RESEARCH STATEMENT

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1. Background

My work lies in an area at the intersection of geometry, higher category theory and theoretical physics, which more precisely concerns the study of **topological quantum field theories** (TQFT). From the mathematical point of view, a TQFT allows to relate a tensor category of topological or geometrical nature, namely a **cobordism category**, to a tensor category of algebraic nature, namely vector spaces, modules over algebras, or in general a **symmetric monoidal category**. This allows on one side to obtain topological invariants of manifolds, and on the other to prove results in algebra using topological or geometric arguments. More precisely, the Atiyah-Segal axioms [Ati89, Seg88] formulate a n-dimensional TQFT as a symmetric monoidal functor from Cob_n, the cobordism category with objects closed $n-1$-manifolds and morphisms given by n-bordisms, to $\mathrm{Vect}_{\mathbb{C}}$, the category of vector spaces. As a consequence of these axioms, a TQFT assigns a numerical invariant to a closed n-manifold. Particularly important examples of topological quantum field theories include the 3-dimensional Reshetikhin-Turaev TQFTs [RT91, Tu94] associated to a modular tensor category and Turaev-Viro TQFTs [TV92] associated to spherical fusion categories. Recently [Mor13], Dijkgraaf-Witten theory for a finite group G has been constructed as a TQFT via a linearisation of the stack of G-bundles over manifolds, which can be seen as a quantisation procedure of a classical theory, as suggested in [FHLT09]. All the above are furthermore examples of **extended** topological quantum field theories.

An extended topological quantum field theory attaches data not only to n-manifolds and $n-1$-manifolds, but also to lower dimensional manifolds which appear as *corners*. To properly formulate this requires the introduction of **higher** categories, which are a generalisation of the notion of a category, where one considers not just objects and morphisms, but also morphisms between morphisms, and so on. Examples of higher categories include the 2-categories of 2-vector spaces [KV94], of algebras, bimodules and intertwiners, and of bundle gerbes over a manifold [W07]. A TQFT which assigns data to all lower dimensional corners up to the point is said to be **fully extended**. Fully extended TQFTs have been completely classified in [Lu09] using the language of ∞-categories, where it is shown that a fully extended n-dimensional framed TQFT is determined by the object assigned to the point, and that any *fully dualisable* object in the target (∞, n)-category gives rise to such a TQFT. This is a remarkble result which proves what is known as the *Cobordism Hypothesis*, a conjecture formulated by John Baez and James Dolan [BD95] concerning the algebraic structure of the higher category of n-dimensional cobordisms, namely that the n-category of completely extended cobordism Cob_n is the free symmetric monoidal n-category with duals generated by a single object.

My interest in this field lies in the investigation of **boundary conditions** and **defects** in TQFTs. In recent developments [FFRS07, Kap10, KS11], it has become clear that the study of boundary conditions and defects will provide the tools for a deeper understanding of the mathematical structures required to describe symmetries and dualities of quantum field theories. On the other hand, many possible applications ranging from quantum computing to impurity problems in condensed matter physics require to consider quantum field theories on manifolds with *physical boundaries*, or on manifolds which are separated by *domain walls* into different parts on which different phases of the quantum field theory are realized. Furthermore, recent progress has been made concerning the relation between the study of boundary conditions and anomalies in quantum field theories [FT12].

2. Work to date

In a recent paper with my collaborators Christoph Schweigert and Jürgen Fuchs [FSV12], I have investigated the 2-category of boundary conditions and surface defects for 3d TQFTs of Reshetikhin-Turaev type, given a modular tensor category $\mathcal{C}$, consistent boundary conditions for the associated TQFT can be imposed only if $\mathcal{C}$ is braided equivalent to the Drinfeld center of a fusion category $\mathcal{A}^b$. Similarly, we show that given two modular tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$, surface defects between the associated TQFTs exist only if $\mathcal{C}_1^{rev} \boxtimes \mathcal{C}_2$ is braided equivalent to the Drinfeld center of a fusion category $\mathcal{A}^d$. When boundary conditions can be imposed, their 2-category is equivalent to the 2-category of **module categories** over the fusion category $\mathcal{A}^b$. In the same way, when surface defects between two TQFTs do exist, their 2-category is equivalent to the 2-category of **module categories** over the fusion category $\mathcal{A}^d$. We then observe that the obstruction for a TQFT to admit boundary conditions lies in the *Witt group* of fusion categories, which has been subject to investigation in recent works [DNO13].
Finally, we also construct (noncanonically) from a surface defect S a special symmetric commutative Frobenius algebra in $\mathcal{C}$, which corresponds to a 2d RCFT, as predicted in [FFRS07].
The results above agree with those arising in more specific and concrete situations, for instance in the study of toroidal Chern-Simons theory [KS11]: in this case the gauge theory is governed by a metric group (D, q), and its associated modular tensor category $\mathcal{C}(D, q)$. From the mathematical point of view, the results described above provide a useful bridge between algebraic results in terms of module categories and their classification, and topological results in terms of TQFTs.
In [FSV13], via an educated guess motivated by the classification of module categories over $\mathrm{Vect}^\omega(G)$, where $\omega \in Z^3_{grp}(G, \mathbb{C}^*)$, we give a functorial construction of Dijkgraaf-Witten theory based on (G, ω) including boundary conditions and surface defects in terms of **relative** G**-bundles**. More specifically, in the simplest case, for $H \subset G$, a relative (G, H)-bundle on a relative manifold (M, Y) consists of a G-bundle over M which admits a restriction of its structure group to $H \subset G$ when restricted over Y. When appropriate morphisms are introduced, relative bundles form a groupoid for each relative manifold, and give rise to a stack over the site of *decorated* manifolds. Using a linearisation procedure [Mor13, FHLT09], we present the categories associated to one-dimensional manifolds with boundaries and defects points, and we show that the results agree with the prediction of the general formalism developed in [FSV12], taking into account the results of [Ost03]. In particular, we generalise the "parmesan" combinatoric techniques from [Will08] concerning the trasgression of n-cocyles for the cohomology of the classifying space, or rather stack BG to be able to transgress along 1-dimensional manifolds with possible boundaries and codimension 1 defects, i.e. marked points.

3. My research plan

The relationship between symmetries and topological defects in topological quantum field theories is a very interesting one, and yet not completely explored. In particular a complete definition of a symmetry for a 3d extended TQFT in the functorial sense should involve natural tansformations of 2-functors and their modifications on a side, and the category of automorphisms of the object the theory assigns to the circle. Moreover, there are results in 2 dimensions [FFRS07] which suggest that all symmetries should be realisable in terms of invertible defects.
Using the constructions in the previous section, in future work I plan to investigate symmetries of Dijkgraaf-Witten theories and their relation to invertible surface defects, in particular to relate the automorphisms of the stack of G-bundles with the category of braided equivalences of the Drinfeld center of the category G-vector spaces. In particular, one expects the 2-group describing the symmetries of Dijkgraaf-Witten as an extended 3d TQFT to be strictly "bigger"

than just the automorphism 2-group of the stack of G-bundles, which corresponds to the *classical* symmetries: from gauge theoretic intuition, indeed, I expect that nongeometric symmetries, as for instance the analog of electric-magnetic duality, should arise only at the quantum level, but should be neverthless captured by invertible surface defects. As a start, in a forthcoming work [FPSV14], I have concentrated on verifying these expectations in the simplified case when G is an Abelian group, and the 3-cocycle ω vanishes.
As pointed out in [DW90], a given 3-cocyle $\omega \in Z^3_{grp}(G, U(1))$ gives rise to a Chern-Simons 2-gerbe over $\mathrm{Bun}(G)$, the stack of G-bundles. In this sense, a Dijkgraaf-Witten theory based on (G, ω) can be regarded as a nonlinear sigma model on the background given by $\mathrm{Bun}(G)$ and the Chern-Simons 2-gerbe ω. It is important then to first formulate the correct notion of symmetry, or automorphism of such a background, since we are dealing with higher categorical objects. I plan to study in which sense an automorphism of $\mathrm{Bun}(G)$ must be compatible with the Chern-Simons 2-gerbe ω, and to show that such automorphisms indeed produce invertible module categories over the category of ω-twisted G-graded vector space $\mathrm{Vect}^{\omega}(G)$, as expected by [FSV12, FSV13, FPSV14].
As argued in [FSV12], module categories play a relevant role in general Reshetikhin-Turaev and Turaev-Viro theories. In recent work [ENO10], the **Brauer-Picard groupoid** $\underline{\mathrm{BrPic}}(\mathcal{A})$ of bimodule categories over a fusion category $\mathcal{A}$ has been introduced: this is a 2-group with objects given by invertible bimodule categories over $\mathcal{A}$ and with morphisms given by equivalent classes of invertible bimodule functors. One of the main results in [ENO10] is to provide an equivalence between $\underline{\mathrm{BrPic}}(\mathcal{A})$ and the 2-group of braided equivalences $\underline{\mathrm{EqBr}}(Z(\mathcal{A}))$ of the Drinfeld center $Z(\mathcal{A})$ of $\mathcal{A}$. I plan to provide a TQFT description of these results, motivated by the following argument, and results in [FPSV14]. Very recently it has been shown [DSPS13] that fusion categories provide fully dualizable objects in the 3-category of finite tensor categories TC: by the cobordism hypothesis, to any fusion category $\mathcal{A}$ there corresponds a 3-dimensional (framed) TQFT with target TC, which assigns the Drinfeld center $Z(\mathcal{A})$ to the framed circle. Moreover, in [Lu09] it is shown that any choice of a $\mathcal{A}$-bimodule $\mathcal{M}$ will produce a TQFT F defined on a cobordism category *with* defects. Consider then a cylinder with an embedded defect circle "in the middle": the theory F must then assign to such a cobordism a functor between $Z(\mathcal{A})$ and itself. If $\mathcal{M}$ is invertible, by the general properties of TQFTs we expect such a functor to be an equivalence and, moreover, to be braided. Motivated by this, I conjecture that this assignment of an invertible bimodule category to a braided equivalence *coincides* with the equivalence constructed in [ENO10], providing a novel topological interpretation of such an algebraic result, and a sign of a fruitful interaction among different fields of mathematics.
Another project I am currently involved in, with Domenico Fiorenza[1], concerns the study of TQFTs with boundary conditions and the description of anomalies.[2]. It is well known that a 3d TQFT gives rise to a 2d modular functor, which can be seen as a representation of the groupoid with objects given by oriented 2-manifolds without boundaries, and morphisms given by isotopy classes of orientation preserving homeomorphisms. Moreover, for any surface Σ, a modular funtor provides a representation of $MCG(\Sigma)$, the mapping class group of Σ. Theories with anomalies will give rise only to *projective* representation of the mapping class group of surfaces. This is usually encoded in the fact that a 3d TQFT with anomalies is a sort of *lax functor* from a cobordism category of extended surfaces. Generalizing ideas from [FT12], we are working on a construction of 2d (projective) modular functors arising from invertible extended 4d TQFTs with boundaries, using the higher categorical techniques developed in [Lu09]. In this picture, the anomalous 3d TQFT is seen as a boundary theory for a 4d topological field theory, and its behaviour is governed precisely by the boundary conditions. Using completely different techniques, these circle of ideas has been exploited in [Wal91], and we are working towards a detailed comparison of these results.
The circle of ideas surrounding the project above can be seen as a manifestation of a *holographic*

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[2]Some of our observations have been studied and expanded in [Nui13].

principle, which can be expressed as the fact that a n-dimensional field theory presenting anomalies can be understood as a boundary theory for a well behaved $n+1$-dimensional field theory. We expect the ideas and techniques developed in this project to have concrete applications, for instance to the study of boundary conditions in 4-dimensional quantum field theories, as those studied in [KW07], where the category of boundary conditions, or *D-branes*, is related to a category arising in the Langland program.

The activity surrounding the study of topological quantum field theories in various dimensions, and in particular boundary conditions and defects, opens the possibility for a fruitful interaction between deep mathematical results in geometry, topology, and algebra, at the same time providing applications for simple, yet interesting models in mathematical physics.

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