of factorization algebra.
factorization algebra = a global version of VOA.

3. Chas-Sullivan:
string topology = certain algebra structure on the homology of the free loop space (BV-algebra, homological CFT).

A direct construction of CFT is, however, less interesting to many geometers because CFT itself is a mysterious object and its connection to geometry is not very clear.

Most of math works (until recent years) are focusing on studying some interesting ingredients of a special kind of CFT called non-linear sigma model. Examples of these ingredients are super-conformal algebras, chiral rings, A-branes, B-branes, partition functions, etc. They are directly connected to mirror symmetry, quantum cohomology, Qcoh(X), Fukaya category, elliptic genus, etc. This approach has been very successful so far.

The disadvantage of studying only ingredients is that we might lose the global picture. What global picture we gain if we look at an entire CFT instead of its ingredients?

It is already apparent in the study of mirror symmetry and works by string theorists that the paradigm of geometry established before the advent of string theory is inadequate.

An “Auguries of Innocence” (K., 5/2007):

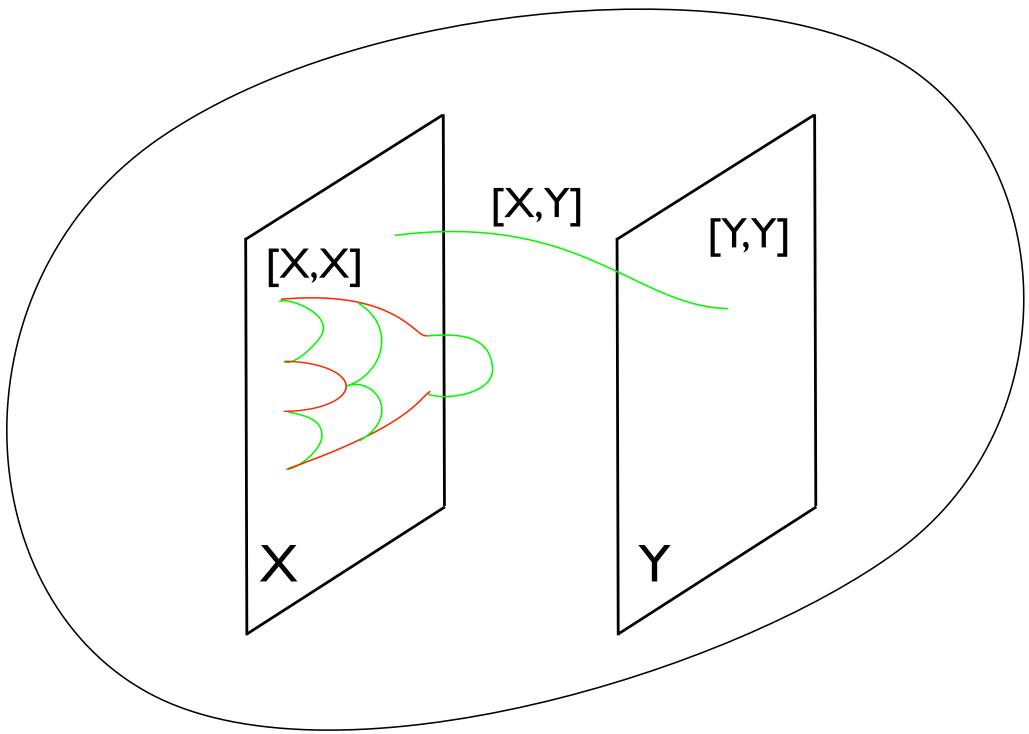
CFT provides an entirely new foundation of geometry!

There are two evidences for such a new geometry from CFT:

1. The closed CFT is indeed a stringy generalization of commutative ring. More precisely, the closed CFT is a commutative associative algebra in certain braided tensor category.

2. D-branes have been used by physicists to probe the geometry of target manifold. As boundary conditions for open strings, they indeed behave like generalized points or sub-varieties. Algebraically, they are certain ``chiral modules'' over closed CFT.

Boundary condition: X,Y are chiral modules over a closed CFT;
Open CFTs: [X,X], [Y,Y];   [X,Y] is a [Y,Y]-[X,X]-bimdule.



Conjecture: an open CFT determines a closed CFT by taking center.

Classical AG	Stringy generalization
a commutative ring A	a closed CFT C
Spec(A) = the set of prime ideals of A	Spec(C) = the category of D-branes

This geometry has the following new features:

1. Categorical instead of set-theoretical: CFT or QFT in general emphasizes the space as a network of interesting subspaces instead of the usual sheaf-theoretical point of view.

2. Holographic Principle: intuitively, if the boundary condition is just a point, the based loop space has certain generalized algebra structure (open CFT) and determines the free loop space as its ``center".

Hints from string topology (Chas-Sullivan) which can be viewed as a homological CFT (Godin):

Let N be a submanifold of M, $C_*(\mathcal{P}_{N,N})$ be the space of singular chains on the path space $\mathcal{P}_{N,N}$. As a dg algebra, $C_*(\mathcal{P}_{N,N})$ is quasi-equivalent to the open string topology introduced by Sullivan. Then we have

$$HH^*(C_*(\mathcal{P}_{N,N}), C_*(\mathcal{P}_{N,N})) \cong H_*(LM)$$

when M is simple connected and closed and N is:

1. N is a point in M (Burghelea, Goodwillie, ... 80's),
2. N=M (Jones, 80's)
3. many other cases (Blumberg-Cohen-Teleman, 2009)

$HH^*(A)$ is nothing but a derived center of A .

To see a world in a grain of D-brane,
And a heaven in a wild flower,
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3. a unification of algebraic geometry with metric.

A CFT is a module over Virasoro algebra (super-conformal algebra) which can be viewed as Laplacian (Dirac operator) on loop space. Therefore, this new geometry should

contain spectral geometry as an ingredient. In particular, a proper completion of super-CFT leads to Connes' (pre-)spectral triple.

A few approaches to new geometry associated to D-branes:

1. Douglas' D-geometry.
2. Gómez-Sharpe's generalized scheme theory.
3. Aspinwall's stringy geometry as the moduli space of D0-branes.
4. Liu-Yau's non-commutative algebraic geometry formulation of D-branes largely motivated by moduli problems.
5. Derived Algebraic Geometry

Derived Algebraic Geometry:

Take an associative algebra A, then take the Hochschild cohomology HH*(A) (derived center of A) as the replacement of commutative ring. Such HH*(A) is also a closed topological conformal field theory.

DAG is parallel to our program of SAG.

In general, A = A-infinity algebra, monoidal category, tensor-infinity category, E_n-algebra, ..., etc.

To physicists:

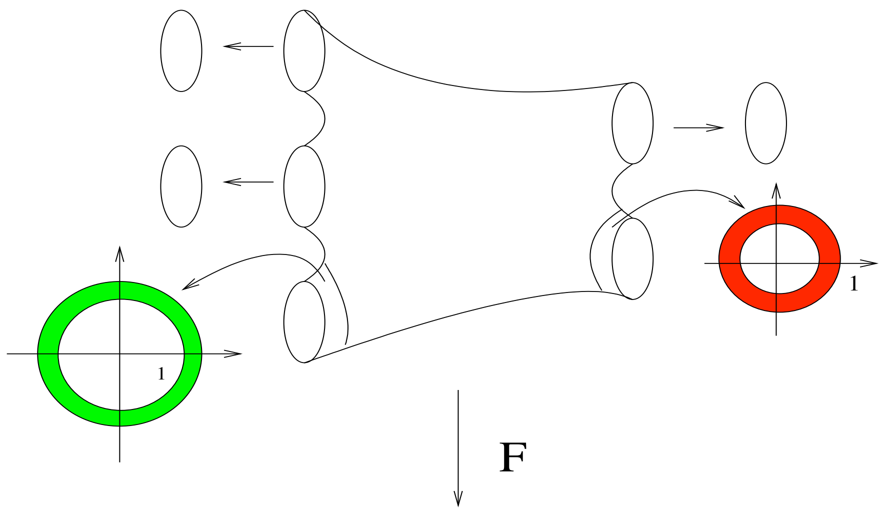
The physical demand for a new geometry is due to the vision that space-time is emergent. One should be able to recover the space-time from the observable algebras.

It is believed that gravity can be rederived from Holographic Principle. For us, Holographic Principle just says that a boundary theory uniquely determine the bulk theory. Conversely, a bulk theory does not determine boundary theory uniquely. This ambiguity of non-uniqueness is nothing but the spectrum of an entirely new/old geometry.

Part II. The Definition of CFT:

1987, Kontsevich and Segal independently gave a definition of CFT as a symmetric projective monoidal functor from the category RS_b of finite ordered set with hom-set being the moduli space of Riemann surfaces with parametrized boundaries to the category of complete locally convex topological vector spaces.

An element in the Hom set of $\mathrm{RS_b}$:



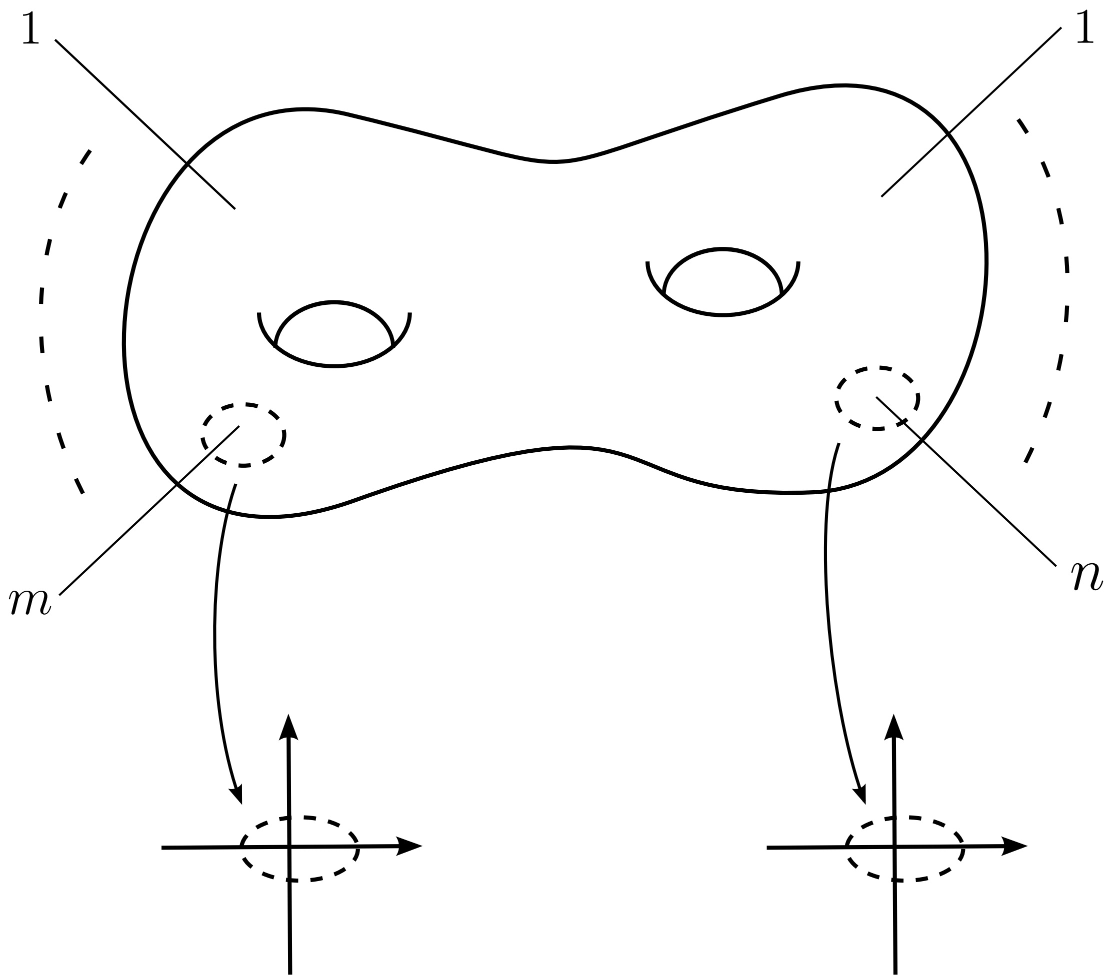
$$H_{cl}^{\otimes 3} \longrightarrow H_{cl}^{\otimes 2}$$

Two disadvantages of Kontsevich-Segal's definition:

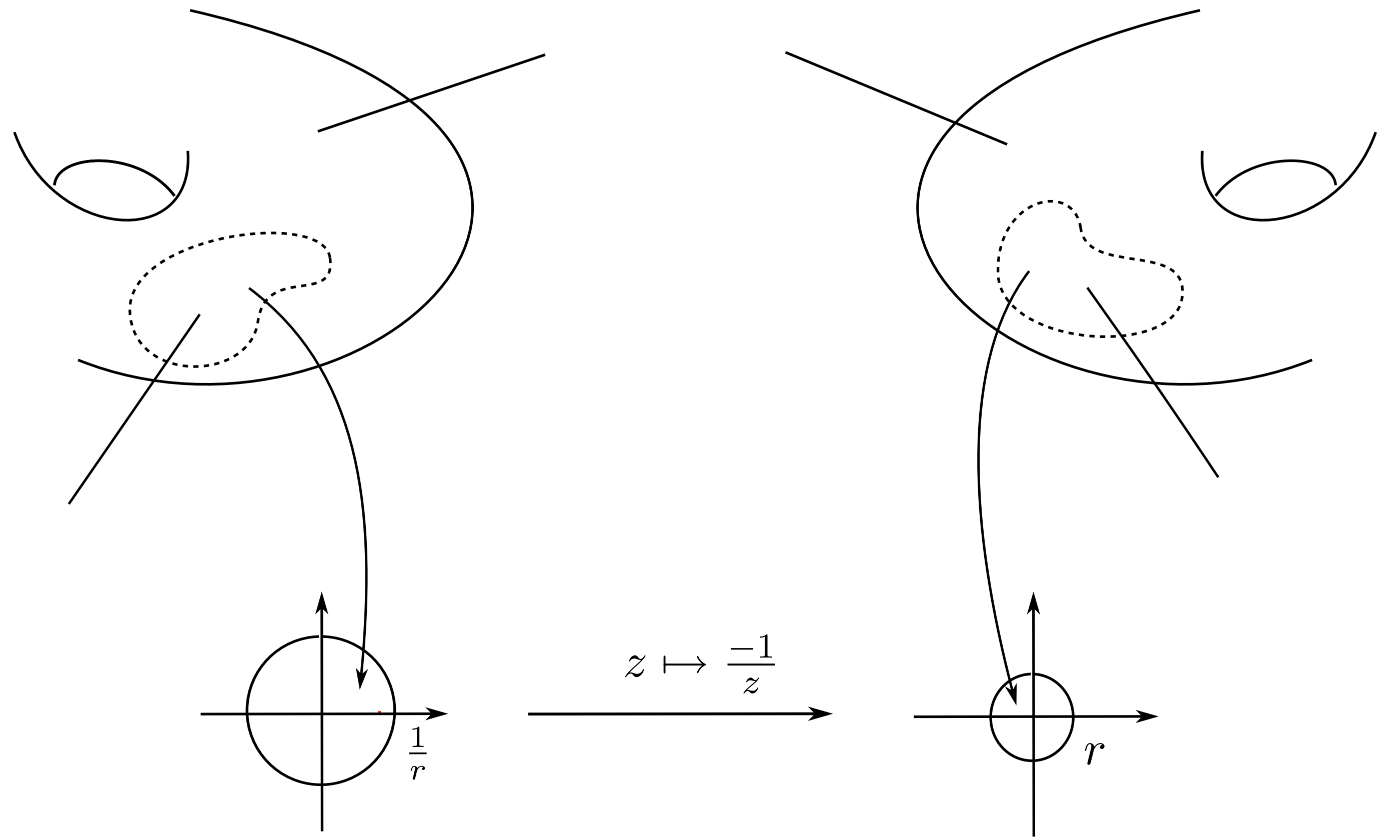
1. The quantum fields in physics usually are associated to a point in the space-time. They do not live in Kontsevich-Segal's definition in an obvious way. This suggests to change RS_b to RS_p (Riemann surfaces with parametrized punctures).

2. A complete topological vector space is very hard to construct. A dense subspace of it is much easier to deal with. This suggests to use the category GVS of graded vector spaces instead.

An element in the Hom set of $\mathsf{RS_P}$:



Sewing operations are not always well-defined:



The category GVS:

1. An object A is a graded vector space over $\mathbb{C}$ with homogeneous spaces being finite dimensional: $A = \oplus_n A_{(n)}$

2. $\mathrm{Hom}_{GVS}(A, B) = \mathrm{Hom}_{\mathbb{C}}(A, \overline{B})$ where $\overline{B} := \prod_n B_{(n)}$

3. $A \xrightarrow{f} \overline{B} \qquad B \xrightarrow{g} \overline{C}$

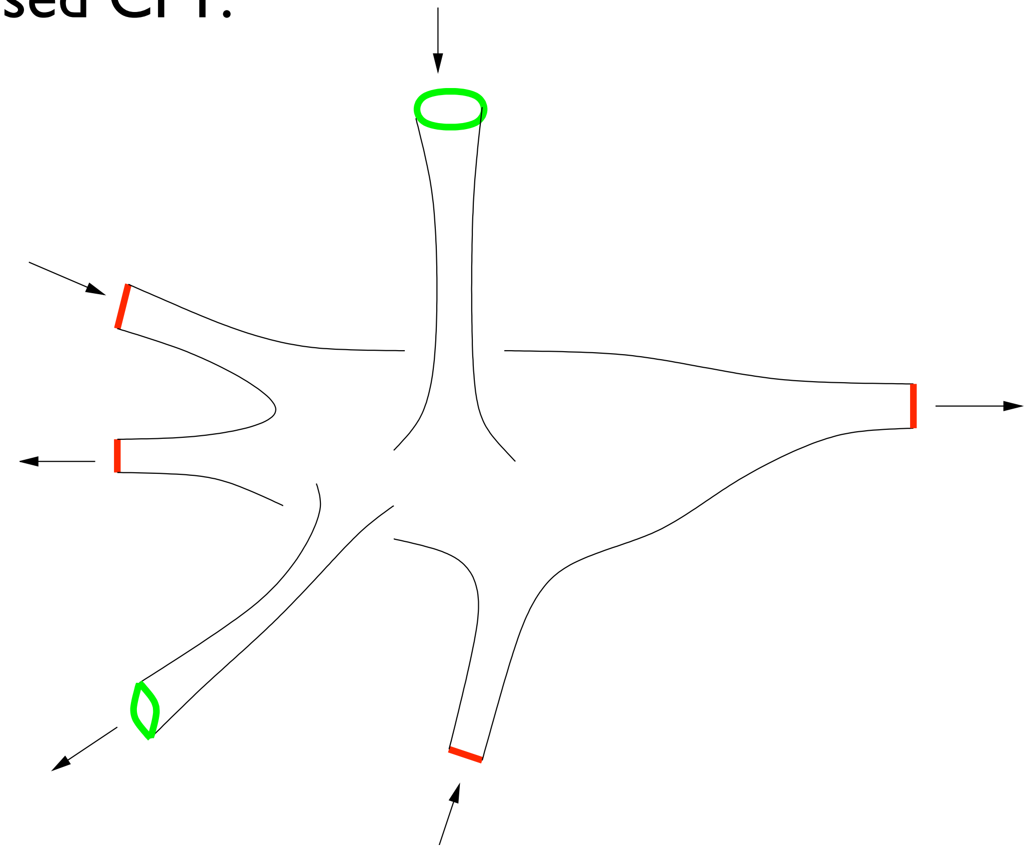
$$\langle c', g \circ f(a) \rangle := \sum_n \langle c', g(P_n f(a)) \rangle, \qquad P_n : \overline{B} \to B_{(n)}$$

CFT_p is a projective symmetric monoidal functor F from RS_p to GVS.

Since the compositions in both categories are only partially defined. A functor from RS_p to GVS requires the following condition:

If two morphisms S and T in RS_p is composable, i.e.
$S \# T$ exists, then $F(S) \circ F(T)$ exists and
$F(S \# T) = F(S)\ \ F(T)$.

Open-closed CFT:



Part III. A classification of rational open-closed CFTs:

Theorem (Huang): The structure of

(F({1}), F(*genus-0 surfaces with only one out-going puncture*))

+ F being holomorphic

+ additional natural conditions such as integer grading, etc

= a vertex operator algebra (VOA).

Theorem (Huang): The category of modules over a rational VOA is a modular tensor category.

Basic structures of a modular tensor category $\mathcal{C}$:

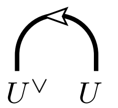
1. tensor product funtor: $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

2. associativity: $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$

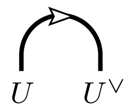
3. tensor unit: $\mathbf{1} \in \mathcal{C}$, $\mathbf{1} \otimes A \cong A \cong A \otimes \mathbf{1}$

4. braiding: $c_{A,B} : A \otimes B \to B \otimes A$, for $A, B \in \mathcal{C}$

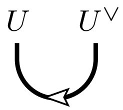
5. rigidity:



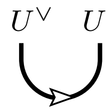
$= d_U : U^\vee \otimes U \to \mathbf{1}$,



$= \tilde{d}_U : U \otimes U^\vee \to \mathbf{1}$,

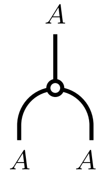


$= b_U : \mathbf{1} \to U \otimes U^\vee$,

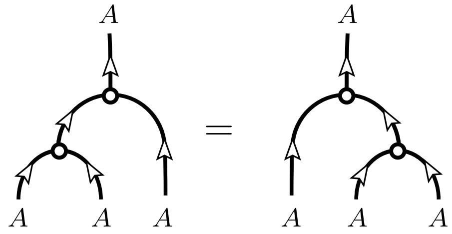


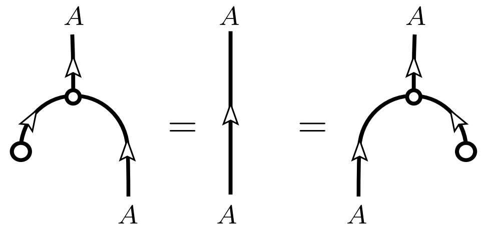
$= \tilde{b}_U : \mathbf{1} \to U^\vee \otimes U$,

1. An algebra in $\mathcal{C}$: $A = (A, m, \eta)$

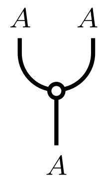
$m =$ 

, $\eta =$ 

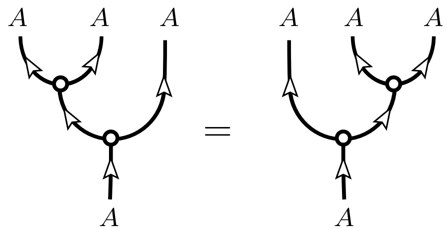


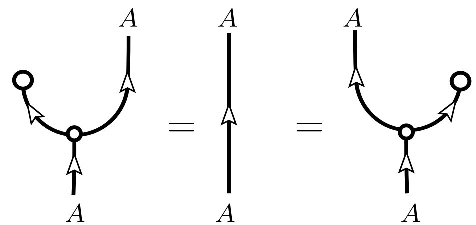


2. A coalgebra in $\mathcal{C}$: $A = (A, \Delta, \varepsilon)$

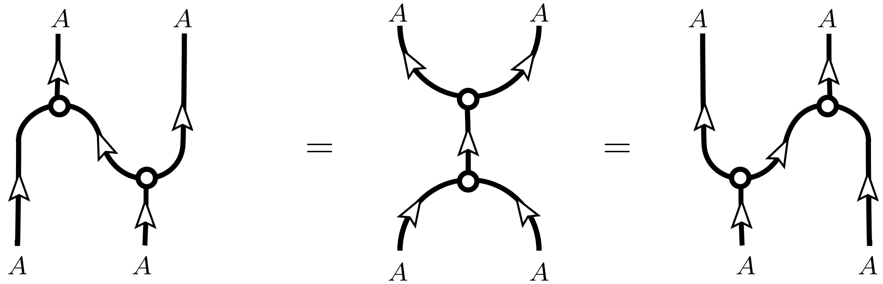
$\Delta =$ 

, $\varepsilon =$ 

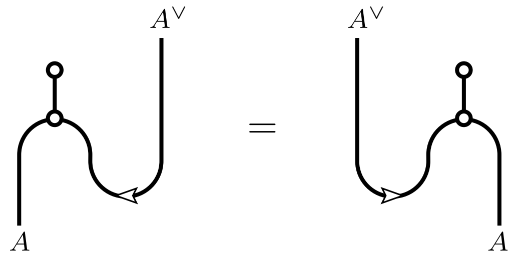




A Frobenius algebra in $\mathcal{C}$: $A = (A, m, \eta, \Delta, \varepsilon)$ satisfying



A Frobenius algebra A is called symmetric if :



For a tensor category C, the monoidal center Z(C) of C is a braided tensor category.

$$Z(A)=\mathrm{Hom}_{A|A}(A, A) \quad \dashrightarrow \quad Z(C):=\mathrm{Fun}_{C|C}(C,C)$$

A forgetful functor F: Z(C) --> C, R= the right adjoint of F.

For an algebra A in C, the center Z(A) of A is defined as an object in Z(C).

$$Z(A) = \underline{\mathrm{Hom}}_{A|A}(A, A) \text{ in } Z(C)$$

The center of a monoidal category

For a tensor category $\mathcal{C}$, one can define its monoidal center $Z(\mathcal{C})$:

$$Z(\mathcal{C}) = \{(X, X \otimes - \xrightarrow{c_X \cong} - \otimes X) | c_{X \otimes Y} = c_X \circ c_Y, \ c_X(\mathbf{1}) = \mathrm{id}_X\}.$$

$Z(\mathcal{C})$ is a braided tensor category with tensor product $\otimes$ given by

$$(X, c_X) \otimes (Y, c_Y) := (X \otimes Y, c_X \circ c_Y).$$

and the braiding given by

$$(X, c_X) \otimes (Y, c_Y) \xrightarrow{c_X(Y)} (Y, c_Y) \otimes (X, c_X).$$

The forgetful functor $F : Z(\mathcal{C}) \to \mathcal{C}$ is monoidal and its right adjoint $F^\vee$ (if exists) is lax and colax monoidal.

The center of an algebra

1. An algebra (A, m, η) in a braided tensor category is called commutative if $m \circ c_{A,A} = m$.

2. Given an algebra A in $\mathcal{C}$, the center $Z(A)$ of A is an object in $Z(\mathcal{C})$ with $Z(A) \xrightarrow{\iota} A$ in $\mathcal{C}$ which is terminal among all pairs (Z, ξ) satisfying

$$\begin{tikzcd}
Z \otimes A \arrow[r, "\xi 1"] \arrow[d, "z_A"] & A \otimes A \arrow[dr, "m"] & \\
A \otimes Z \arrow[r, "1\xi"] & A \otimes A \arrow[r, "m"] & A
\end{tikzcd}$$

In the case of MTC,

$$Z(A) := C_l(F^\vee(A))$$

where $F^\vee(A)$ is an algebra and $C_l(-)$ is the left center.

Theorem(genus=0)/Conjecture(genus>0) (Kong):

An open-closed CFT over V is equivalent to a triple

$$(A_{\mathrm{op}}|A_{\mathrm{cl}}, \iota_{\mathrm{cl-op}})$$

where

1. A_{cl} is a commutative symmetric Frobenius algebra in $\mathcal{Z}(\mathcal{C}_V)$,

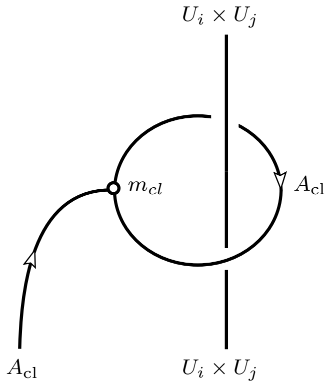
2. A_{op} is a symmetric Frobenius algebra in $\mathcal{C}_V$,

3. $\iota_{\mathrm{cl-op}} : A_{\mathrm{cl}} \to Z(A_{\mathrm{op}})$ is an algebra homomorphism,

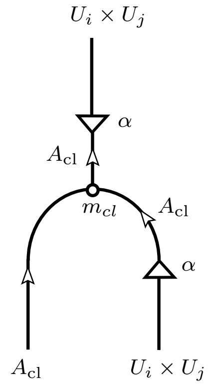
satisfying two axioms:

1. the modular invariance of A_{cl} :

$$\frac{\dim U_i \, \dim U_j}{\dim \mathcal{C}}$$

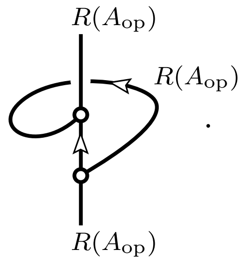


$$= \sum_{\alpha}$$



2. Cardy condition:

$$\iota_{\text{cl-op}} \circ \iota^{*}_{\text{cl-op}} =$$



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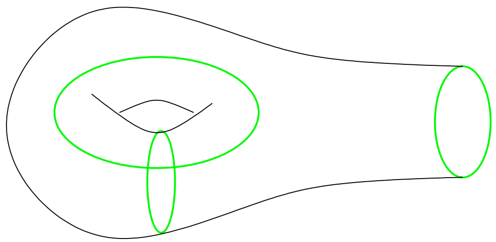
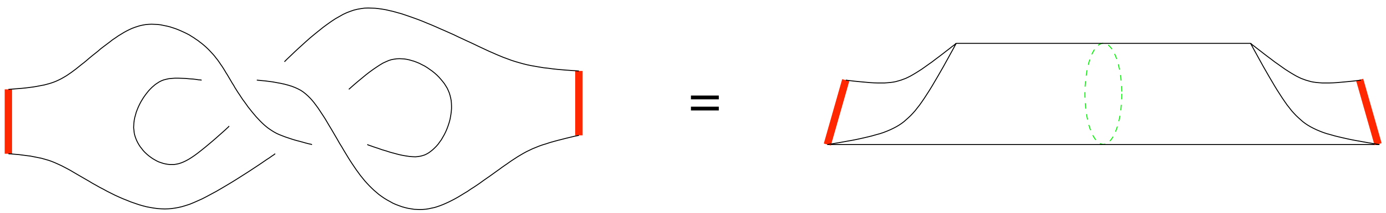
Geometric meaning of these two conditions:

1. Modular invariance of 1-pt correlation functions on torus:



2. Cardy condition:



Relation to 2-d TFT: when $V = \mathbb{C}$,

2-d open-closed CFT over V = 2-d open-closed TFT over $\mathbb{C}$

Other approaches:

1. Fuchs-Runkel-Schweigert obtained similar results for open-closed rational CFT independently in an approach based on 3-dim TFT.

2. A similar classification result for CFT is obtained independently by Longo and Rehren in an approach based on Möbius covariant net on circle.

Construction:

A Frobenius algebra $A = (A, m, \eta, \Delta, \varepsilon)$ is special if $m \circ \Delta \propto \mathrm{id}_A$ and $\epsilon \circ \eta \propto \mathrm{id}_{\mathbf{1}}$.

Theorem (Fuchs-Runkel-Schweigert, K.-Runkel):
Given a special symmetric Frobenius algebra A in $\mathcal{C}_V$,
then $(A|Z(A), \mathrm{id}_{Z(A)})$ gives an open-closed CFT over V.

Example: $A = \mathbf{1} \in \mathcal{C}_V$, $Z(\mathbf{1}) = \oplus_i U_i^\vee \otimes_{\mathbb{C}} U_i$.
For any V-module X, $A = X \otimes X^\vee$, $Z(A) = Z(\mathbf{1}) = \oplus_i U_i^\vee \otimes_{\mathbb{C}} U_i$.

Part IV. Boundary-bulk duality and defects

- Holographic Principle and boundary-bulk duality
- dualities = invertible defects

Holographic Principle:

Theorem (Fjelstad-Fuchs-Runkel-Schweigert, K.-Runkel):

Given an open-closed CFT $(A_{\mathrm{op}}|A_{\mathrm{cl}}, \iota_{\mathrm{cl-op}})$ over V,
if A_{op} is simple and $\dim A_{\mathrm{op}} \neq 0$, then the bulk theory A_{cl} is isomorphic to the center of A_{op}.

Conversely, in rational CFT, the bulk theory does not uniquely determine the boundary theory, but the boundary theories are unique up to Morita equivalence.

Theorem (K.-Runkel):

For two simple special symmetric Frobenius algebras,

$$A \cong_{\text{Morita}} B \text{ iff } Z(A) \cong_{\text{algebra}} Z(B) \ .$$

Parallel results of other QFTs:

1. 3-d Turaev-Viro theory: (Kitaev-Mueger, Etingof-Nikshych-Ostrik)
boundary: A --- a finite fusion category,
bulk: Z(A) --- monoidal center.

2. open-closed TCFT (Costello),
boundary: A --- a Calabi-Yau category,
bulk: Z(A)=HH*(A) --- a derived center.

3. $\mathcal{E}_n$-operad generalization of Deligne conjecture (Konstevich, Lurie),
boundary: A --- an $\mathcal{E}_n$-algebra,
bulk: Z(A) --- an $\mathcal{E}_{n+1}$-algebra .

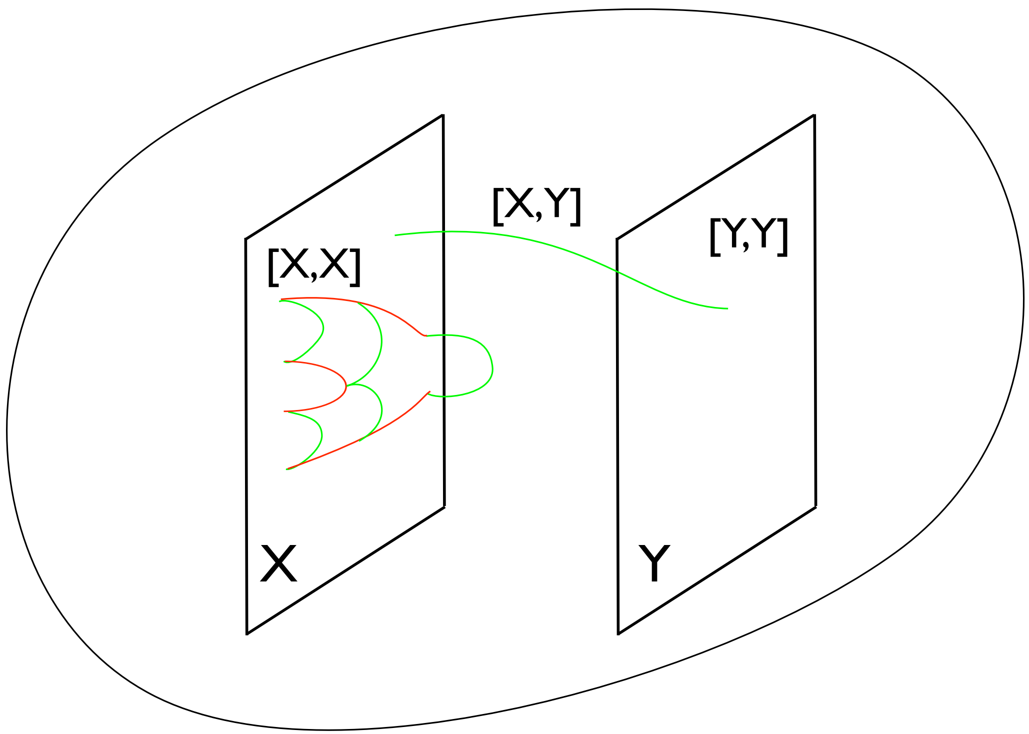
Geometric interpretation:

a closed CFT = a free loop space LX,

an open CFT = the space of paths with two ends ending on a fixed subspace (D-branes) of X.

For a given closed CFT A_{cl} , a D-brane is a pair $(A_{\mathrm{op}}, A_{\mathrm{cl}} \xrightarrow{\iota_{\mathrm{cl-op}}} Z(A_{\mathrm{op}}))$ such that $(A_{\mathrm{op}}|A_{\mathrm{cl}}, \iota_{\mathrm{cl-op}})$ gives an open-closed CFT.

Boundary condition: X,Y are chiral modules over a closed CFT;
Open CFTs: [X,X], [Y,Y]; [X,Y] is a [Y,Y]-[X,X]-bimdule.



Conjecture: an open CFT determines a closed CFT by taking center.

Conformal invariant D-branes:

Open-closed CFT over V means it satisfies a so-called V-invariant boundary condition, which says that boundary is transparent to V. D-branes in this context is called V-invariant D-branes. But they are too few to recover classical geometry.

But we only need Vir-invariant boundary condition, where Vir is the smallest sub-VOA of V containing only the Virasoro element (or the energy-momentum tensor). That is why it is also called conformal invariant D-branes. Such D-branes are rich enough to recover all points in the target manifolds and much more.

How the notion of manifold emerges?

1. the moduli space of D0-branes;
2. in the large volume limit (Kontsevich-Soibelman);
3. other classical limit.

How the notion of time emerges?

Connes observed that a type-III factor contains a God given 1-dimensional (outer)-automorphism subgroup which should be interpreted as time flow.

Question: will time emerges from the automorphism group of a CFT?

Dualities=invertible defects:

The automorphism group of a bulk theory is equivalent to the Picard group of the invertible bimodules of a boundary theory.

Theorem:

1. rational CFT case: (Davydov-K.-Runkel)

$$\mathrm{Aut}(Z(A_{\mathrm{op}})) = \mathrm{Pic}(A_{\mathrm{op}})$$

2. Turaev-Viro 3-d TFT: $\mathrm{Aut}(Z(\mathcal{C})) = \mathrm{Pic}(\mathcal{C})$
(Kitaev-K., Etingof-Nikshych-Ostrik, Drinfeld)

Thank you !

classical geometry	classical algebra	stringy algebraic geometry (SAG)	stringy algebra
affine scheme: M	commutative ring	loop space LM	closed CFT C
points in M: x,y	prime ideals: x, y	sub-manifolds of M: X, Y	D-branes: X, Y
no structure b/w a, b	no structures b/w a,b	the path space between sub-manifolds A and B	[X,Y]: [Y,Y]-[X,X]-bimodule
		Laplacian (Dirac) operators on LM	Virasoro algebra (super version)