

HERIOT-WATT UNIVERSITY

K-Theory, D-Branes and Ramond-Ramond Fields

Alessandro Valentino

SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY IN MATHEMATICS
ON COMPLETION OF RESEARCH IN THE
DEPARTMENT OF MATHEMATICS,
SCHOOL OF MATHEMATICAL AND COMPUTING SCIENCES.

This copy of the thesis has been supplied on the condition that anyone who consults it is understood to recognise that the copyright rests with the author and that no quotation from the thesis and no information derived from it may be published without the written consent of the author or the University (as may be appropriate).

Abstract

This thesis is dedicated to the study of K-theoretical properties of D-branes and Ramond-Ramond fields.

We construct abelian groups which define a homology theory on the category of CW-complexes, and prove that this homology theory is equivalent to the bordism representation of KO-homology, the dual theory to KO-theory. We construct an isomorphism between our geometric representation and the analytic representation of KO-homology, which induces a natural equivalence of homology functors. We apply this framework to describe mathematical properties of D-branes in type I String theory.

We investigate the gauge theory of Ramond-Ramond fields arising from type II String theory defined on global orbifolds. We use the machinery of Bredon cohomology and the equivariant Chern character to construct abelian groups which generalize the properties of differential K-theory defined by Hopkins and Singer to the equivariant setting, and can be considered as a differential extension of equivariant K-theory for finite groups. We show that the Dirac quantization condition for Ramond-Ramond fieldstrengths on a good orbifold is dictated by equivariant K-theory and the equivariant Chern character, and study the group of flat Ramond-Ramond fields in the particular case of linear orbifolds in terms of our orbifold differential K-theory.

Acknowledgements

I want to thank my supervisor Prof. Richard Szabo for the support and patience during these years, for the many discussions we had, and for allowing me to always express and develop my own point of view in any aspect of my research.

I want to thank my colleague and friend Rui Reis for the uncountable discussions on maths and science in general, which made me better appreciate many aspects of these subjects.

A particular thanks goes to my examiners Jacek Brodzki, José Figueroa-O’Farrill, and Des Johnston, whose questions and comments helped improve the quality of the present work.

I want to thank U.Bunke, J.Figueroa-O’Farrill, D.Freed, J.Greenlees, J.Howie, A.Konechny, W.Lück, T.Schick, P.Turner, and S.Willerton for helpful suggestions and correspondence.

I want to thank Dr. Mark Lawson for the discussions on maths, physics, and British culture.

I want to thank Michele Cirafici and Mauro Riccardi for the many conversations we had in front of a pint of ale.

I want to thank Fedele Lizzi for the support during difficult moments, and Patrizia Vitale for the long time interest in my scientific developments.

I want to thank Giorgos, Henry, and Yorgos for being such great mates, and for the many hours of fun we had.

A thank you goes to my officemates Emma, Kenny, Sally, and Singyee, for being the best officemates I could have ever hoped for.

I want to thank Christine, Claire and Pat for helping with bureaucratic matters of any sort.

A special thanks goes to my parents, for letting me always pick any decision by my own, which highly contributed to the person I am now.

Last, but in no way least, I want to thank Antonella. Unfortunately, words are useless to describe my deep gratitude, and her relevance to the very existence of this work. They say that “behind every great man there is always a great woman”: my case trivially suggests that a feminine “version” of the previous statement could not possibly hold.

The present work of thesis is based on the following articles

- R.J.Szabo and A.Valentino, “Ramond-Ramond Fields, Fractional Branes and Orbifold Differential K-theory”, arXiv:0710.2773
- R.M.G.Reis, R.J.Szabo, and A.Valentino, “KO-Homology and Type I String Theory”, arXiv:hep-th/0610177

Contents

Introduction	1
1 Generalities on String Theory	8
1.1 The Bosonic String	8
1.2 The Supersymmetric String	10
1.3 Quantum aspects	12
1.4 Background fields and low energy limit	15
1.5 Some comments on the B-field	17
2 D-branes and Ramond-Ramond fields	19
2.1 D-branes as boundary conditions	19
2.2 The spectrum of open strings on Dp-branes	21
2.3 Chan-Paton factors and Adjoint bundles	22
2.4 D-branes and Supersymmetry	25
2.5 D-branes and Ramond-Ramond charges	27
2.5.1 Generalized electromagnetism and sources	27
2.5.2 Ramond-Ramond charges and anomalies	30
2.6 D-brane decay and Sen's conjectures	33
3 K-Theory, an introduction	36
3.1 The group $K^0(X)$	36
3.2 Relative K-theory and higher K-groups	40
3.3 Multiplicative structures on K-theory	42
3.4 K-theory and classifying spaces	45
3.4.1 Examples: K-theory of spheres and tori	47
3.5 The Atiyah-Bott-Shapiro isomorphism	49
3.6 K-theory and Spin^c manifolds	54
3.6.1 K-orientation and Thom isomorphism	55

3.6.2	Chern Character and Gysin homomorphism	58
3.7	K-theory and type IIA/B D-branes	61
3.8	KO-theory and Type I D-branes: torsion effects	65
4	KO-homology and Type I D-branes	67
4.1	Dual theories and spectral KO-homology	68
4.2	KKO-theory and analytic KO-homology	69
4.2.1	Real C*-algebras	70
4.2.2	Hilbert Modules	74
4.2.3	Kasparov's formalism for KKO-theory	76
4.2.4	Analytic KO-homology	78
4.3	Geometric KO-homology	81
4.3.1	Homological properties of KO_{\sharp}^t	85
4.3.2	Products and Poincaré Duality	88
4.4	K-homology and Index Theorems	89
4.5	The equivalence between KO_{\sharp}^t and KO_{\sharp}^a	92
4.5.1	The natural transformation μ^a	93
4.5.2	The analytic index map ind_n^a	97
4.5.3	The topological index map ind_n^t	99
4.5.4	The Isomorphism Theorem	102
4.6	The Real Chern Character	103
4.7	Cohomological Index formulas	105
4.8	D-branes and K-homology	107
4.9	The group $KO_{\sharp}^t(\text{pt})$ and torsion branes	109
5	Abelian Gauge Theories and Differential Cohomology	115
5.1	An example: the electromagnetic case	115
5.2	Differential Cohomology	118
5.2.1	Cheeger-Simons characters	118
5.2.2	Deligne cohomology	125
5.3	Ramond-Ramond fields and charge quantization	132
5.4	Generalized differential cohomology	135
6	Ramond-Ramond fields and Orbifold differential K-theory	142
6.1	G-CW complexes and equivariant cohomology theories	143
6.2	The equivariant Chern character	148
6.2.1	Bredon cohomology	149

6.2.2	Chern character in equivariant K-theory	154
6.3	String theory on orbifolds	156
6.3.1	D-branes and equivariant K-cycles	158
6.3.2	Delocalization and Ramond-Ramond fields	162
6.3.3	Delocalization of the equivariant Chern character	165
6.3.4	Ramond-Ramond couplings with D-branes	166
6.4	An equivariant Riemann-Roch formula	169
6.5	Orbifold differential K-theory and flux quantization	173
6.5.1	Orbifold differential K-groups	173
6.5.2	Flux quantization of orbifold Ramond-Ramond fields	178
6.6	K-homology and flat fields in Type II String theory	183
Appendix A	186
Appendix B	188
Appendix C	195
Appendix D	199
Bibliography	204

Introduction

“The most powerful method of advance that can be suggested at present is to employ all the resource of pure mathematics in attempts to perfect and generalise the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities”

P.A.M. Dirac, 1931

Since the dawn of science, the interaction between physics and mathematics has always been an interesting and debated one, given the apparent difference between these two disciplines. Indeed, on one hand, physics deals with natural phenomena as they happen in an “objective reality”, external to the observer, and the aim of physics is then to understand and formulate the laws they obey. On the other hand, mathematics appears to deal with a reality which is internal to the human being, populated by objects which need only obey the laws of logic and consistency. Furthermore, their methodology seems rather different: physics proceeds by experiments and particular cases, while mathematics proceeds by deduction and chains of logical statements. As it always happens when different disciplines come in contact, the interaction of these two human activities is bound to enlarge our knowledge of the reality we live in, and of the nature of the human being himself. In this respect, the 20th century has been of a crucial importance. In particular, most part of it has been dominated by the influence of the advances in modern mathematics, such as differential geometry, functional analysis, and algebra in the formulation and understanding of fundamental physical theories, such as General Relativity, Quantum Mechanics, and Quantum Field Theory. This influence was so prominent and surprising that Eugene Wigner was lead to celebrate it in the now classic paper “The Unreasonable Effectiveness of Mathematics in the Natural Sciences” [93]. This fruitful interaction has become even stronger in the late part of the last century. Interestingly, though, we have somehow witnessed the shift of influence from mathematics to physics, in such a way that we are lead to wonder about the “unreasonable effectiveness of physics on mathematics”.

The description of Jones' polynomials via quantum field theoretical techniques, which earned Edward Witten a Fields Medal in 1990, is only the most evident result of the above interaction. This is not something new: after all, differential calculus was born mainly out of the necessity to solve problems in mechanics. However, mathematics in the 20th century, in particular in its second half, has evolved independently of any application, following more abstract and formal principles. This makes the possibility of its relation with physics even more exciting and surprising.

Despite the difficulties and controversies surrounding it, it is undeniable that String theory has played a major role in these developments, generating an incredible number of interesting problems in different fields of mathematics, in particular algebra, geometry and topology. The rich structure of String theory has lead mathematical physicists and pure mathematicians to work on the same problems, albeit with different attitudes and motivations, allowing both communities to better appreciate the extent of their own subject of study.

The present work of thesis belongs to the above trend, finding its place at the interface of geometry, topology, and String theory. Loosely speaking, the "middle point" is represented by the circle of ideas surrounding K-theory, a generalized cohomology theory developed by Atiyah, Hirzebruch, and Grothendieck, among others, which can be defined in terms of complex vector bundles on topological spaces. The relevance of K-theory in String theory rests on the fact that D-branes, extended objects present in the theory, have charges which are not classified by the homology cycle of their worldvolumes, as expected, but rather by the K-theory of their normal bundles in the spacetime manifold. In the first part of this thesis, we will exploit the fact that a more natural description of D-branes can be given in terms of K-homology, which is the generalized homology theory dual to K-theory. The interesting mathematical aspect concerning K-homology is the fact that it can very naturally be constructed by using both a geometric and an analytic representation, respectively in terms of Spin^c -manifolds and vector bundles, and of Fredholm modules. In particular, Baum and Douglas [14] showed that an isomorphism can be constructed at the level of the representative cycles, and that such an isomorphism induces a natural equivalence between the geometric and analytic K-homology functors. The authors then showed that the existence of such an isomorphism is equivalent to the Index Theorem of Atiyah and Singer for the canonical Dirac operator, and conjectured that this is the case for any flavour of K-theory.

In this thesis we will support this conjecture, by constructing a natural equivalence

between geometrical and analytical KO-homology, the dual theory to the K-theory of *real* vector bundles, elucidating its relation with a suitable Index Theorem.

The difficult aspect of the above analysis lies in the fact that the index homomorphism takes values in abelian groups with torsion, which does not allow a local description in terms of characteristic classes. Since KO-homology describes D-branes in type I superstring theory, we will be able to construct D-branes with torsion charges which are not associated to any spacetime form field.

Indeed, D-branes can also be realised as currents for Ramond-Ramond fields. These “dual” objects are gauge fields locally described by differential forms of different degrees, depending on the type of String theory considered. Their fieldstrengths satisfy generalized Maxwell equations, and they interact with D-branes via the usual integral coupling. Since D-brane charges are classified by K-theory/K-homology, it is certainly expected that K-theory plays some role in the Ramond-Ramond gauge theory behaviour. This is indeed the case: the suitable mathematical formalism to describe these fields in nontrivial backgrounds is known as differential K-theory, which is an example of generalized differential cohomology theories recently developed in [54]. In particular, the main ingredient is the Chern character homomorphism from K-theory to the even part of the deRham cohomology ring of the spacetime manifolds, which roughly realizes the Ramond-Ramond current generated by D-branes.

In the second part of this thesis, we will generalize the arguments leading to the above conclusions to the case of type II String theory on orbifolds. An orbifold can be loosely described as a singular object which is locally isomorphic to the quotient of an Euclidean space by a finite subgroup of the group of linear transformations. Despite the presence of the singularities, String theory behaves well on such objects: in particular, D-branes can be introduced, and their charges are classified by equivariant (or orbifold) K-theory, as proposed by [94]. We will analyse the Ramond-Ramond gauge theory arising from the closed type II String theory on global orbifolds, and demonstrate the role played by equivariant K-theory. In particular, we will construct some abelian groups satisfying all the expected properties to be considered a generalization of differential K-theory for global orbifolds. This will require the introduction of an equivariant cohomology theory not very popular in the physical literature, developed by Bredon in [22], which nevertheless, as we will argue, captures all the relevant physical properties of Ramond-Ramond fields on orbifolds.

Plan of the Work

We will now give a brief description of the contents of this thesis, trying to highlight the main new results in mathematics and String theory.

In Chapter 1 we introduce some generalities about String theory, focusing on the structure of its quantum spectrum and on the low energy limit. We have tried throughout to express all the relevant notions in a precise mathematical framework whenever possible, and we have avoided details of the constructions involved, referring the reader to the extensive literature on the subject. In this way, the basic concepts relevant to this work of thesis should be accessible to a mathematically minded audience acquainted with the basics of field theory and quantum mechanics.

In Chapter 2 we introduce D-branes as boundary conditions for open String theory, emphasising the geometric and topological properties of such objects. Indeed, we will focus mainly on properties of D-branes that are expected in any topological nontrivial background, such as the behaviour of the Chan-Paton vector bundle. The main properties of supersymmetric D-branes are briefly reviewed via the analysis of the spinor bundle on the D-brane worldvolume, which is a recurrent ingredient throughout this thesis. We will introduce the gauge theory of Ramond-Ramond fields, and discuss Ramond-Ramond charges and the anomalous couplings with D-branes. Also in this case, we will avoid the rather lengthy computations regarding the inflow mechanism, since these techniques will not play any relevant role in this work of thesis. Finally, we state Sen's conjectures on D-brane decay and mention Witten proposal on D-brane charge classification.

Chapter 3 consists of a quick introduction to topological K-theory. For a given CW-complex X , we define the group $K^0(X)$ as the Grothendieck group associated to the monoid of vector bundles over X , and define the higher K-groups via suspension. We describe the multiplicative structure possessed by K-theory, and discuss the Atiyah-Bott-Shapiro isomorphism in some details. We then restrict ourselves to the category of Spin^c -manifolds, discussing the concept of K-orientation, Thom isomorphisms, and the Chern character. Finally, we illustrate how the K-theoretical machinery is used in the classification of D-brane charges in type II and type I String theory. This chapter is of great importance, since the concepts therein will be tacitly assumed throughout the rest of the thesis.

Chapter 4 consists mostly of original material regarding KO-homology. After a brief introduction on dual theories and spectral KO-homology, we will recall the basic notions about the theory of *real* C^* -algebras, emphasising the differences with the

analogous results in the theory of ordinary complex C^* -algebras. We will then introduce Kasparov's formalism for KKO-theory, which is based on the notions of Hilbert modules and generalized Fredholm modules, and define analytic KO-homology for a topological space X via the C^* -algebra of *real* functions on X . We proceed to construct geometric KO-homology in terms of Spin manifolds and real vector bundles. By comparison with the bordism description of spectral K-homology developed in [58], we prove that our construction is a generalized homology theory dual to KO-theory, and discuss the various relevant homological properties. We then introduce the main mathematical result of this chapter, given by the construction of an isomorphism μ between the geometric and analytic representation of KO-homology which induces a natural equivalence between the geometric and analytic KO-homology functors. To this aim we introduce some index homomorphism on geometric and analytic KO-homology, and prove that the Index theorem for a suitable Dirac operator implies that μ is indeed an isomorphism. We point out that the proof of the above theorem appears also in [15], albeit it is completely different from the one we present here, which is more suitable for the applications we present later. We also construct a homological real Chern character, and use it to give an alternative derivation of cohomological index formulas for the canonical Cl_n -linear Atiyah-Singer operator. From the physical point of view, we introduce the concept of *wrapping charge* of a *wrapped D-brane*, showing that in type I String theory it is a genuinely different notion from that of an ordinary D-brane. Finally, we construct nontrivial generators for the KO-homology of a point, and interpret these in terms of wrapped D-branes.

In Chapter 5 we give a detailed account on (generalized) differential cohomology theories. We motivate this mathematical formalism by discussing the properties of ordinary electromagnetism with Dirac quantization of charges in topologically non-trivial backgrounds. We give a rather extensive treatment of both Cheeger-Simons groups and Deligne, in order to build a solid intuition for these mathematical objects. We then discuss the Moore and Witten argument regarding the charge quantization of Ramond-Ramond fields, as proposed in [74]. We conclude the chapter by explaining the construction of differential K-theory of Hopkins and Singer, which will be generalized later in the thesis.

In Chapter 6 we present new physical and mathematical results regarding String theory on global orbifolds. We first give a brief introduction to equivariant cohomology theories on the category of G-CW complexes, and consider equivariant K-theory as an example. We then proceed to define Bredon cohomology in terms of natural transformations between functors on the orbit category of a finite group. This is the

main ingredient used to construct the equivariant Chern character defined in [67], which has the unique property of inducing an isomorphism on rational equivariant K-theory. One of the main new physical results of this chapter is to demonstrate that the Dirac quantization of Ramond-Ramond fields on orbifolds is dictated by the above Chern character, which, after a process of *delocalization*, can be used to describe the couplings of Ramond-Ramond field on global orbifold with *fractional* D-branes, which we will describe in terms of equivariant K-homology. We check this statement on linear orbifolds, which are the usual cases studied in the physics literature. Our approach has the main advantage of being applicable to the case of nonabelian orbifolds and for quotients of general Spin manifolds by finite groups. From the mathematical point of view, we construct abelian groups which have all the desired properties for a generalization of differential K-theory to global orbifolds. This is mathematically needed, since the general results of Hopkins and Singer in [54] hold only for generalized cohomology theories on the category of manifolds. Far from reaching the generality of [54], we find this an important step towards the general construction of differential extensions of equivariant generalized cohomology theories. We will use our equivariant (or orbifold) differential cohomology theory to describe Ramond-Ramond fields on orbifolds, and study in particular flat Ramond-Ramond potentials.

We conclude the work with Appendices which aim to settle the notations for some of the standard notions used throughout the thesis.

“Freedom is the freedom to say that two plus two make four. If that is granted, all else follows.”

G.Orwell, **1984**

Chapter 1

Generalities on String Theory

In this chapter we will collect some basic results in String theory that will be used as the starting point for the rest of the thesis. We will review some well known aspects of the perturbative formulation of bosonic and supersymmetric string theories. We direct the reader to [49],[79],[33] for more information about these constructions.

1.1 The Bosonic String

The action functional for a string propagating in spacetime is a direct generalization of the functional describing the motion of relativistic point particle. In the case of a point particle, the action functional for a given curve $\gamma : [0, 1] \rightarrow M$, where M is a d -dimensional Lorentzian manifold, is given by

$$S[\gamma] := \int_{\gamma} \mu_{g|_{\gamma}} \quad (1.1.1)$$

where $\mu_{g|_{\gamma}}$ is the invariant volume form for the spacetime metric restricted to the curve γ . The action S computes the length of the curve γ , the worldline of the point particle, and its stationary points are the geodesics for the metric g on M .

It is natural to generalize this action for an extend p -dimensional object propagating in M as the “volume” of the surface swept by the object while propagating. In other words, given $f : \Sigma \rightarrow M$, where Σ is a p -dimensional manifold, called the *worldsheet*, and f is a smooth immersion, we have that

$$S[f] := \int_{\Sigma} \mu_{f^*g} \quad (1.1.2)$$

The case of a free string propagating in M is given by $p=2$, and by $\Sigma \simeq S^1 \times \mathbb{R}$ for closed string, and $\Sigma \simeq [0, 1] \times \mathbb{R}$ for open strings.

The functional S is called the *Nambu-Goto action* and in a local system of coordinates $\{\sigma^i\}$ over Σ , the functional S can be represented as

$$S[f] = \int_{\Sigma} d^p \sigma \sqrt{-\det(g_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^i} \frac{\partial x^\nu}{\partial \sigma^j})} \quad (1.1.3)$$

where $x^\mu(\sigma)$ are local representatives for the function f .

Unfortunately, this action is non-polynomial in the x^μ and its derivatives, making its quantization difficult to define unambiguously, even in flat spacetime.

For this reason, it is generally preferred to use the classically equivalent *Polyakov action* given by

$$S[f, \gamma] := k \int_{\Sigma} \mu_{\gamma} \langle \gamma, f^* g \rangle \quad (1.1.4)$$

In the above expression, γ is a Lorentzian metric on Σ , \langle, \rangle denotes $\|df\|^2$, defined by considering df as an element in $\Omega^1(\Sigma; f^*TM)$, and using both γ on $T^*\Sigma$ and g on TM , and k is called the *tension*. Notice that now the intrinsic metric γ is a dynamical variable, while the metric tensor g is considered as a “background”, i.e. it is a nondynamical quantity. When also γ is held fixed, the action S describes a *nonlinear sigma model*, and its stationary points when $p=2$ are called *harmonic maps*. In the local system of coordinates as above the Polyakov action can be represented as

$$S[f, \gamma] = k \int_{\Sigma} d^p \sigma \sqrt{-\det(\gamma)} \gamma^{ij} (g_{\mu\nu} \frac{\partial x^\mu}{\partial \sigma^i} \frac{\partial x^\nu}{\partial \sigma^j}) \quad (1.1.5)$$

By construction, the Polyakov action is invariant under $Diff^+(\Sigma)$, the group of orientation preserving diffeomorphisms, and under $ISO(g)$, the group of isometries of g . Moreover, only in the peculiar case $p=2$ the functional S is invariant under $C_+^\infty(\Sigma)$, the group of smooth positive functions on Σ , acting as *Weyl rescaling* of the metric γ , i.e. transformations of the type $\gamma \rightarrow \rho \cdot \gamma$.

Hence the full group of symmetries of the string action is given by $Diff^+(\Sigma) \times C_+^\infty(\Sigma) \times ISO(g)$, where \times means semidirect product.

The invariance under diffeomorphisms and Weyl scalings is crucial in giving the Polyakov action a more tractable form, allowing to use canonical quantization techniques, and as this is possible only for $p=2$, this is seen as a reason to rule out higher dimensional extended objects other than strings.

Indeed, a 2-dimensional manifold is always locally *conformally flat*, i.e. there always exist local coordinates (u, v) in which the metric γ can be expressed as $\rho \cdot \eta_{ab}$, where η_{ab} is the usual Minkowski metric in 2 dimensions. Diffeomorphisms and Weyl invariance then allow to “pick a gauge”, called the *conformal gauge*, in which the functional S is the action functional for a vibrating string in a curved spacetime.

When M is the d -dimensional Minkowski spacetime, the equations of motion of the string in the conformal gauge can be completely solved, and its canonical quantization carried on thanks to an additional symmetry enjoyed by the string functional.

Indeed, conformal transformations of the Minkowski metric in two dimensions do not spoil the (local) conformal gauge, implying that String theory (at any fixed γ) is a two dimensional conformal field theory, hence completely integrable by symmetry considerations alone, as the conformal algebra in two dimensions is infinite dimensional.

Conformal invariance is so important that it is required to hold also at the quantum level, where an anomaly could possibly spoil it: the cancellation of such a conformal anomaly, indeed, fixes the spacetime dimensionality to $d=26$, called the *critical dimension*.

Remark During this thesis, we will mainly consider both the worldsheet and spacetime manifolds as being equipped with a Riemannian as opposed to a Lorentzian metric tensor. Even if this is not per se physically realistic, in most of the cases one can perform a *Wick rotation* on the spacetime manifold and obtain the relativistic description we have introduced so far.

Moreover, we will also consider different topologies for the worldsheet, and in particular we will regard Σ as a Riemann surface of genus g : this is essentially due to how *interactions* are introduced in String theory at the quantum level. This perturbative formulation of String theory also forces the use of Riemannian worldsheets, as opposed to Lorentzian, since a compact manifold admits a Lorentzian metric if and only if the Euler number vanishes. For example, the transition amplitude for the propagation of a quantum string will be given by

$$A \sim \sum_{\text{topologies of } \Sigma} \int_{\text{Met}(\Sigma)} \mathcal{D}\gamma \int_{\text{Map}(\Sigma, M)} \mathcal{D}f e^{-S[f, \gamma]}$$

where $\mathcal{D}\gamma$ and $\mathcal{D}f$ are “path integral measures” over a space of metrics and maps, respectively. More precisely, the above expression should be “gauge fixed”: indeed, the path integral over the space of metrics reduces to an integral over a finite dimensional moduli space.

1.2 The Supersymmetric String

Despite the rich structure of its symmetries, the bosonic string has some fundamental flaws. The most relevant ones, apart from the high dimensionality of the spacetimes

allowed, are the presence of a tachyonic state, which is a strong signal towards instability of the quantum theory, and the absence of fermions, which is not a feature of an inconsistent theory, but nevertheless fermions are required for physical reasons.

A way out of these two problems is the so called *Neveu-Schwarz-Ramond (NSR) formulation* of String theory, which consists in introducing additional fermionic degree on the worldsheet.

More precisely, the Polyakov action can be at first generalized as

$$S[f, \gamma, \psi] := k \int_{\Sigma} \mu_{\gamma} \{ \langle \gamma, f^* g \rangle + \bar{\psi} \mathcal{D} \psi \} \quad (1.2.1)$$

where ψ is a section of $S \otimes f^* \text{TM}$, with S the spin bundle over Σ for a given spin structure, and \mathcal{D} is the Dirac operator associated to γ , coupled to $f^* \text{TM}$.

Notice first that Σ is a spin manifold, being 2-dimensional, and it admits, at any genus g , 2^{2g} inequivalent spin structures which can be distinguished by a \pm sign along the homology cycles of Σ . Moreover in even dimensions the spin bundle S decomposes according to the chirality operator as $S^+ \oplus S^-$.

One also requires ψ to be a Majorana spinor field to ensure that the full action is real. Majorana spinors exist on worldsheets with Lorentzian signature, but not on those with Euclidean signature. Anyway, in the Euclidean case one can use an ordinary chiral spinor field ψ_+ , and choose ψ_- to be its complex conjugate, in order to preserve the degrees of freedom.

More concretely, ψ_+ will be a section of $K^{1/2}$, the square root of the canonical bundle on Σ , and ψ_- a section of $\bar{K}^{1/2}$ for the same spin structure.

In the free (closed) superstring case, the topology of Σ admits a single homology cycle, and hence there are two spin structures, which are conventionally referred to as *Ramond(R)* and *Neveu-Schwarz(NS)*, and can be characterized by

$$\begin{aligned} \psi_{\pm}(\tau, \sigma + 2\pi) &= +\psi_{\pm}(\tau, \sigma) & \text{R: periodic conditions} \\ \psi_{\pm}(\tau, \sigma + 2\pi) &= -\psi_{\pm}(\tau, \sigma) & \text{NS: anti-periodic conditions} \end{aligned}$$

On a flat spacetime, the free superstring can be quantized using canonical quantization: the main difference with the bosonic string is given by the appearance of new sectors, NS and R, given by the boundary conditions for the spinor fields ψ .

In particular, the states in the Fock space \mathcal{F}^{NS} for Neveu-Schwarz degrees of freedom can be shown to be spacetime bosons, while the states in the Fock space \mathcal{F}^{R} for Ramond degrees of freedom can be shown to be spacetime fermions.

Unfortunately, both the above spaces contain *negative norm states* that are not eliminated by any kind of symmetry, in contrast to what happens in the bosonic string

case thanks to the Virasoro algebra constraints. To ensure this, one modifies the action (1.2.1) by introducing a *gravitino*, a spin 3/2 field, and an interaction term in order for the action to be invariant under *local worldsheet supersymmetry*. In analogy with the bosonic case, the modified superstring action will define a *superconformal* field theory: the preservation of the super conformal symmetry at the quantum level requires the spacetime dimensionality d to be 10.

Even with these modifications, the theory is still inconsistent, as there is still a tachyonic state in the NS sector, with no supersymmetric partner state, making it impossible to have spacetime supersymmetry.

To overcome this final problem, Gliozzi-Scherk and Olive proposed a procedure for a truncation of the RNS String theory that produces a spectrum with spacetime supersymmetry. This truncation is called the *GSO projection*, which can be thought as a projection on the space of invariant states for the operator $(-1)^F$, which assigns to each state the number of fermions present modulo 2.

This operator can be carefully defined in both the Ramond and Neveu-Schwarz sector, obtaining that the free open GSO projected string is N=1 supersymmetric, while the free closed GSO projected string is N=2 supersymmetric.

Remark In the functional integration formulation, the GSO projection corresponds to a weighted sum over the spin structures of the worldsheet, with the weight chosen in such a way that the resulting amplitudes are invariant under the action of the modular group of Σ .

1.3 Quantum aspects

In this section we will briefly recall the (massless) content of the supersymmetric string space of states, in flat 10-dimensional Minkowski spacetime [49, 33].

Denote by \mathcal{F}_k and $\tilde{\mathcal{F}}_k$ the Fock spaces for the bosonic degrees of freedom at momentum k for *left* and *right* movers, obtained upon a holomorphic decomposition of the fields in local complex worldsheet coordinates.

Then the full RNS Fock space for open and closed strings is given by

$$\begin{aligned} \mathcal{F}_{\text{open}} &:= \bigoplus_k \mathcal{F}_k^{RNS} \\ \mathcal{F}_{\text{closed}} &:= \bigoplus_k \mathcal{F}_k^{RNS} \otimes \tilde{\mathcal{F}}_k^{RNS} \quad k \in \mathbb{R}^{10} \end{aligned} \tag{1.3.1}$$

where

$$\mathcal{F}_k^{RNS} := \mathcal{F}_k \otimes (\mathcal{F}^R \oplus \mathcal{F}^{NS})$$

and the same for $\tilde{\mathcal{F}}_k^{RNS}$.

After imposing the physical constraints required by the superconformal symmetry, we will obtain two positive-definite Hilbert spaces for open and closed strings, denoted by $\mathcal{F}_{open}^{phys}$ and $\mathcal{F}_{closed}^{phys}$, respectively.

By (1.3.1), the open string Fock space contains two sectors, Ramond and Neveu-Schwarz. The ground state in the Ramond sector satisfies a massless Dirac equation, suggesting that it is single particle state for a spacetime fermion field, while the ground state in the Neveu-Schwarz sector is a bosonic tachyon, obtained by a spacetime scalar field.

Moreover, the first excited state is a *massless vector state*, with its degree of freedom suggesting it is a one particle state for a Yang-Mills field [79, 49].

These, and an infinite tower of massive states, are contained in the left-moving sector, but by (1.3.1) this suffices to construct the Hilbert space of the open string.

For the closed string, instead, we need to consider a tensor product for left and right movers Hilbert spaces: we will have four sectors, characterized by the spin structure choice for left and right-moving degrees of freedom.

The ground state in the NS-NS sector is a tachyon, as for the open string, while the massless states contains, in the same sense as before, a *graviton*, associated to a symmetric spacetime tensor of type (2,0), the *B-field*, coming from a spacetime two form, and a *dilaton*, a spacetime scalar field. In particular, the graviton state satisfies linearized Einstein equations, hence its name.

The R-NS (and NS-R) ground state is a massless state, which is reducible into a spinor state coming from a spinor field λ , called the *dilatino*, and a *gravitino* state, coming from a spinor-vector field. These are the superpartners of the dilaton and graviton, respectively.

Finally, the ground state in the R-R sector is massless, and can be reduced in states that are one particle states for spacetime differential forms of degree $0, \dots, 10$, called *Ramond-Ramond fields*. In a certain sense, this is the “most important” sector for the content of this thesis: indeed, most of the next chapters will be devoted to explore the rich mathematical properties of these objects, and the interaction with their “sources”, called *D-branes*.

Of course, all the sectors described above contain also a (infinite) number of massive excited states.

As was pointed out in the previous section, the theory is still inconsistent, due to the presence of the tachyon state in the NS sector, and does not present spacetime supersymmetry, required by the consistency of the coupling of the *massless* gravitino appearing in the spectrum.

This enforces the use of the GSO projection, both for the open and closed string. We are left then with three different spacetime supersymmetric string theories, called *type IIA*, *type IIB*, and *type I*, which uses unoriented worldsheets. Moreover, the “I” and “II” refers to the fact that the theory is spacetime supersymmetric with one and two supercharge generators, respectively.

Type IIA and type IIB are superstring theories constructed from GSO projecting the Fock space of closed strings: in this case, the GSO projection requires a choice of chirality for the R ground state in the left and right moving sector, but by spacetime parity symmetry the theories obtained by the same choice of chirality coincide. Indeed, under exchange of left and right movers, type IIA is a non-chiral theory, while type IIB is chiral. Moreover, it’s a theory of oriented closed strings.

The GSO projection modifies, among other things, the content of the massless R-R sector: in type IIA there will be one particle states coming from differential form of *odd* degree, while in type IIB there are states associated to differential forms of *even* degree. Moreover, in type IIB the 5-form fieldstrength is required to be selfdual.

Type I is a superstring theory constructed from open *and* closed strings, and will be discussed further on, when we will introduce additional degrees of freedom for open strings, called “Chan-Paton” factors, which are essential in obtaining massless Yang-Mills states, i.e the *standard model gauge interactions*.

To end, we should mention the Heterotic String theory, which is a hybrid theory obtained by combining right movers of type II String theory with bosonic left movers. We will not discuss this theory in this thesis.

Remark As pointed out at the beginning of this section, the discussion above refers to the spectrum of a string propagating in 10-dimensional Minkowski spacetime. This is one of the very few cases in which the quantization of the theory can be done “accurately”, even if in particular gauges and with a particular choices of spacetime coordinates, and the spectrum can be found explicitly¹. In particular, the associated *classical* fields for the various particle states can be inferred thanks to the Poincaré symmetry of Minkowski spacetime. It is generally “assumed” that the field content

¹In case of interests, the choice of a gauge does not affect the quantum theory, as gauge invariance is restored at the quantum level.

of the effective theory does not change for a more generic choice of spacetime manifold: this is a point of view we will adopt in the development of the following chapters.

1.4 Background fields and low energy limit

As we mentioned in the previous section, the closed string Fock space contains states in its NS-NS sector that can be associated to a symmetric, an antisymmetric and scalar massless field. These fields can arise as a modification of the Polyakov action, describing a string propagating in *background fields*. By background field we mean a (spacetime) field which is not affected by the presence of the propagating string, and that does not represent a dynamical variable; moreover, background fields are not integrated over in the path integral.

The bosonic part of the modified action is given by

$$\frac{1}{8\pi l_s^2} \left\{ \int_{\Sigma} \mu_{\gamma} \langle \gamma, f^* g \rangle + \int_{\Sigma} f^* B + \int_{\Sigma} \mu_{\gamma} R_{\gamma} f^* \Phi \right\} \quad (1.4.1)$$

where g is the metric tensor on the spacetime M , B is locally an element in $\Omega^2(M; \mathbb{R})$, $\Phi \in C^{\infty}(M; \mathbb{R})$, and R_{γ} is the Ricci scalar for the worldsheet metric. Moreover we have expressed the tension k in term of the worldsheet length scale l_s .

Notice that the first term in the modified string action “coincides” with the definition (1.1.4), in the sense that in the formulation of the string dynamics as a nonlinear sigma model we have tacitly assumed the presence of a gravitational background.

The second term contains the B-field, and for closed strings, i.e in the case $\partial\Sigma = \emptyset$, it is invariant under the gauge transformation $B \rightarrow B + d\lambda$, with $\lambda \in \Omega^1(M; \mathbb{R})$.

The last term plays an important role, as for any Riemann surface Σ of genus g one has

$$\chi(\Sigma) = \int_{\Sigma} \mu_{\gamma} R_{\gamma}$$

where $\chi(\Sigma) = 2(1 - g)$ is the *Euler number* of Σ , a topological invariant.

This makes the expansion in powers of l , called the *low energy expansion*, somehow complicated, as the term containing the dilaton is directly dependent on the loop order.

Moreover, the action (1.4.1) is invariant under $Diff^+(\Sigma)$ and $Diff(M)$, but not under Weyl rescaling. Indeed, enforcing these (super) symmetries constrains the choice for the above background fields, providing equations they have to obey at the lowest order in l , which are essentially given by the vanishing of β -functions governing the

Weyl scaling of the model described by (1.4.1).

These equations can be obtained as Euler-Lagrange equations for the following action [79, 49]

$$2 \int_M \mu_g e^{-2\Phi} (R_g + 4\langle D\Phi, D\Phi \rangle - \frac{1}{2} H \wedge \star H) \quad (1.4.2)$$

where D the usual covariant derivative, H is the 3-form field strength for the B-field, and \star denotes the Hodge operator on M . Moreover, the critical dimension d of M is again 10.

The important aspect about the above action is that it corresponds to (part of) the bosonic part of $N=2$ supergravity. More precisely, the above action has to be complemented with an additional set of fields coming from the low energy approximation of String theory, but that cannot be incorporated as background fields. These are essentially the field strengths for the Ramond-Ramond fields, giving a type IIA or type IIB supergravity theory. We will discuss these additional contributions in the next chapters, as they play a prominent role in the study of Ramond-Ramond fields as a (generalized) gauge theory.

We conclude this section by briefly mentioning the main differences between the perturbative analysis around general field configurations in String theory and quantum field theory. In quantum field theory, one usually starts with a set of canonical fields ϕ and a classical action $S[\phi]$ which is independent of any quantum parameter, such as \hbar . In the functional integral formalism, perturbation theory in the loop expansion is carried out by choosing a classical field configuration ϕ_0 , and expanding the fields around this configuration as $\phi = \phi_0 + \sqrt{\hbar}\varphi$. The Feynman rules are then obtained by expanding the action $S[\phi]$ in powers of $\sqrt{\hbar}$: the propagator for the field φ and the interaction vertices will then depend upon the choice of the background field ϕ_0 . As ϕ_0 is a classical solution, all the Feynman graphs usually referred to as *tadpoles*, corresponding to the linear term in $\sqrt{\hbar}$, vanish. This works also in the other way, in the sense that the vanishing of all the tadpoles graphs allows one to infer that ϕ_0 is a classical solution.

As stated, this procedure has no analogue in String theory, simply because there is no equivalent for the canonical fields ϕ : indeed, the S-matrix in String theory is not obtained by an action governing string interactions, but it is defined perturbatively through a generalization of Feynman diagrams. Hence it is not possible to check directly that a given configuration for the fields g , B and Φ (and possibly other fields) is a background configuration, as the string action does not include their equations of motion. One can then think of using the vanishing of tadpole graphs as a criterion

to decide when a set of fields defines a background solution.

It is remarkable, then, that asking for the action (1.4.1) to define a conformal field theory of a given central charge, which is the case in which one is assured that there exists a Hilbert space of states (no nonnegative and zero length vectors), forces the tadpole graphs (of certain fields) to vanish, giving us the equations of motion that the background fields have to satisfy.

1.5 Some comments on the B-field

As we have seen in the previous section, an additional term containing a two form B can be consistently added to the (bosonic) string action. We have also mentioned that the action so extended enjoys a “gauge symmetry” with respect to the field B . This gauge symmetry gives the B-field a very rich mathematical interpretation: indeed, even if the B-field “plays no active role” for the extent of this thesis, we find it nevertheless important to give some informations about its mathematical nature.

We start by noticing that the description of the B field as a two form on the spacetime M need not hold globally, as the B field is defined up to the transformation $B \rightarrow B + d\lambda$, since the action (1.4.1) is invariant under this process. A local description, then, can be obtained as follows.

Let $\{\mathcal{U}_\alpha\}$ be a open cover of M enjoying the property that \mathcal{U}_α and all its intersections are contractible sets. Such a cover is called a *good cover*, and any manifold can be equipped with one.² Moreover, denote with $\mathcal{U}_{\alpha\beta\dots\tau}$ the intersection $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \dots \cap \mathcal{U}_\tau$. As in any open set \mathcal{U}_α the form B_α is only defined up to the transformation above, we have that on any $\mathcal{U}_{\alpha\beta}$ the following equation holds

$$B_\beta - B_\alpha = d\lambda_{\alpha\beta} \quad \lambda_{\alpha\beta} \in \Omega^1(\mathcal{U}_{\alpha\beta}; \mathbb{R}) \quad (1.5.1)$$

These equations imply that on a triple overlap $\mathcal{U}_{\alpha\beta\gamma}$ one has

$$d(\lambda_{\alpha\beta} + \lambda_{\beta\gamma} + \lambda_{\gamma\alpha}) = 0 \quad (1.5.2)$$

Then by Poincaré’s lemma one has that

$$\lambda_{\alpha\beta} + \lambda_{\beta\gamma} + \lambda_{\gamma\alpha} = df_{\alpha\beta\gamma} \quad f_{\alpha\beta\gamma} \in C^\infty(M; \mathbb{R}) \quad (1.5.3)$$

for some smooth real function $f_{\alpha\beta\gamma}$.

Finally, on a quadruple intersections $\mathcal{U}_{\alpha\beta\gamma\sigma}$

$$f_{\beta\gamma\sigma} - f_{\alpha\gamma\sigma} + f_{\alpha\beta\sigma} - f_{\beta\gamma\alpha} = 2\pi \omega_{\alpha\beta\gamma\sigma} \quad (1.5.4)$$

²Just consider a cover made of geodesically complete open sets for some Riemannian metric.

for some real numbers $\omega_{\alpha\beta\gamma\sigma}$.

Notice, at this point, that the local field strengths defined as $H_\alpha := dB_\alpha$ can be “glued” together to form a globally defined three form on M . If we require H to have integral periods, then one can choose $\omega_{\alpha\beta\gamma\sigma}$ to be integer numbers. According to this description, then, a B-field configuration is defined as a triple $(B_\alpha, \lambda_{\alpha\beta}, f_{\alpha\beta\gamma})$ satisfying the above equations.

In a mathematical language, the triple above defines a *(abelian) gerbe with connection*, which is a generalization of the idea of principal bundle with connection. As in the case of a connection on a principal bundle, the field strength H is related to the cohomology of the manifold M : indeed, the class $[H]$ represented by H in $H^3(M; \mathbb{R})$ is the image of the class $[\omega_{\alpha\beta\gamma\sigma}] \in \check{H}^3(M; \mathbb{R})$ under the isomorphism between DeRham cohomology and Čech cohomology with coefficients in the sheaf of locally constant real functions.

For the rest of this thesis, we will always assume that the B-field is “turned off”, meaning that we will always work under the hypothesis that $B = 0$. We will mention, though, the major modifications that the presence of a nontrivial B-field induces.

Notice, at this point, that the above definition of a gerbe has been given in a particular choice of a good cover: this raises the question of the independence of this definition from any particular choice.

A more elegant formulation, which manifestly does not rely on any particular local choice, will be the subject of the next chapters.

Chapter 2

D-branes and Ramond-Ramond fields

In this chapter we will introduce *D-branes* and analyze the main features of open strings in the presence of such objects. We will review the generalized electromagnetic theory of Ramond-Ramond fields, and discuss their anomalous coupling with D-branes. Finally, we will discuss Sen's conjectures, and motivate the relevance of K-theory in the description of D-brane charges.

2.1 D-branes as boundary conditions

As we have mentioned in the previous chapter, the bosonic string propagating in a flat 26-dimensional Minkowski spacetime describes a two dimensional conformal field theory, in which the spacetime cartesian coordinates are fields defined over a surface of a given genus. We have also seen that String theory can be defined, apart from instabilities and other unpleasant effects, both for closed and open strings: in the latter case, the worldsheet Σ has a nonempty boundary $\partial\Sigma$. Since Σ has a boundary, we are faced with the task of specifying boundary conditions for the conformal fields defined on Σ . Moreover, one has to require that these boundary conditions preserve the conformal invariance of the worldsheet theory.

For a general conformal field theory, the classification of local boundary conditions preserving conformal invariance is very complicated to obtain; moreover, very often the boundary conditions have no geometrical interpretation, making it more difficult to understand their relation with the fundamental strings described in the previous chapter.

For this reason, we will for the moment focus on those boundary conditions that have

a clear geometrical interpretation, postponing the discussion of more general cases to the next chapters.

Consider the Polyakov action (1.1.4) in the case in which Σ is a Riemann surface of genus g with boundary $\partial\Sigma$. Recall that the dynamical variables are given by maps from the worldsheet to the spacetime M , and the worldsheet metric γ .

At this point we can consider the action restricted to a subspace of $\text{Maps}(\Sigma, M)$ specified by the condition

$$f(\partial\Sigma) \subset Q \tag{2.1.1}$$

where $Q \subset M$ is a submanifold of the target spacetime.

We have then for the moment the following geometrical working definition

Definition *A D-brane is a (physical) object whose dynamical evolution is described by a submanifold Q of the spacetime containing the worldlines of the end points of open strings.*

In the above definition we are also supposing that the manifold Q is chosen in such a way that the worldsheet field theory is still conformal, and are usually referred to as *D-submanifold*.

The “D” in D-brane refers to the fact that some of the coordinates representatives for the map f are locally subject to Dirichlet boundary conditions. More precisely, consider a set of coordinates $\{x^\mu\}$ in M in such a way that Q can locally be represented by the condition $x^\alpha = 0, \alpha = p + 1, \dots, 26$, with $p = \dim Q$, and choose coordinates $\{\sigma, \tau\}$ on Σ such that τ is the coordinate along the boundary, for $\sigma = 0, \pi$.

Then condition (2.1.1) implies that

$$x^\mu(\tau, \sigma)|_{\sigma=0} = x^\mu(\tau, \sigma)|_{\sigma=\pi} = 0, \quad \mu = p + 1, \dots, 26 \tag{2.1.2}$$

which are referred to as Dirichlet boundary conditions.

Usually, (2.1.2) are supplemented with local Neumann conditions for the fields representing the coordinates on Q . Briefly, these can be enforced by requiring that

$$df|_{(\partial\Sigma)_p^\perp} = 0, \quad p \in \partial\Sigma \tag{2.1.3}$$

where $(\partial\Sigma)_p^\perp$ denotes the space of normal vectors to the boundary in p . In other words, the end points are free to move on the submanifold Q .

It is common use to refer to a D-brane represented by a $p+1$ -dimensional manifold Q as *Dp-brane*. More precisely, in the following we will refer to Q as the *worldvolume* of the Dp-brane, suggesting a conceptual difference between a D-brane and the submanifold it “wraps”, i.e. it is represented with. This difference will play an important role in

the next chapters of this thesis.

We conclude this section by stressing out that the above boundary conditions do not exhaust all the possible conditions for a boundary conformal field theory, and they represent a particular well-behaved subset of boundary conditions. Nevertheless, their presence introduces remarkable additional features to String theory, as we will see in the next sections. Also, notice that D-branes can be formally added also to the theory of closed strings, since the condition (2.1.1) does not apply. In particular, D-branes do not interact if no open strings are present.

2.2 The spectrum of open strings on Dp-branes

As usual in String theory, open strings in the presence of a Dp-brane can be quantized only in very specific settings. Indeed, one usually considers open strings propagating in flat d -dimensional Minkowski spacetime in the presence of *hyperplanar* D-branes, i.e. D-branes whose worldvolume is specified by linear conditions in (spatial) cartesian coordinates. Then, the spectrum content of the effective theory and its properties are assumed to be the same for more general D-brane configurations.

Notice, at this point, that introducing a D-brane generally reduces the spacetime symmetries of the conformal field theory. For instance, in the flat case above, introducing a fixed $p+1$ -dimensional hyperplane in Minkowski spacetime destroys the translational symmetry, and reduces the $SO(1, d)$ action to that of $SO(1, p) \times SO(d - p - 1)$. This is analogous to the fact that introducing an electron in spacetime breaks its translational symmetry, and is not surprising. The analogy is even deeper, in view of the fact that D-branes can arise as solitonic solutions of a low energy effective theory.

Consider then a Dp-brane in d -dimensional Minkowski spacetime represented by the hyperplane $x^\mu = \bar{x}^\mu$, $\mu = p + 1, \dots, d$, and where x^0 is the time coordinate. In this situation, the Fock space of the quantized open string contains, as in the case without any D-brane, states that can be recognised as one particle states for a quantized classical field. Moreover, these classical fields are constrained to “live” on the worldvolume representing the Dp-brane, in the sense that they can be naturally interpreted as objects defined only on the brane itself.¹ Notice, however, that statements about the support of these fields are usually dependent on the quantization procedure, and different procedures can yield in principle different answers.

¹This is due essentially to the fact that they are states generated from the ground state by the action of creation operators associated to the momenta corresponding to the coordinate describing points on the brane.

In the following, we will just list the first few states arising from the ground state, referring the reader to [96] for more details.

As in the case of a bosonic open string without the presence of a D-brane, the ground state contains tachyonic states, which arise by a scalar field on the brane, and constitute a section of the normal bundle.

The next states are massless states, arising by a vector field on the D-brane, with the degrees of freedom of a gauge field, i.e. they can be described by a 1-form A , which can be shifted by an exact 1-form. This is usually interpreted as a Maxwell field defined on the worldvolume of the Dp -brane. Actually, this field has a deeper geometrical and topological description, as we will see in the next chapters.

Finally, there are massless states for each normal direction to the Dp -brane, coming from scalar fields defined on the brane itself. These are usually interpreted as the fields generating the excitations of the Dp -brane, describing fluctuations in the Dp -brane position in spacetime.

In the case of general spacetimes and brane configurations, we will assume that the picture described above is still valid locally for any given neighborhood of a point in the Dp -brane worldvolume.

2.3 Chan-Paton factors and Adjoint bundles

In this section we will consider how the description in the previous section is modified when we allow for the presence of n Dp -branes represented by the *same* worldvolume Q . This is indeed possible, due to the fact D-branes represent boundary conditions for the worldsheet conformal field theory, and hence they are distinguished by the open string boundary. In this case, the n D-branes are said to be *coincident*.

Then, any quantum state of an open string constrained on the worldvolume of a set of n coincident Dp -branes can be labelled by $|k; ij \rangle$, where k is a given collection of quantum numbers, and $i, j = 1, \dots, n$ label the branes which the end points of the strings are constrained to. Allowing a general superposition of this states, one has that a general state assumes the form $|k; a \rangle = \sum_{i,j} \lambda_{ij}^a |k; ij \rangle$: the coefficients λ_{ij}^a are called the *Chan-Paton factors* of the open strings.

For physical reasons ([79],[33]), the Chan-Paton factors $\lambda^a = (\lambda_{ij}^a)$ are constrained to be antihermitian, i.e. $\lambda^{a\dagger} = -\lambda^a$; moreover, they are conserved in interactions. Hence, the Fock space for an open string in the presence of n coincident Dp -branes is

given by

$$\mathcal{F}_{\text{open}}^{\text{D}} \otimes \mathfrak{u}(n)$$

where \mathcal{F}^{D} is the space of an open string in the presence of a single Dp -brane, and $\mathfrak{u}(n)$ is the Lie algebra of $U(n)$. Consequently, all the massless fields described in the previous section will take values, at least locally on the brane, in $\mathfrak{u}(n)$.

As the Chan-Paton factors are conserved in interactions, their contribution in the interaction amplitude at any level is of the form of a trace of matrix products. As the trace of a matrix is preserved under the adjoint action of the group of invertible matrices, the transformation $\lambda^a \rightarrow g\lambda^a g^\dagger$, with $g \in U(n)$ is a symmetry of the system.² Hence the Lie algebra $\mathfrak{u}(n)$ can be seen as the adjoint representation for the matrix group $U(n)$.

Consider now a set of n Dp -branes wrapping a submanifold Q of the spacetime, and let $\{\mathcal{U}_\alpha\}$ be a good cover for Q . Recall that at low energy the spectrum produces a $\mathfrak{u}(n)$ -valued 1-form A_α , for any α . Consistently with the symmetry of Chan-Paton factors and the properties of A_α , on a double intersection $\mathcal{U}_{\alpha\beta}$ we *can* have

$$A_\beta = g_{\alpha\beta}(p)A_\alpha g_{\alpha\beta}^{-1}(p) + g_{\alpha\beta}^{-1}(p)dg_{\alpha\beta}(p) \quad (2.3.1)$$

where $g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \rightarrow U(n)$ is a smooth function. At the same time, other fields will obey similar properties. For instance, the $\mathfrak{u}(n)$ -valued scalar field T describing the tachyon will satisfy on $\mathcal{U}_{\alpha\beta}$

$$T_\beta = g_{\alpha\beta}(p)T_\alpha g_{\alpha\beta}^{-1}(p) \quad (2.3.2)$$

The function $g_{\alpha\beta}$ must be the same as it arises from a symmetry of the transition amplitude.

Consistency on triple overlaps requires that

$$\text{Ad}(g_{\alpha\beta})\text{Ad}(g_{\beta\gamma})\text{Ad}(g_{\gamma\alpha}) = 1 \quad (2.3.3)$$

The above condition implies that

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = \omega_{\alpha\beta\gamma}1 \quad (2.3.4)$$

where $\omega_{\alpha\beta\gamma} \in U(1)$, i.e. in the kernel of the adjoint representation. At this point one can ask if it is possible to redefine the functions $g_{\alpha\beta}$ in such a way that $\omega_{\alpha\beta\gamma} = 1$, i.e. if the vector bundle E^{adj} , of which the tachyon field is a section, is the adjoint bundle associated to a principle bundle P over Q defined by the transition functions

² $U(n)$ is singled out because it preserves the antihermitian property of λ^a .

$g_{\alpha\beta}$. In this case, the $\mathfrak{u}(n)$ -valued forms A_α can be seen as the local representatives of a connection defined on the principal bundle P , hence as a Yang-Mills field over Q . This “lifting” process is not always possible for general vector bundles: anyway, we will always assume that this is possible. With this we mean that choosing functions $g_{\alpha\beta}$ behaving as transition functions of a principal bundle is *consistent* with the path integral for open strings in the presence of D-branes being well defined and gauge invariant. In this case, the path integral is supplemented with a factor

$$\text{hol}_{\partial\Sigma}(f^*A) \tag{2.3.5}$$

which only makes sense when A is a connection on some principal bundle.

In the case in which a topologically nontrivial B-field is present, though, the equation (2.3.1) has to be modified in order to take into account the gauge transformations for the B-field discussed in section 1.5: this, in turn, forces a redefinition of the function $g_{\alpha\beta}$, which will now obey condition (2.3.4), with the cocycle $\omega_{\alpha\beta\gamma}$ directly determined by the topological properties of the B-field. See [60] for more details.

Hence, in absence of a B-field, the bundle E^{adj} over Q is chosen to be decomposable as

$$E^{adj} \simeq E \otimes \bar{E}$$

where E is a $U(n)$ -vector bundle, and \bar{E} is its complex conjugate. It is customary to refer to the vector bundle E as the *Chan-Paton bundle*, as from this last we can obtain the bundle of Chan-Paton factors E^{adj} .

We can then summarize the crucial aspects of this section by the following

Fact *In the absence of a nontrivial B-field and in the low energy approximation, a set of n coincident D-branes with worldvolume Q gives rise to a $U(n)$ -vector bundle $E \rightarrow Q$ equipped with a linear connection.*

Notice that the topology of the vector bundle E is not determined by the string dynamics, and has to be specified *a priori* when introducing a set of D-branes. Also for a single D-brane, we have to assign a line bundle over its worldvolume: in the case of hyperplanar D-branes discussed in the previous section, this is hidden by the fact that any line bundle over such a hypersurface is topologically trivial.

2.4 D-branes and Supersymmetry

In this section we will consider the effect of the presence of a D-brane in superstring theories; we will focus, in particular, on the case of type IIA and IIB superstring theories.

Recall that superstring theories, in the RNS formalism, are obtained by introducing degrees of freedom on the worldsheet Σ which are sections of $S(T\Sigma) \otimes f^*TM$, where $S(T\Sigma)$ is the spinor bundle associated to $T\Sigma$, and that the GSO projection ensures spacetime supersymmetry. For this reason, we will require M to be a spin manifold, where M is a 10-dimensional euclidean spacetime.

We have seen in the previous sections that the defining property of a Dp -brane with worldvolume Q is that of constraining the maps f to satisfy $f(\partial\Sigma) \subset Q$, and we have argued that this requirement modifies the spectrum of the open bosonic string. Indeed, the presence of a Dp -brane modifies also the fermionic sector for the open supersymmetric string [90, 79].

Notice, first, that for open strings whose end points are constrained to the submanifold Q , one has $f^*TM \simeq f^*TM|_Q$, where $TM|_Q$ is the restriction of TM to Q . Moreover, as Q is a submanifold of M , the following exact sequence holds

$$0 \rightarrow TQ \rightarrow TM|_Q \rightarrow \nu_Q \rightarrow 0 \quad (2.4.1)$$

and with a choice of a metric on TM it splits orthogonally as $TM|_Q \simeq TQ \oplus \nu_Q$, where ν_Q is the normal bundle to Q in M . Hence, the ground state in the Ramond sector will give rise to a section of

$$S(TM|_Q) \simeq S(TQ) \otimes S(\nu_Q) \quad (2.4.2)$$

i.e. a worldvolume spinor charged under an internal $SO(9-p)$ symmetry, the structure group of ν_Q ³.

To be more precise, notice that in general neither $S(TQ)$ nor $S(\nu_Q)$ are *well defined* as vector bundles, even if their tensor product is. Anyway, we will for the moment require that the normal bundle ν_Q admits a spin structure: together with the requirement that M is a spin manifold, one has that TQ also admits a spin structure, i.e. Q is a spin manifold.

We refer the reader to Appendix B for more details on spin manifolds and Clifford algebras.

³Actually, the decomposition (2.4.2) depends on the parity of the codimension of Q in M , since in general $Cl(V \oplus W) \simeq Cl(V) \hat{\otimes} Cl(W)$, where $\hat{\otimes}$ denotes the \mathbb{Z}_2 -graded product.

Because of the properties of a system of n coincident D-branes discussed in section 2.3, the worldvolume spinors obtained in the low energy approximation are sections of $S(TQ) \otimes S(\nu_Q) \otimes E^{adj}$, and hence they are charged under the gauge field localized on the worldvolume. Then, it is natural to think that, in a certain sense, Dp -branes in the low energy approximation are described by the $p+1$ -dimensional gauge theories localized on their worldvolume.

As we have mentioned in section 2.2, the introduction of a D-brane can reduce the spacetime symmetries of the vacuum configuration, and hence affect the properties of the open and closed string spectrum. As should be expected, a D-brane affects also the amount of supersymmetry present in the theory. Indeed, the introduction of a D-brane in an interacting String theory requires the presence of open string states in the spectrum: this is due to the fact that a closed string, though not affected by the boundary conditions, can *break* on the worldvolume of the D-brane into an open string, and open strings have at most $N=1$ supersymmetry [79, 16, 90].

A D-brane is said to be *supersymmetric* if the spectrum of open and closed superstrings in its presence is supersymmetric. Moreover, a D-brane is said to be *stable* if the spectrum does not contain a tachyonic state. In this sense, all D-branes in bosonic String theory are unstable. Supersymmetry and stability of a D-brane are very difficult features to determine for general brane configurations and in nontrivial backgrounds, but are well understood in the case of hyperplanar D-branes in Minkowski spacetime. Indeed, one can show that flat Dp -branes in type IIA on a Minkowski spacetime are supersymmetric for p even, while p must be odd in type IIB. This can be inferred by analyzing the conserved supersymmetry in each case, and by using *T-duality*[79, 90, 16]. As usual, we will assume these properties of D-branes to hold for a more general type II configurations.

As suggested by the terminology used above, a D-brane appears not only to be a geometrical region enforcing the boundary conditions for the worldsheet conformal field theory, but also to enjoy particle-like properties, like mass, charge, etc., detected through the behavior of the strings propagating around it. Even if a full theory of quantum D-branes, which would be the right framework to describe quantum properties like decay, etc., has not yet been developed, the superstrings in the presence of a D-brane give useful information about the dynamics of these extended objects.

Remark The definition of supersymmetric and stable D-brane given above are sensible in the low energy approximation, and in the case in which we are considering a D-brane that is *fixed*, i.e. we are neglecting its dynamics: indeed, to take into account

that a D-brane is a dynamical object, and to describe some aspects of its quantum behavior, a formalism known as the *boundary state formalism* is more suitable. In this formalism, the boundary conditions characterizing the D-brane are imposed *after* the theory of closed superstrings has been quantized: this suggests to identify states of a *quantum* D-brane as coherent states in the Fock space of the closed superstring, making it more precise the investigation of stability, supersymmetry, etc. of these states. Moreover, it makes it possible to introduce D-branes even when the boundary condition it represents has no clear geometrical description.

As in this thesis we will be concerned only with a semiclassical and geometrical description of D-branes, we will not invoke this formalism, directing the reader to [59, 79], and references therein, for a complete review of these topics.

2.5 D-branes and Ramond-Ramond charges

As we have mentioned in section 1.3, the low energy approximation of type IIA and type IIB superstring theory contains particle states described by a gauge theory of differential forms $C^{(p)}$ defined on a d -dimensional spacetime manifold M , with p odd in type IIA and p even in type IIB.

Moreover, if we denote by $F^p = dC^{(p)}$ the Ramond-Ramond field strength, the GSO projection imposes the constraint $\star F^p = F^{d-p}$, representing the fact that not all the forms $C^{(p)}$ are *independent* degrees of freedom [79].

It follows that the Ramond-Ramond field strengths satisfy the linearized equations

$$\begin{aligned} d \star F^{(p)} &= 0 \\ dF^{(p)} &= 0 \end{aligned}$$

These equations are a generalization of the Maxwell equations in 4 dimensions.

As the field theory they describe plays a prominent role in this thesis, we will digress slightly to review some of its aspects, following [29, 42, 41].

2.5.1 Generalized electromagnetism and sources

In analogy with Maxwell electromagnetism in 4 dimensions, consider a theory of differential forms $A \in \Omega^n(M; \mathbb{R})$, where $M = \mathbb{R} \times Y$ is a d -dimensional oriented Lorentzian manifold with Y the spatial slice, and whose equations of motion are

given by

$$\begin{aligned} d \star G &= 0 \\ dG &= 0 \end{aligned}$$

with $G = dA$.

Such a theory is usually referred to as *generalized electromagnetism*, as the Maxwell equations (in vacuum) are obtained for $n=1$ and $d=4$. The equations of motion are invariant under the transformation $A \rightarrow A + \omega$, with ω a closed n -form, describing an abelian gauge theory of n -forms. It is natural, then, to introduce an *electric* source for the field A in complete analogy with the electromagnetic case, i.e. by modifying the equations of motion as

$$\begin{aligned} d \star G &= j_e \\ dG &= 0 \end{aligned} \tag{2.5.1}$$

where $j_e \in \Omega_s^{d-n}(M; \mathbb{R})$ is a $(d-n)$ -form with compact support on the spatial slice, called the *electric current distribution*.

The equations of motion imply that $dj_e = 0$, hence j_e represents a class

$$[j_e] \in H_s^{d-n}(M; \mathbb{R})$$

in the cohomology of M with real coefficients, and compact support on Y .

Notice that the equations of motion *do not* imply that the class $[j_e]$ is vanishing, as $\star G$ is not required to have compact support on the spatial slice.

More precisely, if we denote with $i_t : Y \rightarrow M$ the map $Y \rightarrow \{t\} \times Y \subset M$, the class

$$Q_e := [i_t^* j_e] \in H_{\text{cpt}}^{d-n}(Y; \mathbb{R}) \tag{2.5.2}$$

is in general non vanishing, and it is called the *total charge* of the electric current distribution. Indeed, the fact that j_e is a closed form implies⁴ that the class Q_e does not depend upon the choice of i_t , hence it is a conserved quantity for the equations of motion.

This cohomological interpretation of the electric charge may sound “exotic” at first: but notice that in the ordinary electromagnetism case, with $n=1$, $d=4$, and $Y = \mathbb{R}^3$, one has $H_{\text{cpt}}^3(\mathbb{R}^3; \mathbb{R}) \simeq \mathbb{R}$, and the total electric charge is given by a real number, as usual.

Equations (2.5.1) can be obtained from the functional

$$S[A] := -\frac{1}{2} \int_M G \wedge \star G - \frac{1}{2} \int_M j_e \wedge A \tag{2.5.3}$$

⁴This is also due to the particular choice made for M .

via a variational principle. This functional is *not* gauge invariant, but one can show that the equations of motion obtained do not depend on the particular choice of a gauge. Moreover, also the coupling term in (2.5.3) can be given a cohomological interpretation. As j_e is a closed form with compact support on M , it will induce a homomorphism

$$\psi_{j_e} : H^n(M; \mathbb{R}) \rightarrow \mathbb{R}$$

defined as

$$\psi_{j_e}([a]) := \langle [j_e] \cup [a], [M] \rangle = \int_M j_e \wedge a$$

for any *closed* n -form a .

As in cohomology with real coefficients we have that $\text{Hom}(H^n(M; \mathbb{R})) \simeq H_n(M; \mathbb{R})$, one has that there exists a class in $[Q] \in H_n(M; \mathbb{R})$ represented by a compact oriented submanifold Q such that

$$\psi_{j_e}([a]) = \langle [a], [Q] \rangle = \int_Q a$$

The class $[Q]$ is called the *Poincaré dual* of $[j_e]$.

Indeed, using this class we could rewrite the coupling term in (2.5.3) as

$$\int_M j_e \wedge A = \int_Q A|_Q \tag{2.5.4}$$

which gives rise to the straightforward interpretation of Q as the worldvolume of an extended object acting as a source for the field A , with coupling given by (2.5.4). Moreover, in the usual physical case in which $Q \simeq \mathbb{R} \times \tilde{Q}$, the total charge of the distribution is represented by $[\tilde{Q}]$.

Unfortunately, as it is stated, equation (2.5.4) is not quite true, as A is in general *not* closed.

The mathematical solution to this problem is to regard j_e as a *current* [29, 50], i.e. as a distribution valued differential form, with support on Q . This indeed allows equation (2.5.4) to hold for a non-closed differential form A ; notice that this is perfectly analogous to the electromagnetic case, where an electron current is described with a Dirac delta distribution having support on the worldline of the electron itself.

To summarize, in this section we have argued that a generalized abelian gauge theory of n -forms admits extended objects as electric sources, whose worldvolume Q are submanifolds of the spacetime, and whose charges can be identified with the homology classes represented by Q in de Rham homology.

2.5.2 Ramond-Ramond charges and anomalies

As we have seen in the previous section, a generalized electromagnetic theory naturally admits extended objects as its electric sources. Since the dynamics of Ramond-Ramond fields are governed by such a gauge theory, it is natural to ask for their sources. Recall that Ramond-Ramond fields are given by p -forms, with p odd in type IIA String theory, and p even in type IIB: sources for Ramond-Ramond fields in type IIA are necessarily odd-dimensional submanifolds, while in type IIB they need to be even-dimensional. Hence, the natural candidates for the role of sources are the supersymmetric stable D-branes of type II String theory: indeed, we have seen in section 2.4 that they occur with the right dimension, i.e. as odd-dimensional in type IIA, and as even-dimensional in type IIB. This would indeed take into account the stability of D-branes of certain dimensions.

As it is stated here, this could be merely a coincidence, in the sense that the only property of having the right dimension does not necessarily identify D-branes with the sources of Ramond-Ramond fields.

On the contrary, in the seminal paper [78] strong support was given to the fact that D-branes *do* interact with Ramond-Ramond fields: this was achieved by scattering closed strings with D-branes, and showing that the interaction amplitude contains a term that can be manipulated to a coupling of the form (2.5.4). Moreover, it was shown that D-branes coupling to Ramond-Ramond fields enjoy the *BPS* property: their mass, that in the case of D-branes is called the *tension*, coincides with their charge. D-branes in type II with the “wrong” dimension do not couple to Ramond-Ramond fields, and they are unstable: they are called *non-BPS* D-branes.

Hence, let $Q \subset M$ be a D_p -brane: the term

$$\int_Q C^{(p+1)}|_Q \tag{2.5.5}$$

is called the *Ramond coupling*, and the class $[Q]$ is called the *Ramond charge* of the D_p -brane Q .

It turns out [71, 48], though, that (2.5.5) is not yet the right coupling term, and has to be modified in order to take into account some peculiar properties of D-branes.

Before explaining the reason for this modification, we add a comment on the coupling (2.5.5). Indeed, in the above discussion we have willingly neglected the fact that the different Ramond-Ramond fields are dependent upon the relation $\star F^{(p)} = F^{(d-p)}$. This means that any electric source for the field $C^{(p)}$ is a magnetic source for $C^{(d-p)}$, which requires a shift in the meaning of $C^{(p)}$ themselves, for the appropriate

values of p . More precisely, in the presence of a D p -brane, $C^{(d-p-1)}$ cannot be a globally defined differential $d - p - 1$ -form, as the Bianchi identities for its field strength do not hold any longer. Moreover, this poses a serious threat to the quantization of the Ramond-Ramond abelian gauge theory. For the moment, we will continue to neglect the problems posed by the duality constraints.

Recall from section 2.4 that in the low energy approximation the worldvolume of a set of n D p -branes carries a dimensionally reduced Yang-Mills theory coupled with a spinor field charged under some additional internal symmetries; moreover, if the D p -brane is supersymmetric, the worldvolume gauge theory is an N=1 super Yang-Mills theory, and the tachyon field is absent.

A consistency requirement on the gauge theory supported by the D-brane is that the theory does not present any “quantum” anomaly, which would spoil its gauge invariance after quantization. Recall, indeed, that this gauge theory is obtained by considering only the massless states in the open string spectrum in the presence of a D-brane, which should represent one particle states for the full quantized gauge theory: an anomaly would render this an inconsistent procedure.

It turns out that the gauge theory on the D-brane suffers from two anomalies. The *abelian* anomaly is related to the fact that the spinor fields on the worldvolume are charged under an internal symmetry, represented by the tensor product $S(\nu_Q) \otimes E^{adj}$, where Q is the D-brane worldvolume. More precisely, in the type IIB case, where Q is an even-dimensional (spin) manifold, the dynamics for the spinor fields is governed by the Dirac operator

$$\not{D} : S^+(\text{TM}|_Q) \otimes E^{adj} \rightarrow S^-(\text{TM}|_Q) \otimes E^{adj} \quad (2.5.6)$$

where the \pm splitting is given by the chiral decomposition, since M even-dimensional. From equation (2.4.2) one can see that

$$S^\pm(\text{TM}|_Q) \simeq [(S^+(\text{TQ}) \otimes S^\pm(\nu_Q)) \oplus (S^-(\text{TQ}) \otimes S^\mp(\nu_Q))]$$

Hence, from the worldvolume perspective the spinor fields do not have a definite chirality: this, in general, leads to the so called “chiral” anomaly, and is measured by the index of the Dirac operator in (2.5.6).

The other possible source of anomaly is given by the intersection of two D-branes, usually referred to as *I-brane*. The spectrum of the open string in the presence of an I-brane is different from the one obtained in the presence of a D-brane wrapping the same worldvolume [79]. In particular, the fermion sector on the I-brane develops itself an anomaly, depending on the dimension of the two intersecting branes.

Both the anomalies need to be cancelled: the mechanism used is known as the *inflow mechanism*, and essentially consists in modifying the coupling (2.5.5) in such a way that its variation under gauge transformations cancel the quantum anomalies above. We refer the reader to [48, 29, 90] for a detailed exposition, as the computations involved are rather lengthy, and the techniques used therein will not play any relevant role for the rest of this thesis.

The coupling (2.5.5) is modified as

$$\int_{\mathbb{Q}} i^* C \wedge \text{ch}(E) i^* \sqrt{\hat{A}(\text{TM})} \frac{1}{\hat{A}(\text{TQ})} \quad (2.5.7)$$

where $i : \mathbb{Q} \rightarrow \text{M}$ is the embedding map, ch denotes the (total) Chern character, and \hat{A} denotes the A-roof genus⁵. See Appendix C for details on characteristic classes of vector bundles.

Moreover, in the coupling (2.5.7), C is the *total* Ramond-Ramond field, defined as the formal sum $C := C^{(i)} + C^{(i+2)} + \dots$, with $i = 0, 1$ in type IIB, type IIA, respectively, and the integration is understood to be 0 when the degrees do not match.

The coupling term (2.5.5) is referred to as the Wess-Zumino, or Chern-Simons term for D-branes: we will refer to it as the *anomalous coupling*.

The anomalous coupling forces us to rethink the charge classification of D-branes: indeed, the current generated by such extended objects is no longer given by the Poincaré class dual to the homology cycle of their worldvolume, but needs to be corrected according to the equations of motion for the (total) Ramond-Ramond field induced by (2.5.7). We will refer to the charge of this current as the *D-brane charge*. The modern interpretation for the charge of a D-brane was first exploited in [71], where it was proposed that the correct mathematical tool to represent D-brane charges is not de Rham or singular cohomology, but K-theory, which represents, together with some of its “flavours”, the main mathematical subject of this thesis.

We conclude this section by noticing that the anomalous coupling (2.5.7) suffers from the same, and actually worse, problems of the coupling (2.5.5). Indeed, the anomalous coupling requires that *all* the Ramond-Ramond differential forms should be defined on the worldvolume \mathbb{Q} of the D-brane. This is not the case when we take into account the fact that D-branes are electric *and* magnetic sources: indeed, the usual description for a gauge field A in the presence of a magnetic distribution prescribes that A should be defined on the *complement* of the distribution’s support, which renders the expression (2.5.7) ill defined [42]. Again, its correct description requires a

⁵More precisely, these are the Chern-Weil forms representing such characteristic classes.

more powerful formalism: we will address a possible solution for this problem in later chapters.

2.6 D-brane decay and Sen's conjectures

The fact that D p -branes are charged under Ramond-Ramond fields allows the introduction of *anti-D-branes*. Of course, such objects should arise in a proper quantum description of D-branes, which we currently lack. In any case, one can give the following semiclassical definition:

Definition *An anti-D-brane for a charged D p -brane Q supporting a Chan-Paton line bundle E is given by a submanifold Q' carrying the opposite D p -brane charge, and supporting a line bundle E' which is topologically equivalent to E .*

As D-brane charges are conserved during the dynamics, charged D p -branes are stable, and hence cannot decay. Again, the description of the decay process would require a proper quantum theory of D-branes: in this context, we will say that a D p -brane has *decayed* if the theory of open and closed strings in the presence of the D p -brane is *equivalent*, in some sense, to a theory of only closed strings in the absence of the D p -brane itself. This is believed to be described at low energy by the dynamics of the tachyon field living on the worldvolume of the D p -brane, which mimics the “slow rolling” dynamics of the Higgs field in the standard model of elementary particles. In particular, the decay process of the D p -brane ends when the tachyon field reaches a stable minimum of the potential modelling its dynamics [96]. Even if at the present stage such a process cannot be described in a satisfactory way, it has greatly contributed to a more basic understanding of D-branes and their charges. More precisely, it is at the base of the so called *Sen's conjectures*.

Recall that in the bosonic String theory all D-branes are unstable, as the open string spectrum contains a tachyon. In particular, one has a spacetime filling D25-brane which is unstable. In [85, 84] Sen stated the following

Conjectures(Sen) *The open bosonic String theory in the presence of a spacetime filling D25-brane is such that*

- a) *The tachyon field potential has a stable local minimum. Moreover, the energy density of this minimum as measured with respect to that of the initial unstable point is equal with opposite sign to the tension of the D25-brane;*
- b) *Lower dimensional D p -branes can be obtained as solitonic solutions of the field*

theory living on the D25-brane worldvolume;

- c) The stable minimum of the tachyon potential corresponds to the closed string vacuum with no open string excitations*

The above conjecture can be stated also in the case of a lower dimensional unstable Dp -brane: the advantage of using a spacetime filling D-brane consists in the fact that the spacetime symmetries are not broken.

Moreover, the conjecture can be adapted to superstring theory on a 10-dimensional spacetime, but some modifications are needed. For instance, in type IIB String theory the spacetime filling D9-brane is stable, and hence does not obey Sen's conjecture. In any case, an instability appears in a system of a coincident Dp -brane and its anti-D-brane, denoted as $D\bar{p}$ -brane, wrapped on a submanifold Q : at an intuitive level, the system has no conserved D-brane charge, and hence it should be able to annihilate by Sen's conjectures. This is supported by the fact that the spectrum of the open strings "stretching" between the Dp -brane and the $D\bar{p}$ -brane contains a tachyonic state, which is not projected out by the GSO projection. This extends to a system of n Dp -branes and n $D\bar{p}$ -branes wrapping the same submanifold of the spacetime.

Notice that a system of n Dp -branes with Chan-Paton bundle E and m anti-D-branes with Chan-Paton bundle F wrapping the submanifold Q has Chan-Paton bundle $E \oplus F$, as the D-branes and the anti-D-branes are distinguished by the open string endpoints. In particular, at low energy the surviving open string tachyon field is described by a section of $E \otimes F^*$, where F^* denotes the dual bundle.

Despite the resemblances, there is a major difference between an unstable bosonic Dp -brane and a brane-antibrane system: indeed, while by Sen's conjectures *all* the bosonic Dp -branes will eventually decay to the vacuum state, a generic brane-antibrane system can decay to a stable Dp -brane, if the starting configuration has a net D-brane charge different from "zero". In particular, in [83] Sen was able to construct a Dp -brane as a decay product of a system of a $Dp+2$ -brane and a coincident $D\overline{p+2}$ -brane, identifying the worldvolume wrapped by the Dp -brane as a "vortex" for the tachyon field living on the brane-antibrane worldvolume. Indeed, recall that in this case the tachyon field T is a section of the complex line bundle $E \otimes F^*$ defined on the brane-antibrane worldvolume: in Sen's construction, the Dp -brane is identified with the submanifold representing the Poincaré dual class to the (first) Chern class of $E \otimes F^*$, the zero loci for the section T .⁶

A somewhat simple observation was used by Witten in the seminal paper [94] to give

⁶We are supposing that the Chern class of $E \otimes F^*$ is *nontorsion* and that T is a transversal section.

an elegant mathematical description of the various D-brane configurations. More precisely, consider in type IIB String theory a system of n spacetime filling D9-branes with Chan-Paton bundle E , and a system of n $D\bar{9}$ -branes carrying a Chan-Paton bundle F ;⁷ label such a configuration by the pair (E,F) . As the process of brane-antibrane creation and annihilation does not change the D-brane charge, we can add any collection of m D9-branes and m $D\bar{9}$ -branes with Chan-Paton bundle H .⁸

Hence, the configuration (E,F) should be considered to have the same D-brane charge as the configuration $(E \oplus H, F \oplus H)$: pairs of vector bundles with such an equivalence relation constitute, in a nutshell, the basic ingredients for the K-theory group $K(M)$ of the spacetime.

In particular, Witten showed that the construction used by Sen can be generalized to arbitrary spacetime and D-brane configurations, and that such a generalization perfectly corresponds to mathematical properties of K-theory. More importantly, Witten's classification of D-branes via K-theory applies not only to type IIB/A superstring theories, but also to type I and to String theory on *orbifolds*: in these latter cases, indeed, the K-theory description makes new and unexpected predictions, as we will see in the following chapters.

⁷The number of branes and antibranes needs to be the same in type IIB to cancel the Ramond-Ramond tadpole.

⁸We use the same notation H for both the gauge bundle on the D-brane and that on the anti-D-brane, as they are by definition topologically equivalent.

Chapter 3

K-Theory, an introduction

3.1 The group $K^0(X)$

In this section we introduce the basic definitions used to construct the group $K^0(X)$ of a topological space X . In particular, we will restrict ourselves to compact Hausdorff topological spaces which can carry a structure of a finite CW-complex. Even if most of the constructions can be extended to more general topological spaces, the choice of CW-complexes is not restrictive for the aim of this thesis, as we will mainly be interested in the K-theory groups of finite dimensional manifolds, which naturally carry a canonical CW-complex structure. In the following exposition we tacitly refer to [4, 61], unless otherwise stated.

Let $\text{Vect}^F(X)$ denote the set of isomorphism classes of topological F -vector bundles over X , where $F = \mathbb{C}, \mathbb{R}$. The direct (Whitney) sum of vector bundles gives $\text{Vect}^F(X)$ the structure of an abelian monoid: in a nutshell, the group $K^0(X)$ is constructed via a procedure that consists intuitively in adding “inverses” to $\text{Vect}^F(X)$. In particular, the procedure can be generalized to any abelian monoid, which is the case we will illustrate in the following.

Let \mathcal{A} be an abelian monoid. We can then associate to \mathcal{A} an abelian group $K^0(\mathcal{A})$ and a monoid homomorphism $\alpha : \mathcal{A} \rightarrow K^0(\mathcal{A})$ with the the following *universal* property. For any group G , and any morphism of the underlying monoids $f : \mathcal{A} \rightarrow G$, there is a unique group homomorphism \tilde{f} such $\tilde{f}\alpha = f$. Because of the uniqueness property, it is an immediate consequence that if such a $K^0(\mathcal{A})$ exists, then it is unique up to isomorphism. The group $K^0(\mathcal{A})$ is known usually as the *Grothendieck group* of \mathcal{A} .

A possible construction for the group $K^0(\mathcal{A})$ is the following. Denote with $F(\mathcal{A})$ the

free abelian group generated by the elements of \mathcal{A} , and let $E(\mathcal{A})$ be the subgroup of $F(\mathcal{A})$ generated by those elements of the form $a + b - (a \oplus b)$, with $a, b \in \mathcal{A}$, and \oplus denoting the addition in \mathcal{A} . Then for $K^0(\mathcal{A}) := F(\mathcal{A})/E(\mathcal{A})$, the universality property above holds, with $\alpha : \mathcal{A} \rightarrow K^0(\mathcal{A})$ the obvious map.

A different and sometimes convenient construction of $K^0(\mathcal{A})$ is the following. Consider the diagonal homomorphism of monoids $\Delta : \mathcal{A} \rightarrow \mathcal{A} \times \mathcal{A}$, i.e the map $\Delta(a) = (a, a)$, and denote with $K^0(\mathcal{A})$ the set of cosets of $\Delta(\mathcal{A})$ in $\mathcal{A} \times \mathcal{A}$. It is clearly a quotient monoid, but the interchange of factors induces inverses in $K^0(\mathcal{A})$, giving it a group structure. If we define $\alpha : \mathcal{A} \rightarrow K^0(\mathcal{A})$ to be the composition of $a \rightarrow (a, 0)$ with the natural projection $\mathcal{A} \times \mathcal{A} \rightarrow K^0(\mathcal{A})$, then the universality property above holds.

The association to a monoid \mathcal{A} of its Grothendieck group as defined above induces in an obvious way a unique covariant functor K^0 from the category \mathcal{A} of abelian monoids with monoid morphisms to the category \mathcal{G} of abelian groups with group morphisms. Indeed, to any morphism $f : \mathcal{A} \rightarrow \mathcal{B}$, the functor K^0 associates the morphism $K^0(f) : K^0(\mathcal{A}) \rightarrow K^0(\mathcal{B})$ satisfying $\alpha_{\mathcal{B}} \circ f = K^0(f) \circ \alpha_{\mathcal{A}}$, and such a morphism is unique by the universality property.

Example 3.1. Consider $\mathcal{A} := (\mathbb{N}_0, +)$. Then $K^0(\mathcal{A}) \simeq \mathbb{Z}$

Example 3.2. Consider $\mathcal{A} := (\mathbb{Z} - \{0\}, \cdot)$. Then $K^0(\mathcal{A}) \simeq \mathbb{Q} - \{0\}$.

A fundamental example of the above construction that allows a higher degree of generality is the following. Let \mathcal{C} an additive category, and denote with \dot{E} the isomorphism class of the object E . Then the set $\Phi(\mathcal{C})$ of such classes can be provided with a structure of an abelian monoid if one defines $\dot{E} + \dot{F}$ to be $E \dot{\oplus} F$: the well-definedness of the $+$ operation and the algebraic identities derive from the additivity of the category \mathcal{C} . In this case one denotes with $K^0(\mathcal{C})$ the group $K^0(\Phi(\mathcal{C}))$, called the *Grothendieck group of the category \mathcal{C}* . Moreover, if $\varphi : \mathcal{C} \rightarrow \mathcal{C}'$ is an additive functor, then φ naturally induces a monoid homomorphism $\Phi(\mathcal{C}) \rightarrow \Phi(\mathcal{C}')$, hence a group homomorphism $K^0(\Phi(\mathcal{C})) \rightarrow K^0(\Phi(\mathcal{C}'))$, denoted with φ_* . The usual composition rule follows.

We will now specialize to the case $\mathcal{A} = \text{Vect}^F(X)$, for X a topological space. We denote with $K^0(X)$ the group $K^0(\text{Vect}^F(X))$, or equivalently the Grothendieck group of the additive category of topological F -vector bundles on X , that we will denote with \mathcal{V}^F . To follow the notations usually found in literature, we will use $K^0(X)$ for $F = \mathbb{C}$, and $KO^0(X)$ for $F = \mathbb{R}$. When the results *do not* depend on the choice of F , we will use the notation for $F = \mathbb{C}$.

The basic constructions for the K^0 group described above allow to prove the following basic, but very useful results.

Proposition 3.3. *Let X be a compact space. Then every element $x \in K^0(X)$ can be written in the form $[E] - [F]$, with E, F vector bundles on X . Moreover, $[E] - [F] = [E'] - [F']$ in $K^0(X)$ if and only if there exists a vector bundle G on X such that $E \oplus F' \oplus G \simeq E' \oplus F \oplus G$*

Proof. By definition, $x = [(\dot{E}, \dot{F})]$, for some vector bundles E, F . Then we have that $[(\dot{E}, \dot{F})] = [(\dot{E}, 0) + (0, \dot{F})] = [(\dot{E}, 0)] + [(0, \dot{F})] = [E] - [F]$, where with $[E], [F]$ we denote the composition of $E \rightarrow \dot{E}$ with $\dot{E} \rightarrow [\dot{E}]$.

Let $[E] - [F] = [E'] - [F']$. Then one has that $[E \oplus F'] = [E' \oplus F]$, which implies $\dot{E} + \dot{F}' + \dot{G} = \dot{E}' + \dot{F} + \dot{G}$, for some G . Hence $E \oplus F' \oplus G \simeq E' \oplus F \oplus G$. \square

The above proposition allows almost immediately to prove the following

Proposition 3.4. *Let E and F be vector bundles over X . Then $[E] = [F]$ if and only if $E \oplus \theta_n \simeq F \oplus \theta_n$, for some θ_n a trivial bundle of rank n .*

Proof. Recall that for any vector bundle G on a compact space X there exists a vector bundle G' such that $G \oplus G' \simeq \theta_n$, for some n , i.e. any vector bundle is a *projective* object in \mathcal{V}^F .

By Proposition 3.3, if $[E] = [F]$, then $E \oplus G \simeq F \oplus G$, for some vector bundle G . By adding G' chosen as above, one gets that $E \oplus \theta_n \simeq F \oplus \theta_n$, which yields the desired result. \square

An easy consequence of Proposition 3.3 is that any element $x \in K^0(X)$ can be written as $[H] - [\theta_n]$, for some vector bundle H and some θ_n .

As we have seen, the group $K^0(\mathcal{A})$ “depends” covariantly on the monoid \mathcal{A} . However, $K^0(X)$ depends contravariantly on the topological space X , i.e. K^0 is a contravariant functor from the category \mathcal{S} of topological spaces with continuous maps to \mathcal{G} .

More precisely, if we let $f : X \rightarrow Y$ be a continuous map, then f induces a monoid morphism $f^* : \text{Vect}^F(Y) \rightarrow \text{Vect}^F(X)$ via the pullback of vector bundles, hence a map $K^0(Y) \rightarrow K^0(X)$, which we still denote with f^* .

By the homotopy theory of vector bundles one has immediately the following

Theorem 3.5. *Let X and Y be compact topological spaces, and let $f_0, f_1 : X \rightarrow Y$ be continuous and homotopic maps. Then f_0 and f_1 induces the same homomorphism $K^0(Y) \rightarrow K^0(X)$.*

Hence, the group $K^0(X)$ is a topological invariant, in the sense that two isomorphic spaces have isomorphic K^0 groups. More importantly, if the two spaces can be “deformed” into each other, they have isomorphic K^0 : this property is typical of a cohomology theory, and it is intuitively a first indication that the group $K^0(X)$ is a building block for a cohomology theory. In this sense, it is important to define the *reduced group* $\tilde{K}^0(X)$.

First notice that $K^0(\{pt\}) \simeq \mathbb{Z}$, as a vector bundle on a point is uniquely characterized by the dimension of its typical fiber. Then, the inclusion $i : x_0 \rightarrow X$ induces the homomorphism

$$i^* : K^0(X) \rightarrow K^0(\{x_0\}) \simeq \mathbb{Z}$$

The reduced group $\tilde{K}^0(X)$ is defined as the kernel of i^* . Moreover, the following exact sequence

$$0 \rightarrow \tilde{K}^0(X) \rightarrow K^0(X) \xrightarrow{i^*} K^0(\{x_0\}) \rightarrow 0$$

canonically splits, i.e. $K^0(X) \simeq \mathbb{Z} \oplus \tilde{K}^0(X)$.

Given a vector bundle $E \rightarrow X$ with E_x the fiber of E over x , we can define the *rank function of E* $\text{rk}(E) : X \rightarrow \mathbb{N}_0$. As E is locally trivial as a vector bundle over X , its rank function is a locally constant function over X with values in \mathbb{N}_0 , i.e. an element of the abelian monoid $H^0(X; \mathbb{N}_0)$. Hence, the rank map extends naturally as the homomorphism

$$\begin{aligned} \text{rk} : K^0(X) &\rightarrow H^0(X; \mathbb{Z}) \\ [E] - [F] &\rightarrow \text{rk}(E) - \text{rk}(F) \end{aligned} \tag{3.1.1}$$

In the case in which X is a connected space, the integer (3.1.1) is called the *virtual dimension* of the class $[(E, F)]$ in $K^0(X)$. Moreover, in the same case, the group $\tilde{K}^0(X)$ is isomorphic to the kernel of the rank homomorphism, hence it consists of the subgroup of elements whose virtual dimension is zero, classes $[(E, F)]$ with E and F of equal rank.

We conclude this section with a proposition which states how the K-theory group behaves under disjoint unions.

Proposition 3.6. *Let $X = \coprod_{i=1}^n X_i$. Then the inclusions of the X_i in X induces the decomposition $K^0(X) \simeq K^0(X_1) \oplus K^0(X_2) \oplus \cdots \oplus K^0(X_n)$.*

Proof. Use the fact that any vector bundle on X is characterized by its restrictions on the X_i . □

Remark Notice that this last proposition is *not* true for the functor $\tilde{K}^0(X)$. Indeed, if X is the disjoint union of two points $\{x_0\}$ and $\{x_1\}$, then $\tilde{K}^0(X) \simeq \mathbb{Z}$, but $\tilde{K}^0(\{x_i\}) = 0$, for $i = 0, 1$.

3.2 Relative K-theory and higher K-groups

Starting with the group $K^0(X)$ one can naturally define the *relative* K-theory and the *higher* K-theory groups.

Let \mathcal{T}_P denote the category of *compact pairs* (X, Y) , where X, Y are topological spaces, and X is such that it can be equipped with a structure of a CW-complex such that Y is a CW-subcomplex. The morphisms $(X, Y) \rightarrow (X', Y')$ are given by *relative maps*, i.e. continuous functions $f : X \rightarrow X'$ such that $f(Y) \subset Y'$.

We define the *relative K-theory group* as

$$K^0(X, Y) := \tilde{K}^0(X/Y) \tag{3.2.1}$$

where X/Y is obtained from X by shrinking Y to a point, with respect to which the reduced K-group is defined. In the case when $Y = \{\emptyset\}$, we define X/Y as $X^+ := X \amalg \{pt\}$. Hence, for a base-pointed space X one has

$$K^0(X, \emptyset) := \tilde{K}^0(X^+) \simeq K^0(X) \tag{3.2.2}$$

Moreover, for $\{x_0\} \subset X$ we have $K^0(X, \{x_0\}) = \tilde{K}^0(X)$.

Let us denote with $\pi : X \rightarrow X/Y$. The map π induces the obvious relative map

$$\pi : (X, Y) \rightarrow (X/Y, \{pt\})$$

hence the map

$$\pi^* : K^0(X/Y, \{pt\}) \rightarrow K^0(X, Y)$$

We have the following *excision theorem*

Theorem 3.7. *The map π^* is an isomorphism.*

It is instructive, at this point, to explain a useful way to describe vector bundles on the space X/Y , given a vector bundle E on X with typical fiber F . Suppose E

is trivializable on Y , i.e. there exists a homeomorphism $\alpha : E|_Y \xrightarrow{\cong} Y \times F$ which is linear on the fibres. Consider the equivalence relation on $E|_Y$ given by

$$v \sim v' \Leftrightarrow p \circ \alpha(v) = p \circ \alpha(v')$$

where $p : Y \times F \rightarrow Y$ is the canonical projection. The relation is then extended to the whole E . The corresponding set of equivalence classes can be shown to be the total space of a vector bundle over X/Y ; moreover, every vector bundle on X which is trivial when restricted to Y is isomorphic to the pullback of a vector bundle over X/Y .

This construction, in particular, allows to prove that the following sequence

$$K^0(X, Y) \xrightarrow{\rho^*} K^0(X) \xrightarrow{i^*} K^0(Y) \quad (3.2.3)$$

is exact, where the homomorphism ρ^* is induced by the map $\rho : X \rightarrow X/Y$, and $i : Y \hookrightarrow X$ realizes Y as a subspace of X [61, 56].

To define the higher K-groups, we need to introduce some well known operations on topological spaces. Let X and Y be compact spaces with base points $\{x_0\}$ and $\{y_0\}$, respectively. The *wedge product* of X and Y is defined as

$$X \vee Y := (X \amalg Y) / \{x_0 \sim y_0\} \quad (3.2.4)$$

while the *smash product* of X and Y is defined as

$$X \wedge Y := X \times Y / \{X \times \{y_0\} \cup \{x_0\} \times Y\} \quad (3.2.5)$$

The two operations above are related. Indeed, one has natural maps

$$X \vee Y \rightarrow X \times Y \rightarrow X \wedge Y$$

which allow to write

$$X \wedge Y = X \times Y / X \vee Y$$

Moreover, the operations \vee and \wedge are associative and commutative, and \wedge is distributive over \vee . This means, for example, that there is a canonical homeomorphism between $X \wedge Y$ and $Y \wedge X$.

An important property of the smash product is that

$$S^n \simeq S^1 \wedge S^1 \wedge \cdots \wedge S^1 \quad n\text{-times} \quad (3.2.6)$$

where S^n is the standard n -sphere with base point.

For a given space X with base point, we define the n -th *reduced suspension* $\Sigma^n(X)$ as

$$\Sigma^n(X) := S^n \wedge X \quad (3.2.7)$$

Because of property (3.2.6), we have that the n -th reduced suspension $\Sigma^n(X)$ is the n -th iterated reduced suspension of X .

We have then the following

Definition 3.8. For (X, Y) a compact pair, and for $n \geq 0$ we define

$$K^{-n}(X, Y) := \tilde{K}^0(\Sigma^n(X/Y))$$

For X a compact space we put

$$K^{-n}(X) := K^{-n}(X, \emptyset) := \tilde{K}^0(\Sigma^n(X^+))$$

and for X a compact space with base point $\{x_0\}$ we put

$$\tilde{K}^{-n}(X) := K^{-n}(X, \{x_0\}) := \tilde{K}^0(\Sigma^n(X))$$

The K^{-n} are contravariant functors, as the reduced suspension induces a covariant functor on \mathcal{T} . The relation between higher and reduced K-theory groups is given canonically by

$$K^{-n}(X) \simeq \tilde{K}^{-n}(X) \oplus \tilde{K}^{-n}(\{x_0\})$$

with $\{x_0\}$ being the base point. By the definitions above $K^{-n}(\{x_0\}) \simeq \tilde{K}^0(S^n)$: we will compute these groups in section 3.4.

The various higher K-groups are linked together by the following semi-infinite long exact sequence

$$\dots K^{-(n+1)}(X) \rightarrow K^{-(n+1)}(Y) \xrightarrow{\delta} K^{-n}(X, Y) \rightarrow K^{-n}(X) \rightarrow K^{-n}(Y) \xrightarrow{\delta} \dots \quad (3.2.8)$$

where δ is the boundary homomorphism. For a definition of δ see [4, 61]. The sequence above is another typical feature of a cohomology theory.

3.3 Multiplicative structures on K-theory

As vector bundles on a space X can be “multiplied” together via tensor product, it is natural to look for multiplicative structures on the K-groups.

The tensor product of vector bundles on X induces a multiplication

$$K^0(X) \otimes_{\mathbb{Z}} K^0(X) \rightarrow K^0(X)$$

defined as

$$[(E, F)] \otimes [(E', F')] := [(E \otimes E' \oplus F \otimes F', E \otimes F' \oplus F \otimes E')] \quad (3.3.1)$$

The expression above comes from writing $[(E, F)]$ as $[E] - [F]$, and formally imposing distributivity of the tensor product on virtual bundles. Hence, $K^0(X)$ is actually a ring.

There is another product, usually called the external tensor product or *cup product*, which is a homomorphism

$$\cup : K^0(X) \otimes_{\mathbb{Z}} K^0(Y) \rightarrow K^0(X \times Y) \quad (3.3.2)$$

defined as follows. Denote with $\pi_X : X \times Y \rightarrow X$ and $\pi_Y : X \times Y \rightarrow Y$ the canonical projections.

Hence, $\cup([E], [F]) \in K^0(X \times Y)$ is the class in $K^0(X \times Y)$ defined by

$$\cup([E], [F]) := \pi_X^*([E]) \otimes \pi_Y^*([F])$$

and extended by (bi)linearity. Notice that when $Y = X$, we have that

$$\Delta^* \cup([(E, F)], [(E', F')]) = [(E, F)] \otimes [(E', F')]$$

where $\Delta : X \rightarrow X \times X$ is the diagonal map.

The above exterior product is crucial when dealing with higher K-groups. In particular, when restricted to reduced K-theory, the cup product induces a homomorphism [4]

$$\tilde{K}^0(X) \otimes_{\mathbb{Z}} \tilde{K}^0(Y) \rightarrow \tilde{K}^0(X \wedge Y)$$

which immediately induces the homomorphism

$$\tilde{K}^0(\Sigma^n(X)) \otimes_{\mathbb{Z}} \tilde{K}^0(\Sigma^m(Y)) \rightarrow \tilde{K}^0(\Sigma^{n+m}(X \wedge Y)) \quad (3.3.3)$$

By substituting in the above expression X^+ and Y^+ for X and Y , respectively, one obtains the homomorphism

$$K^{-n}(X) \otimes_{\mathbb{Z}} K^{-m}(Y) \rightarrow K^{-(n+m)}(X \times Y) \quad (3.3.4)$$

If we denote with $K^{-*}(X) := \bigoplus_{n \geq 0} K^{-n}(X)$, the cup product induces, via (3.3.4), the structure of a graded ring on $K^{-*}(X)$. Moreover, for any space X with base point $\{\text{pt}\}$, the cup product makes $K^{-*}(X)$ into a graded module over $K^{-*}(\text{pt})$.

Notice, at this point, that a priori the ring $K^{-*}(X)$ could be very complicated, even for the case $X = \{\text{pt}\}$. It's an extremely remarkable property of K-theory that this is not the case, as the following results show.

For the first time in this introduction, we have to make a difference between complex and real K-theory to introduce a fundamental and deep result.

Theorem 3.9. (Bott Periodicity) *The ring $K^{-*}(\{\text{pt}\})$ is a polynomial algebra generated by an element $u \in K^{-2}(\{\text{pt}\}) \simeq \tilde{K}^0(S^2)$, i.e. there is a ring isomorphism*

$$K^{-*}(\{\text{pt}\}) \simeq \mathbb{Z}[u] \tag{3.3.5}$$

The element u can be represented as $u = [H] - [\theta_1]$, where H denotes the *tautological* complex line bundle over $S^2 \simeq \mathbb{C}\mathbb{P}^1$. For a proof see [4].

The above theorem, in particular, says that the map $\mu_u : K^{-n}(\{\text{pt}\}) \rightarrow K^{-n-2}(\{\text{pt}\})$ induced by multiplication by u , is an isomorphism for all n .

The previous theorem generalizes as

Theorem 3.10. *Let X be a compact space. Then the map*

$$\mu_u : K^{-n}(X) \xrightarrow{\simeq} K^{-n-2}(X)$$

given by module multiplication by u , is an isomorphism for all $n \geq 0$

The above theorem is usually referred to as the *general Bott periodicity theorem*, and essentially states that there are only two “independent” K functors, namely K^0 and K^{-1} . Moreover, theorem (3.10) can be extended to relative and reduced K-theory. Finally, the original simplest formulation of the periodicity theorem states that for any compact space X , there is an isomorphism between $K^0(X) \otimes K^0(S^2)$ and $K^0(X \times S^2)$.

In the real case things are a bit different, and slightly more complicated. Indeed, one has the following real version of the Bott periodicity theorems.

Theorem 3.11. *The ring $KO^{-*}(\{\text{pt}\})$ is generated by elements*

$$\eta \in KO^{-1}(\{\text{pt}\}), \quad y \in KO^{-4}(\{\text{pt}\}), \quad x \in KO^{-8}(\{\text{pt}\})$$

subject to the relations

$$2\eta = 0, \quad \eta^3 = 0, \quad \eta y = 0, \quad y^2 = 4x$$

i.e. there is a ring isomorphism

$$KO^{-*}(\{\text{pt}\}) \simeq \mathbb{Z}[\eta, y, x] / \langle 2\eta, \eta^3, \eta y, y^2 - 4x \rangle \tag{3.3.6}$$

Theorem 3.12. *Let X be a compact space. Then the map*

$$\mu_x : KO^{-n}(X) \xrightarrow{\simeq} KO^{-n-8}(X)$$

given by module multiplication by x , is an isomorphism for all $n \geq 0$.

The first consequence of theorem (3.11) is that the ring $KO^{-*}(\{\text{pt}\})$ is not freely generated, i.e. it contains torsion subgroups.

Apart from limiting the number of K-groups to compute, the Bott periodicity theorem allows to define K-theory for positive degrees, which is the final important step needed to construct a cohomology theory associated to the K-groups. Indeed, for $n > 0$ one defines $K^n(X, Y)$ as

$$\begin{aligned} K^n(X, Y) &:= K^0(X, Y), & \text{for } n \text{ even} \\ K^n(X, Y) &:= K^{-1}(X, Y), & \text{for } n \text{ odd} \end{aligned}$$

and analogous definitions are given for $KO^n(X, Y)$. In particular, the Bott isomorphism is compatible with the long exact sequence (3.2.8), and hence it can be extended to positive degrees.

Because of Bott periodicity, it is convenient to define for any pair (X, Y)

$$\begin{aligned} K^*(X, Y) &:= K^0(X, Y) \oplus K^{-1}(X, Y) \\ KO^*(X, Y) &:= \bigoplus_{i=0}^7 KO^{-i}(X, Y) \end{aligned}$$

The cohomology theory constructed in this way is referred to as K-theory, and satisfies all the usual axioms of a cohomology theory E on \mathcal{T}_P , but the *dimension axiom*, which requires the cohomology E to satisfy $E^i(\text{pt}) = 0$, for $i \neq 0$, and $E^0(\text{pt}) = \mathbb{Z}$.

Finally, because of Bott periodicity, the long exact sequence (3.2.8) for *complex* K-theory can be truncated to the six-term exact sequence

$$\begin{array}{ccccc} K^{-1}(X) & \longrightarrow & K^{-1}(Y) & \xrightarrow{\delta} & K^0(X, Y) \\ \uparrow & & & & \downarrow \\ K^{-1}(X, Y) & \xleftarrow{\delta} & K^0(Y) & \longleftarrow & K^0(X) \end{array}$$

3.4 K-theory and classifying spaces

As for ordinary cohomology, there is a description of K-theory in terms of classifying spaces. As this approach will play a very important role in later chapters, we will review it in some detail.

Let X a compact CW-complex of finite type. It is a very important result in geometry that F-vector bundles of rank k over X can be classified up to isomorphism. Namely, there exists a connected topological space BF_k , equipped with a rank k F-vector bundle \mathbb{E}_k , called the *universal F-vector bundle*, such that

$$\text{Vect}_k^F(X) \simeq [X, BF_k]$$

where the isomorphism is induced by assigning to the homotopy class $[f]$ the isomorphism class $[f^*\mathbb{E}_k]$ [56].

A model for BF_k can be defined as follows. Denote with $\mathrm{Gr}(k, m; \mathbb{F})$ the *Grassmannian manifold* of \mathbb{F} -linear subspaces in \mathbb{F}^m of dimension k . For $\mathbb{F} = \mathbb{C}$ one has

$$\mathrm{Gr}(k, m; \mathbb{C}) \simeq \frac{\mathrm{U}(m)}{\mathrm{U}(k) \times \mathrm{U}(m-k)}$$

while for $\mathbb{F} = \mathbb{R}$ one has

$$\mathrm{Gr}(k, m; \mathbb{R}) \simeq \frac{\mathrm{O}(m)}{\mathrm{O}(k) \times \mathrm{O}(m-k)}$$

The inclusion $\mathbb{F}^m \subset \mathbb{F}^{m+1}$ induces the natural inclusion $\mathrm{Gr}(k, m; \mathbb{F}) \subset \mathrm{Gr}(k, m+1; \mathbb{F})$. Taking the inductive limit over such inclusions allows to define the *infinite Grassmannian*

$$\mathrm{Gr}(k, \infty; \mathbb{F}) := \lim_{m \rightarrow \infty} \mathrm{Gr}(k, m; \mathbb{F})$$

which can be used as a model for our classifying space [56]. We will use the notation $\mathrm{BU}(k)$ for $\mathrm{Gr}(k, \infty; \mathbb{C})$, and $\mathrm{BO}(k)$ for $\mathrm{Gr}(k, \infty; \mathbb{R})$.

The universal \mathbb{F} -vector bundle $\mathbb{E}_k \rightarrow \mathrm{Gr}(k, \infty; \mathbb{F})$ is given by assigning to a point $x \in \mathrm{Gr}(k, \infty; \mathbb{F})$ the vector space represented by x itself.

To represent K-theory we need a limit of the above construction over the vector bundle rank k . More precisely, denote with $K^0(X)$ the kernel of the rank homomorphism (3.1.1). Recall that when X is connected, $K^0(X)$ is isomorphic to $\tilde{K}^0(X)$. In general one has the decomposition

$$K^0(X) \simeq K^{\prime 0}(X) \oplus H^0(X; \mathbb{Z}) \tag{3.4.1}$$

with $H^0(X; \mathbb{Z}) = [X, \mathbb{Z}]$.

Notice, at this point, that for $[\mathbb{E}_k] \in \mathrm{Vect}_k^{\mathbb{F}}(X)$, we have that $[\mathbb{E}_k] - [\theta_k] \in K^{\prime 0}(X)$. If we define

$$\mathrm{Vect}_{\infty}^{\mathbb{F}}(X) := \lim_{k \rightarrow \infty} \mathrm{Vect}_k^{\mathbb{F}}(X)$$

as the inductive limit over the inclusions $\mathrm{Vect}_k^{\mathbb{F}}(X) \subset \mathrm{Vect}_{k+1}^{\mathbb{F}}(X)$ induced by the maps $\mathbb{E}_k \rightarrow \mathbb{E}_k \oplus \theta_1$, one has that the map $[\mathbb{E}_k] \rightarrow [\mathbb{E}_k] - [\theta_k]$ is a monoid morphism $\mathrm{Vect}_{\infty}^{\mathbb{F}}(X) \rightarrow K^{\prime 0}(X)$, hence $\mathrm{Vect}_{\infty}^{\mathbb{F}}(X)$ can be given the structure of an abelian group [61]. Finally, as

$$\mathrm{Vect}_{\infty}^{\mathbb{F}}(X) = [X, \mathrm{BF}_{\infty}]$$

one has

$$K^0(X) \simeq \mathrm{Vect}_{\infty}^{\mathbb{F}}(X) \oplus [X, \mathbb{Z}] = [X, \mathbb{Z} \times \mathrm{BF}_{\infty}]$$

where $\mathrm{BF}_\infty := \bigcup_k \mathrm{BF}_k$.

Moreover one can obtain reduced K-theory for a base pointed space X as

$$\tilde{K}^0(X) = [X, \mathbb{Z} \times \mathrm{BF}_\infty]_*$$

where the base point in $\mathbb{Z} \times \mathrm{BF}_\infty$ has been chosen to lie in $0 \times \mathrm{BF}_\infty$. Consequently, higher relative K-theory can be obtained as

$$K^{-n}(X, Y) = [\Sigma^n(X/Y), \mathbb{Z} \times \mathrm{BF}_\infty]_* \simeq [X/Y, \Omega^n(\mathbb{Z} \times \mathrm{BF}_\infty)]_*$$

where Ω^n is the iterated loop space functor, and we have used the fact that for CW-complexes X, Y one has $[\Sigma(X), Y]_* = [X, \Omega(Y)]_*$.

In particular, if X is a connected space, then

$$\tilde{K}^0(X) = [X, \mathrm{BF}_\infty]_*$$

In later sections we will see that another classifying space for K-theory is given by the space of Fredholm operators on a infinite dimensional Hilbert space.

3.4.1 Examples: K-theory of spheres and tori

The reduced K-theory of spheres can be deduced immediately by the Bott periodicity theorems (3.9) and (3.11): in this section we want to give a geometric view on the Bott periodicity theorems, at least in the complex case. Indeed, this is actually how the Bott theorem originated.

Consider the standard n -dimensional sphere S^n . We will illustrate a procedure, called the *clutching construction*, to construct vector bundles $E \rightarrow S^n$. Write the n -sphere S^n as the union of the upper and lower hemispheres D_+^n and D_-^n , such that $D_+^n \cap D_-^n \simeq S^{n-1}$. Given a continuous map $f : S^{n-1} \rightarrow \mathrm{GL}_k(\mathbb{C})$, we can define the total space of a complex vector bundle E_f of rank k as

$$E_f := D_+^n \times \mathbb{C}^k \amalg D_-^n \times \mathbb{C}^k / \sim$$

where the identification \sim is between $(x, v) \in \partial D_+^n \times \mathbb{C}^k$ and $(x, f(x)v) \in \partial D_-^n \times \mathbb{C}^k$, and the projection $E_f \rightarrow S^n$ is the obvious one. Essentially, the clutching construction is a special case of the gluing construction for locally trivial vector bundles, which one can see by considering a “small” strip $S^n \times \{-\epsilon, \epsilon\}$, and using the function f on each slice $S^n \times \{t\}$. Moreover, as the structure group for a complex vector bundle of rank k can always be reduced to the unitary group $U(k)$, upon the introduction

of a Hermitian metric, the function f can always be chosen to be a continuous map $f : S^{n-1} \rightarrow U(k)$.

A basic property of the clutching construction is that $E_f \simeq E_g$ if f and g are homotopic maps. This allows to define a map

$$\Phi : [S^{n-1}, U(k)] \rightarrow \text{Vect}_k(S^n)$$

A fundamental result is the following [61]

Theorem 3.13. *The map Φ is a bijection.*

In other words, *all* complex vector bundles on spheres are obtained up to isomorphism by the clutching construction. By taking inductive limits, we finally obtain

$$\text{Vect}_\infty(S^n) = [S^{n-1}, U(\infty)]$$

where $U(\infty)$ denotes the *infinite* unitary group.

Hence we have

$$\tilde{K}^0(S^n) = \pi_{n-1}(U(\infty)) \tag{3.4.2}$$

Because $K^{-n}(\text{pt}) \simeq \tilde{K}^0(S^n)$, we see that the Bott periodicity theorem is a statement about the homotopy of the infinite unitary group: indeed, this is how it was originally stated¹ in [20].

The real case requires some modifications in the clutching construction, and, as it is expected from the structure of $KO^*(\text{pt})$, the relation with the orthogonal groups is more complicated.

Notice, also, that as a topological invariant for spheres, K-theory is a very coarse one, as it cannot detect the sphere dimensionality, in contrast to ordinary cohomology.

The case for tori, instead, is quite different. To compute the complex K-theory groups for tori, we will use the following result [4]: given two base pointed spaces X and Y we have the following isomorphism

$$\tilde{K}^{-n}(X \times Y) \simeq \tilde{K}^{-n}(X \wedge Y) \oplus \tilde{K}^{-n}(X) \oplus \tilde{K}^{-n}(Y) \tag{3.4.3}$$

We now specialize (3.4.3) to the case in which $Y = S^1$, with the usual base point. In this case, we have that

$$\begin{aligned} \tilde{K}^{-n}(X \times S^1) &\simeq \tilde{K}^{-n}(X \wedge S^1) \oplus \tilde{K}^{-n}(X) \oplus \tilde{K}^{-n}(S^1) \\ &\simeq \tilde{K}^{-(n+1)}(X) \oplus \tilde{K}^{-n}(X) \oplus \tilde{K}^{-(n+1)}(\{\text{pt}\}) \end{aligned}$$

¹To be precise, the statement was on the homotopy groups of the unitary groups $U(k)$ in the *stable range*.

where we have used Bott periodicity.

Recalling that for odd n the reduced and unreduced K-theory groups coincide, we have that

$$\begin{aligned} K^{-2n}(X \times S^1) &\simeq K^{-1}(X) \oplus K^0(X) \\ K^{-(2n+1)}(X \times S^1) &\simeq K^0(X) \oplus K^{-1}(X) \end{aligned} \tag{3.4.4}$$

Let $T^n \simeq S^1 \times \cdots \times S^1$ be the standard n -dimensional torus. The K-theory for T^n can be computed using isomorphism (3.4.4) by induction with base $n = 2$.

Indeed, we have that

$$K^0(T^2) \simeq K^{-1}(T^2) \simeq K^0(S^1 \times S^1) \simeq \mathbb{Z} \oplus \mathbb{Z}$$

As $T^n \simeq T^{n-1} \times S^1$, we have

$$K^0(T^n) \simeq K^{-1}(T^n) \simeq K^{-1}(T^{n-1}) \oplus K^0(T^{n-1})$$

Then, by induction we find that there is the following non-canonical isomorphism for $n \geq 2$

$$K^0(T^n) \simeq K^{-1}(T^n) \simeq \mathbb{Z}^{2^{(n-1)}}$$

Hence, for tori the K-theory groups can indeed detect their dimension.

3.5 The Atiyah-Bott-Shapiro isomorphism

In this section we will introduce the *Atiyah-Bott-Shapiro* (ABS) isomorphism, which will give explicit representatives for the generators of the rings $K^*(\{\text{pt}\})$ and $KO^*(\{\text{pt}\})$ via the so called “difference bundle construction”. More importantly, the ABS isomorphism relates complex and real Clifford algebras to K-theory: such a relation is somehow expected, given that the periodicity of K-theory is similar to the periodicity of Clifford algebras. The main reference is the seminal paper [5]; moreover, we will refer to Appendix B for the basic notions of Clifford algebras.

To construct the ABS isomorphism, we need a reformulation of the relative K-theory group $K^0(X, Y)$, which will also prove useful in relation to elliptic operators, and in the description of D-branes, in particular in type IIA String theory. Again, this reformulation is due to [5], and there is no difference between the complex and real case.

Definition 3.14. Let X and Y be CW-complexes of finite type. For $n \geq 1$, denote with $\mathcal{L}_n(X, Y)$ the set of elements $\mathbf{E} = (E_0, E_1, \dots, E_n; \sigma_1, \sigma_2, \dots, \sigma_n)$, where E_i is a

vector bundle over X , $\sigma_i : E_{i-1}|_Y \rightarrow E_i|_Y$ is a bundle morphism defined on Y , such that

$$0 \rightarrow E_0|_Y \xrightarrow{\sigma_1} E_1|_Y \xrightarrow{\sigma_2} \dots \xrightarrow{\sigma_n} E_n|_Y \rightarrow 0$$

is an exact sequence of vector bundles.

We will say that two such elements \mathbf{E} and \mathbf{E}' are *isomorphic* if there are bundle isomorphisms $\varphi_i : E_i \rightarrow E'_i$ over X such that the diagram

$$\begin{array}{ccc} E_{i-1}|_Y & \xrightarrow{\sigma_i} & E_i|_Y \\ \varphi_{i-1} \downarrow & & \downarrow \varphi_i \\ E'_{i-1}|_Y & \xrightarrow{\sigma'_i} & E'_i|_Y \end{array}$$

commutes for every i .

Finally, an element $\mathbf{E} = (E_0, E_1, \dots, E_n; \sigma_1, \sigma_2, \dots, \sigma_n)$ is said to be *elementary* if there is an i such that

- a) $E_i = E_{i-1}$ and $\sigma_i = \text{id}$
- b) $E_j = \{0\}$, for $j \neq i$ or $i - 1$

The Whitney sum \oplus of vector bundles induces naturally an operation on $\mathcal{L}_n(X, Y)$. We define the equivalence relation \sim on $\mathcal{L}_n(X, Y)$ generated by isomorphisms and addition of elementary elements. Namely, we will say that two elements \mathbf{E}, \mathbf{E}' are equivalent if there are elementary elements $\mathbf{P}_1, \mathbf{P}_2, \dots, \mathbf{P}_k, \mathbf{Q}_1, \mathbf{Q}_2, \dots, \mathbf{Q}_l$ in $\mathcal{L}_n(X, Y)$ such that

$$\mathbf{E} \oplus \mathbf{P}_1 \oplus \dots \oplus \mathbf{P}_k \simeq \mathbf{E}' \oplus \mathbf{Q}_1 \oplus \dots \oplus \mathbf{Q}_l$$

The set of all equivalence classes in $\mathcal{L}_n(X, Y)$ under \sim is denoted by $L_n(X, Y)$, and is an abelian group under the operation \oplus . Moreover, if $Y = \emptyset$, we will use the notation $L_n(X)$.

Consider the natural map $L_n(X, Y) \rightarrow L_{n+1}(X, Y)$ which associates to the element $(E_0, E_1, \dots, E_n; \sigma_1, \sigma_2, \dots, \sigma_n)$ the element $(E_0, E_1, \dots, E_n, 0; \sigma_1, \sigma_2, \dots, \sigma_n, 0)$.

We refer to [5] for the proof of the following fundamental result

Proposition 3.15. *For each $n \geq 1$, the induced map $L_n(X, Y) \rightarrow L_{n+1}(X, Y)$ is an isomorphism.*

Hence we can focus on the group $L_1(X, Y)$, whose elements are given by triples $\mathbf{E} = [E_0, E_1; \sigma]$.

The *difference bundle construction* allows to associate to any triple \mathbf{E} an element $\chi(\mathbf{E})$ in $K^0(X, Y)$ in the following way.

First, set $X_k = X \times \{k\}$, for $k = 0, 1$, and consider the space $A = X_0 \cup_Y X_1$, obtained from the disjoint union $X_0 \amalg X_1$ by identifying $y \times 0$ with $y \times 1$ for any $y \in Y$.

Notice that the map

$$\rho : A \rightarrow X_1$$

is a retraction, i.e. $i \circ \rho = id$, $i : X_1 \hookrightarrow A$. In this case, the exact sequence (3.2.3) becomes the split short exact sequence [4]

$$0 \rightarrow K^0(A, X_1) \rightarrow K^0(A) \xrightarrow{i^*} K^0(X_1) \rightarrow 0 \quad (3.5.1)$$

Moreover, the relative map $(X, Y) \rightarrow (A, X_1)$ which identifies X with X_0 induces an isomorphism $\varphi : K^0(A, X_1) \xrightarrow{\cong} K^0(X, Y)$.

Now, from the element $\mathbf{E} = [E_0, E_1; \sigma]$ we construct (up to isomorphism) a vector bundle over A by setting $F|_{X_k} := E_k$, and using σ to identify over Y . Recall, at this point, that the map i^* is given essentially by restricting vector bundles from A to X_1 . Hence, setting $F_1 := \rho^*E_1$, the class $[F] - [F_1]$ is in $\ker(i^*) \subset K^0(A)$. By (3.5.1), there exists a unique element $\chi(\mathbf{E}) \in K^0(X, Y)$ such that

$$\pi^* \varphi^{-1} \chi(\mathbf{E}) = [F] - [F_1]$$

In this way we have defined a homomorphism $\chi : L_1(X, Y) \rightarrow K^0(X, Y)$.

The following result allows the desired reformulation of relative K-theory [5]

Proposition 3.16. *The map χ is an isomorphism.*

The above proposition is easy to prove in the case in which $Y = \emptyset$. In this case, indeed, the map χ satisfies

$$\chi([E_0, E_1]) = [E_0] - [E_1]$$

The surjectivity of χ is obvious.

Suppose that $\chi([E_0, E_1]) = 0$: then there exists a vector bundle G such that $E_0 \oplus G \simeq E_1 \oplus G$. Hence the element $\mathbf{E} \oplus \mathbf{G}$, where \mathbf{G} is the elementary sequence defined by G , is isomorphic to the elementary sequence defined by $E_1 \oplus G$, and hence it represents 0 in $L_1(X)$.

With this reformulation of relative K-theory, we can describe the ABS isomorphism. Again, we refer the reader to Appendix B for the relevant notions of modules of Clifford algebras.

Let D^n denote the unit disk in \mathbb{R}^n , and S^{n-1} the boundary ∂D^n .

Let $W = W^0 \oplus W^1$ be a \mathbb{Z}_2 -graded module over the Clifford algebra $Cl_n := Cl(\mathbb{R}^n)$. To the module W we can associate the element

$$\varphi(W) = [E_0, E_1; \mu] \in K^0(D^n, S^{n-1}) \quad (3.5.2)$$

where $E_k := D^n \times W^k$, and μ is the bundle isomorphism $E_0 \rightarrow E_1$ defined over S^{n-1} by Clifford multiplication

$$\mu(x, v) := (x, x \cdot v)$$

As the element $\varphi(W)$ depends only on the isomorphism class of W , and the map $W \rightarrow \varphi(W)$ is an additive homomorphism, it follows that (3.5.2) induces a homomorphism

$$\varphi : \widehat{\mathfrak{M}}_n^{\mathbb{C}} \rightarrow K^0(D^n, S^{n-1}) \quad (3.5.3)$$

where $\widehat{\mathfrak{M}}_n^{\mathbb{C}}$ is the abelian group freely generated by the irreducible complex graded Cl_n -modules.

Consider now the homomorphism $i^* : \widehat{\mathfrak{M}}_{n+1}^{\mathbb{C}} \rightarrow \widehat{\mathfrak{M}}_n^{\mathbb{C}}$ induced by restricting the action from Cl_{n+1} to Cl_n . Now, let W be a graded Cl_n -module obtained from a Cl_{n+1} -module by restriction. Then, the isomorphism μ defined on S^{n-1} can be extended to all of D^n by setting

$$\tilde{\mu}(x, v) := (x, (x + \sqrt{1 - \|x\|^2} e_{n+1}) \cdot v)$$

where $e_{n+1} \in \mathbb{R}^{n+1}$ is a unit vector orthogonal to \mathbb{R}^n . As E_0 and E_1 are isomorphic bundles over D^n , the element $[E_0, E_1; \mu]$ is 0 in $K^0(D^n, S^{n-1})$.

Hence, the map (3.5.3) descends to the homomorphism

$$\varphi_n : \widehat{\mathfrak{M}}_n^{\mathbb{C}} / i^* \widehat{\mathfrak{M}}_{n+1}^{\mathbb{C}} \rightarrow K^0(D^n, S^{n-1}) \quad (3.5.4)$$

In complete analogy, in the real case we have the homomorphism

$$\varphi_n^{\mathbb{R}} : \widehat{\mathfrak{M}}_n^{\mathbb{R}} / i^* \widehat{\mathfrak{M}}_{n+1}^{\mathbb{R}} \rightarrow KO^0(D^n, S^{n-1}) \quad (3.5.5)$$

Recalling that

$$K^0(D^n, S^{n-1}) = \widetilde{K}^0(D^n/S^{n-1}) \simeq \widetilde{K}^0(S^n)$$

Denote with

$$\widehat{\mathfrak{M}}_*^{\mathbb{C}} / i^* \widehat{\mathfrak{M}}_{*+1}^{\mathbb{C}} := \bigoplus_{n \geq 0} \widehat{\mathfrak{M}}_n^{\mathbb{C}} / i^* \widehat{\mathfrak{M}}_{n+1}^{\mathbb{C}}$$

the graded ring induced by the graded tensor product of Clifford modules, and the same for the real case.

Finally, we can state the following fundamental result in K-theory

Theorem 3.17. Atiyah-Bott-Shapiro Isomorphisms. *The maps φ_n and $\varphi_n^{\mathbb{R}}$ induce graded ring isomorphisms*

$$\begin{aligned}\varphi_* &: \widehat{\mathfrak{M}}_*^{\mathbb{C}}/i^*\widehat{\mathfrak{M}}_{*+1}^{\mathbb{C}} \rightarrow K^{-*}(\{\text{pt}\}) \\ \varphi_*^{\mathbb{R}} &: \widehat{\mathfrak{M}}_*^{\mathbb{R}}/i^*\widehat{\mathfrak{M}}_{*+1}^{\mathbb{R}} \rightarrow KO^{-*}(\{\text{pt}\})\end{aligned}$$

As the periodicity of the quotients $\widehat{\mathfrak{M}}_*^{\mathbb{C}}/i^*\widehat{\mathfrak{M}}_{*+1}^{\mathbb{C}}$ and $\widehat{\mathfrak{M}}_*^{\mathbb{R}}/i^*\widehat{\mathfrak{M}}_{*+1}^{\mathbb{R}}$ can be deduced by the algebraic properties of Clifford algebras, it may seem that the above theorem gives an algebraic proof of the Bott periodicity theorems. This is not the case, as the proof of the above theorem in [5] actually invokes the Bott result.

The ABS isomorphisms can now be used to obtain explicit representatives for $K^{-*}(\{\text{pt}\})$ and $KO^{-*}(\{\text{pt}\})$. Namely, consider $S_{\mathbb{C}} = S_{\mathbb{C}}^+ \oplus S_{\mathbb{C}}^-$, the fundamental \mathbb{Z}_2 -graded complex representation of Cl_{2n} . Then, by fundamental results in the theory of Clifford algebras, the group $\widehat{\mathfrak{M}}_{2n}^{\mathbb{C}} \simeq \mathbb{Z} \oplus \mathbb{Z}$ is generated by $S_{\mathbb{C}}$ and $\tilde{S}_{\mathbb{C}}$, the graded module obtained by interchanging the factors in $S_{\mathbb{C}}$.

Moreover, the homomorphism i^* maps the generator of the group $\widehat{\mathfrak{M}}_{2n+1}^{\mathbb{C}} \simeq \mathbb{Z}$ to $(S_{\mathbb{C}}, \tilde{S}_{\mathbb{C}}) \in \mathbb{Z} \oplus \mathbb{Z}$. Hence, the generator for $K^{-2n}(\{\text{pt}\}) \simeq K^0(D^{2n}, S^{2n-1})$ is given by the element

$$\sigma_{2n}^{\mathbb{C}} := [S_{\mathbb{C}}^+, S_{\mathbb{C}}^-; \mu]$$

In the case $n = 1$, $\sigma_2^{\mathbb{C}}$ is mapped by the isomorphism $K^0(D^2, S^1) \simeq \tilde{K}^0(S^2)$ to the class $[H] - [\theta_1]$, where H is the tautological complex line bundle over $S^2 \simeq \mathbb{C}P^1$.

Due to the presence of torsion, the generators for the ring $KO^{-*}(\{\text{pt}\})$ are more complicated to describe. As an example, we will consider the case $n = 1$, where we have $KO^{-1}(\{\text{pt}\}) \simeq \mathbb{Z}_2$.

First, recall that to any graded module $W = W^0 \oplus W^1$ for Cl_n we can assign the *ungraded* module W^0 for the Clifford algebra $Cl_n^0 \simeq Cl_{n-1}$, where Cl_n^0 denotes the even part of Cl_n . The converse is also true: given an ungraded module W^0 for the Clifford algebra Cl_{n-1} , the module

$$W := Cl_n \otimes_{Cl_n^0} W^0$$

is naturally a graded module for the Clifford algebra Cl_n .

This induces the isomorphism $\widehat{\mathfrak{M}}_n \simeq \mathfrak{M}_{n-1}$, where \mathfrak{M}_{n-1} denotes the ungraded version of $\widehat{\mathfrak{M}}_{n-1}$, and consequently the isomorphism

$$\widehat{\mathfrak{M}}_n/i^*\widehat{\mathfrak{M}}_{n+1} \simeq \mathfrak{M}_{n-1}/i^*\mathfrak{M}_n$$

In the case $n = 1$, $Cl_0 \simeq \mathbb{R}$ and $Cl_1 \simeq \mathbb{C}$: taking the real and complex dimension of the ungraded modules gives the isomorphisms $\mathfrak{M}_0 \simeq \mathbb{Z}$ and $\mathfrak{M}_1 \simeq \mathbb{Z}$, respectively.

The map $i^* : \mathfrak{M}_1 \rightarrow \mathfrak{M}_0$ is given by considering a complex vector space to be a real vector space under restriction of scalars. It induces a homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$ given by multiplication by 2, hence $\mathfrak{M}_0/i^*\mathfrak{M}_1 \simeq \mathbb{Z}_2$. Then, the generator for $KO^{-1}(\{\text{pt}\})$ is given by the element in $KO^0(D^1, S^0)$

$$\sigma_1 := [D^1 \times \mathbb{R}, D^1 \times \mathbb{R}; \mu]$$

where

$$\begin{aligned} \mu(1, v) &= v \\ \mu(-1, v) &= -v \end{aligned}$$

Moreover, $\sigma_1 = [H_1] - [\theta_1] \in \widetilde{KO}^0(S^1)$, where H_1 is the *infinite Möbius bundle* over the circle, i.e. the tautological real line bundle over $\mathbb{R}P^1 \simeq S^1$.

3.6 K-theory and Spin^c manifolds

In this section, we will specialize to the case in which the CW-complex X is a smooth finite dimensional manifold. Being a topological invariant, K-theory does not depend on the presence of any smooth structure: nevertheless, restricting to manifolds provides a strong connection between K-theory and the theory of elliptic operators, and, moreover, it is the right framework for the description of D-branes in String theory. Again, all the following results for which a smooth structure is not explicitly required, are to be considered valid in a more general context.

Before introducing the notions of K-orientation and the Thom isomorphism, we need some basic preliminary results.

In the following we will consider the K-theory of total spaces of vector bundles, which are locally compact spaces. For a locally compact space X we define

$$K_{cpt}^0(X) := \widetilde{K}^0(X^+)$$

and

$$K_{cpt}^{-n}(X) := K_{cpt}^0(X \times \mathbb{R}^n)$$

The functors K_{cpt}^{-n} are defined on the category of locally compact spaces and proper maps, and they constitute the *K-theory with compact support*.

Moreover, we have that

$$K_{cpt}^{-n}(X, Y) := K_{cpt}^0((X - Y) \times \mathbb{R}^n)$$

Analogous definitions can be given for KO-theory, and the Bott periodicity theorems assume the form

$$\begin{aligned} K_{cpt}^0(X) &\simeq K_{cpt}^0(X \times \mathbb{C}) \\ KO_{cpt}^0(X) &\simeq KO_{cpt}^0(X \times \mathbb{R}^8) \end{aligned}$$

Now, let X be a compact space, and let $E \rightarrow X$ be a complex vector bundle. The one point compactification E^+ is called the *Thom space*, or *Thom complex* of E .

By definition, $\tilde{K}^0(E^+) = K_{cpt}^0(E)$.

Consider the (unit) *ball* and *sphere* bundle of E , denoted as $B(E)$ and $S(E)$, respectively, and defined as

$$\begin{aligned} B(E) &:= \{e \in E : \varphi(e, e) \leq 1\} \\ S(E) &:= \{e \in E : \varphi(e, e) = 1\} \end{aligned}$$

for φ a Hermitian metric on E , and the projection is the obvious one.

By noticing that $B(E) - S(E) \simeq E$, we have

$$B(E)/S(E) \simeq E^+$$

and consequently

$$K_{cpt}^0(E) := \tilde{K}^0(E^+) \simeq \tilde{K}^0(B(E)/S(E)) := K^0(B(E), S(E))$$

The isomorphism above does not depend on the choice of the Hermitian metric φ , and a similar construction follows for KO-theory.

In the following, we will often interchange $K_{cpt}^0(E)$ and $K^0(B(E), S(E))$ freely using the isomorphism above.

3.6.1 K-orientation and Thom isomorphism

In the same way that an orientation of a vector bundle $E \rightarrow X$ induces a class $\tau \in H_{cpt}^*(E)$ and a ring isomorphism $H_{cpt}^*(E) \simeq H^*(X)$ via cup product with τ , the existence of certain structures on vector bundles will induce analogous results for K- and KO-theory.

Let $\pi : E \rightarrow X$ be a complex or real vector bundle, and denote with k^* the functor K_{cpt}^* or KO_{cpt}^* , or, more generally, any multiplicative generalized cohomology theory. Define the map

$$j^* : k^*(X \times E) \rightarrow k^*(E)$$

induced by the map $j(e) = (\pi(e), e)$.

Then, the composition

$$k^*(X) \times k^*(E) \xrightarrow{\cup} k^*(X \times E) \xrightarrow{j^*} k^*(E)$$

canonically equips $k^*(E)$ with a structure of left module over $k^*(X)$.

In the following we will give a somewhat general definition of orientation for k^* , as it will become important later in the context of K-homology.

Definition 3.18. An n -dimensional vector bundle $\pi : E \rightarrow X$ is said to be k^* -orientable if there exists a class $\tau \in k^n(E)$ such that $i_x^* \tau$ is a generator of $k^n(E_x)$ for each fiber inclusion $i_x : E_x \hookrightarrow E$. A choice of such a class τ is called a k^* -orientation for π , and π is said to be k^* -oriented by τ .

For a k^* -oriented vector bundle one has the following fundamental general result [38, 88]

Theorem 3.19. Let $\pi : E \rightarrow X$ be an n -dimensional vector bundle which is k^* -oriented by the class $\tau \in k^n(E)$. Then the homomorphism

$$\mathfrak{T}_{X,E} : k^i(X) \rightarrow k^{i+n}(E)$$

defined by

$$\mathfrak{T}_{X,E}(\xi) := \pi^*(\xi) \cup \tau$$

is an isomorphism.

The isomorphism $\mathfrak{T}_{X,E}$ is called the *Thom isomorphism*. As a corollary of theorem (3.19), we have that $k^*(E)$ is a free k^* -module with generator τ , and hence is completely known once $k^*(X)$ is.

Given a vector bundle, one is interested in finding *sufficient*, if not necessary conditions for orientation classes to exist, and, more generally, to investigate which are the possible obstructions to such an existence.

For this, consider the trivial bundle

$$E = X \times \mathbb{C}^m \xrightarrow{\pi} X$$

and consider the class

$$\tau(E) := [\Lambda_{\mathbb{C}}^{even} \underline{\mathbb{C}}^m, \Lambda_{\mathbb{C}}^{odd} \underline{\mathbb{C}}^m; \sigma] \in K^0(B(E), S(E))$$

where $\underline{\mathbb{C}}^m = \pi^* E$, and where

$$\sigma_{(x,v)}(\varphi) := v \wedge \varphi - i_{v^*} \varphi$$

with i_{v^*} given by contraction. See Appendix B for details.

Now, the restriction of $\tau(E)$ to the fiber is given by the class

$$[\Lambda_{\mathbb{C}}^{even} \mathbb{C}^m, \Lambda_{\mathbb{C}}^{odd} \mathbb{C}^m; \sigma] \in K_{cpt}^0(\mathbb{C}^m) = K^0(D^{2m}, S^{2m-1})$$

where we have identified $\mathbb{C}^m \simeq \mathbb{R}^{2m}$.

Upon the identification $\Lambda^*\mathbb{C}^m \simeq Cl_{2m}$, and noticing that σ is given by Clifford multiplication by the element v , the class above can be rewritten as

$$[S_{\mathbb{C}}^+, S_{\mathbb{C}}^-; \mu]$$

which by the ABS isomorphism is a generator of $K^0(D^{2m}, S^{2m-1})$.

As any complex vector bundle E is locally trivializable, and there is no obstruction to construct $\Lambda_{\mathbb{C}}^*E$ globally, we have the following

Theorem 3.20. *Let $\pi : E \rightarrow X$ be a complex hermitian vector bundle over a compact space X . Then the class*

$$\tau_K(E) := [\Lambda_{\mathbb{C}}^{even} \pi^*E, \Lambda_{\mathbb{C}}^{odd} \pi^*E; \sigma] \in K_{cpt}^0(E)$$

where $\sigma_e(\varphi) := e \wedge \varphi - i_{e^*}\varphi$, is a K^* -orientation class for π .

Hence, any complex vector bundle is K^* -orientable, and there is no obstruction present.

The case for real vector bundles is instead different. Indeed, consider the $8n$ -dimensional real trivial vector bundle

$$E = X \times \mathbb{R}^{8n} \xrightarrow{\pi} X$$

and consider the class

$$\tau_{KO}(E) := [\pi^*S^+(E), \pi^*S^-(E); \mu] \in KO_{cpt}^0(E)$$

where $S^+(E) \oplus S^-(E) = S(E) = X \times W$, with W the irreducible real graded Cl_{8n} -module, and μ is Clifford multiplication. Again, by Bott periodicity, the class $\tau_{KO}(E)$ gives the generator of $KO_{cpt}^0(\mathbb{R}^{8n}) = KO^0(D^{8n}, S^{8n-1})$, when restricted to the fiber.

Given an arbitrary real vector bundle E , the bundle $S(E)$ will exist if and only if E admits a spin structure.

Hence, we have the following

Theorem 3.21. *Let $\pi : E \rightarrow X$ be a real $8n$ -dimensional vector bundle with a spin structure over a compact space X . Then the class*

$$\tau_{KO}(E) := [\pi^*S^+(E), \pi^*S^-(E); \mu] \in KO_{cpt}^0(E)$$

where $\mu_e(\varphi) = e \cdot \varphi$ is Clifford multiplication, is a KO^* -orientation for π .

In the same way, the following theorem gives sufficient conditions for a real vector bundle to be K*-orientable.

Theorem 3.22. *Let $\pi : E \rightarrow X$ a real $2n$ -dimensional vector bundle with a spin^c structure over a compact space X . Then the class*

$$\tau_K(E) := [\pi^*S_{\mathbb{C}}^+(E), \pi^*S_{\mathbb{C}}^-(E); \mu] \in K_{\text{cpt}}^0(E)$$

where $S_{\mathbb{C}} = S_{\mathbb{C}}^+ \oplus S_{\mathbb{C}}^-$ is the irreducible spinor bundle associated to the spin^c structure on E , and $\mu_e(\varphi) = e \cdot \varphi$ is Clifford multiplication. Then $\tau_K(E)$ is a K*-orientation for π .

Thanks to the results above, we can construct an orientation class for a spin^c vector bundle of arbitrary rank in the following way, with the spin case requiring minor modifications.

Let $\pi : E \rightarrow X$ be a spin^c vector bundle of rank n , and consider the Whitney sum $E \oplus \theta_p$, with p such that $n + p = 2m$.

If we equip θ_p with its canonical spin^c structure, then the sum $E \oplus \theta_p$ is a spin^c vector bundle of rank $2m$.

Hence, by Theorem 3.22 there exists a Thom class

$$\tau_K(E \oplus \theta_p) \in K^0(B(E \oplus \theta_p), S(E \oplus \theta_p))$$

By using the following isomorphisms [38, 88]

$$\begin{aligned} K^0(B(E \oplus \theta_p), S(E \oplus \theta_p)) &:= \widetilde{K}^0(B(E \oplus \theta_p)/S(E \oplus \theta_p)) \\ &\simeq \widetilde{K}^0(\Sigma^p(B(E)/S(E))) \\ &\simeq K^{-p}(B(E), S(E)) \\ &\simeq K^n(B(E), S(E)) \end{aligned}$$

it follows that the class $\tau_K(E \oplus \theta_p)$ is an orientation class for the vector bundle π .

It is important to notice that, in contrast to the results in Theorem 3.22, the construction of the Thom class above is *not* natural, as we have to make several choices in the process, starting from that of a spin^c structure on $E \oplus \theta_p$. Nevertheless, the induced Thom isomorphism will be an essential ingredient in the next chapter, where we will discuss a more natural description of D-branes in terms of K-homology.

3.6.2 Chern Character and Gysin homomorphism

In this section we will briefly introduce the homomorphisms induced by the Chern character and the Thom isomorphism. We refer to Appendix C for the basic notions

on characteristic classes for vector bundles.

Let $E \rightarrow X$ be a complex vector bundle. The Chern character $\text{ch}(E)$ is an element in the rational cohomology $H^{ev}(X; \mathbb{Q})$ constructed from the total Chern class of E . The importance of the Chern character lies in the fact that it respects the (semi)additive and multiplicative structure on $\text{Vect}(X)$. Namely, for E and E' vector bundles on X we have

$$\begin{aligned}\text{ch}(E \oplus E') &= \text{ch}(E) + \text{ch}(E') \\ \text{ch}(E \otimes E') &= \text{ch}(E) \cup \text{ch}(E')\end{aligned}$$

Hence, we can define the following homomorphism

$$\begin{aligned}\text{ch} : K^0(X) &\rightarrow H^{ev}(X; \mathbb{Q}) \\ [E] - [F] &\rightarrow \text{ch}(E) - \text{ch}(F)\end{aligned}\tag{3.6.1}$$

It is easy to show that the Chern character homomorphism (3.6.1) for a class x does not depend on its representation in terms of vector bundles, hence it is a well defined ring homomorphism.

The Chern character homomorphism can be extended to a group homomorphism

$$\text{ch} : K^{-n}(X, Y) \rightarrow H^*(X, Y; \mathbb{Q})$$

by defining

$$\text{ch}(x) := \alpha((\sigma^n)^{-1}\text{ch}(x')) \quad x \in K^{-n}(X, Y)\tag{3.6.2}$$

where x' is the class corresponding to x in $\tilde{K}^0(\Sigma^n(X/Y))$, σ^n is the suspension isomorphism in cohomology, and α is the canonical isomorphism

$$\alpha : \tilde{H}^*(X/Y; \mathbb{Q}) \xrightarrow{\cong} H^*(X, Y; \mathbb{Q})$$

Moreover, one can prove that the homomorphism (3.6.2) is compatible with Bott periodicity: this allows to define $\text{ch}(x)$ for $x \in K^n(X, Y)$, where n is any integer.

Finally, by using a spectral sequence argument, the following fundamental result was proven in [7]

Theorem 3.23. *Let X be a finite CW-complex. Then the homomorphism*

$$\text{ch} \otimes id : K^*(X) \otimes \mathbb{Q} \rightarrow H^*(X; \mathbb{Q})$$

is an isomorphism and maps $K^0(X) \otimes \mathbb{Q}$ onto $H^{ev}(X; \mathbb{Q})$ and $K^1(X) \otimes \mathbb{Q}$ onto $H^{odd}(X; \mathbb{Q})$.

As a corollary of Theorem 3.23, when X is a finite CW-complex, $K^*(X)$ is a finitely generated abelian group. This technical simplification in a sense justifies the

restriction to CW-complexes.

In the case in which X is a smooth manifold, the Chern character homomorphism is closely related to the Gysin homomorphism, which we will illustrate in the following. Consider two locally compact manifolds X and Y , such that $\dim Y - \dim X = 0 \pmod{2}$, and let $f : X \rightarrow Y$ be a proper embedding². Denote with ν_X the normal bundle of X in Y , and suppose it is equipped with a spin^c structure, which is a restriction on the map f . Moreover, identify this normal bundle with a tubular neighbourhood N of X in Y . Then, we can define the *Gysin homomorphism*

$$f_* : K_{cpt}^0(X) \rightarrow K_{cpt}^0(Y)$$

as the composition

$$K_{cpt}^0(X) \xrightarrow{\cong_{X, \nu_X}} K_{cpt}^0(\nu_X) \simeq K_{cpt}^0(N) \xrightarrow{j^*} K_{cpt}^0(Y)$$

where the map j^* is induced by the map $j : Y^+ \rightarrow N^+$ defined as

$$\begin{aligned} j(z) &= z, \forall z \in N \\ j(z) &= \infty, \forall z \notin N \end{aligned}$$

with ∞ denoting the point that compactifies N . Intuitively, j^* extends a class in $K_{cpt}^0(N)$ via trivial bundles, and is usually referred to as “extension by zero”.

The Gysin homomorphism is also called the “wrong way” morphism, as it “goes” in the direction opposite to the contravariance of the K-functor.

The Gysin homomorphism does not depend on the tubular neighbourhood N , and only depends on the homotopy class of f , as the vector bundle ν_X is defined as $f^*(TY)/TX$, and the K-functor is homotopy invariant.

Moreover, the Gysin homomorphism enjoys the following functoriality property, which will be very useful in later chapters.

Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be proper embeddings satisfying the hypothesis above. Then

$$(g \circ f)_* = g_* f_* \tag{3.6.3}$$

As the Thom isomorphism can be extended to higher K-groups, the Gysin homomorphism can be extended to a homomorphism

$$f_* : K_{cpt}^{-1}(X) \rightarrow K_{cpt}^{-1}(Y)$$

²The embedding condition can be relaxed. See [61].

Moreover, by recalling the discussion on orientation classes in section 3.6.1, we can relax the condition that $\dim Y - \dim X = 0 \pmod{2}$. Indeed, in general, we will have a Gysin homomorphism

$$f_* : K_{cpt}^i(X) \rightarrow K_{cpt}^{i+n}(Y)$$

where n is the rank of the normal bundle. Again, the price we pay is that the homomorphism f_* so obtained does not induce a natural transformation.

Finally, along the same lines, one can define a Gysin homomorphism in ordinary cohomology

$$f_*^H : H^*(X; \mathbb{Q}) \rightarrow H^*(Y; \mathbb{Q})$$

As mentioned before, the Chern character homomorphism and the Gysin homomorphism are closely related via the Atiyah-Hirzebruch version of the Riemann-Roch theorem [6].

Consider two locally compact manifolds X and Y and a proper embedding $f : X \rightarrow Y$ as above, and denote with $d(\nu_X)$ the cohomology class defining the spin^c structure on the normal bundle ν_X . Then we have the following

Theorem 3.24. (*Riemann-Roch*) *For each class $x \in K^0(X)$ we have the relation*

$$\text{ch}(f_*(x)) = f_*^H(e^{d(\nu_X)/2} \hat{A}(\nu_X) \text{ch}(x))$$

This form of the Riemann-Roch theorem will be used in the next section, where we will see how the K-theoretic machinery developed so far is related to D-branes in String theory.

3.7 K-theory and type IIA/B D-branes

Having developed the necessary notions in the previous sections, we will describe how K-theory is related to D-branes in superstring theory. In particular, in this section we will consider type II String theory on a 10-dimensional spin manifold M , while in the next section we will investigate type I String theory.

We will present two supporting arguments for the K-theoretic description of D-branes, based on the fact that D-branes are currents for Ramond-Ramond fields, and on Sen's conjectures, respectively.

We first need some comments on the anomalous coupling (2.5.7) between the total Ramond-Ramond potential and a D-brane. Recall that in section 2.4 we made the assumption that the worldvolume Q wrapped by the D-brane is a spin manifold, pointing out that this condition is *not* necessary from a string theoretic point of view.

In presence of a D-brane the partition function of the superstring develops an anomaly, called the *Freed-Witten anomaly* [45]: in a topologically trivial B-field setting, this anomaly cancels exactly when the normal bundle of Q in M admits a spin^c structure. Since M is a spin manifold, the manifold Q necessarily admits a spin^c structure. Moreover, the anomalous coupling is modified as

$$\int_Q i^* C \wedge e^{d(\nu_Q)/2} \text{ch}(E) i^* \sqrt{\hat{\mathcal{A}}(\text{TM})} \frac{1}{\hat{\mathcal{A}}(\text{TQ})} \quad (3.7.1)$$

which coincides with the coupling (2.5.7) when Q is spin manifold, as in this case the class $d(\nu_Q)$ vanishes.

Consider the cohomology class

$$j_Q = e^{d(\nu_Q)/2} \text{ch}(E) i^* \sqrt{\hat{\mathcal{A}}(\text{TM})} \frac{1}{\hat{\mathcal{A}}(\text{TQ})}$$

As we have discussed in section 2.5, the class j_Q represents the charge associated to the Ramond-Ramond current generated by the D-brane wrapping Q . More precisely, for such an interpretation to be valid, the class j_Q needs to be “pushed” into the spacetime manifold M as

$$i_*^H(j_Q) \quad (3.7.2)$$

where we recall that $i : Q \rightarrow M$ is the embedding map. By noticing that

$$\nu_Q \simeq i^* \text{TM} / \text{TQ}$$

and using the fact that the roof genus is a characteristic class, we have the identity

$$\hat{\mathcal{A}}(\nu_Q) = i^* \hat{\mathcal{A}}(\text{TM}) / \hat{\mathcal{A}}(\text{TQ})$$

Hence, we can rewrite (3.7.2) as

$$i_*^H \left(e^{d(\nu_Q)/2} \hat{\mathcal{A}}(\nu_Q) \text{ch}(E) \frac{1}{i^* \sqrt{\hat{\mathcal{A}}(\text{TM})}} \right)$$

By using the following property of the cohomological Gysin homomorphism

$$i_*^H(\alpha \cup i^* \beta) = i_*^H(\alpha) \cup \beta, \quad \forall \alpha \in H^*(Q; \mathbb{Q}), \forall \beta \in H^*(M; \mathbb{Q})$$

we have that

$$i_*^H \left(e^{d(\nu_Q)/2} \hat{\mathcal{A}}(\nu_Q) \text{ch}(E) \frac{1}{i^* \sqrt{\hat{\mathcal{A}}(\text{TM})}} \right) = i_*^H \left(e^{d(\nu_Q)/2} \hat{\mathcal{A}}(\nu_Q) \text{ch}(E) \right) \cup \frac{1}{\sqrt{\hat{\mathcal{A}}(\text{TM})}}$$

Finally, by using the Riemann-Roch theorem we have

$$i_*^H(j_Q) = \text{ch}(i_*[E]) \cup \frac{1}{\sqrt{\hat{\mathcal{A}}(\text{TM})}} \quad (3.7.3)$$

The identity (3.7.3) strongly suggests that the charge for a D-brane wrapping a world-volume Q equipped with a Chan-Paton bundle E is naturally given by the element $i_*[E] \in K^0(M)$.

This argument was developed in [71], and constitutes the first evidence for the importance of K-theory in String theory. However, the argument is strictly speaking only valid over \mathbb{Q} , where the map

$$\text{ch} \cup \frac{1}{\sqrt{\hat{\mathcal{A}}(\text{TM})}} : K^*(M) \otimes \mathbb{Q} \rightarrow H^*(M; \mathbb{Q})$$

is a group isomorphism. In this case, then, the cohomological description of D-branes “coincides” with the K-theoretical one, with the latter having the advantage of being more natural.

An argument which relates the full K-theory groups to D-brane charges is based on Sen’s conjectures, as discussed in section 2.6: of course, the price we pay consists in the fact that we have to invoke (open) string tachyon condensation, which is not yet fully understood, both from the physical and the mathematical point of view.

Recall from section 2.6 that Witten made the observation that in type IIB String theory, a configuration of n D9-branes and n $\overline{\text{D9}}$ -branes (E, F) has to be considered equivalent to the configuration $(E \oplus H, F \oplus H)$, because the brane-antibrane system (H, H) is able to decay in the string vacuum, as conjectured by Sen.

As we have seen in this chapter, the equivalence classes of configurations (E, F) under brane-antibrane annihilation are elements in $K^0(M)$, or more precisely $\tilde{K}^0(M)$, as the bundles E and F have the same rank.

Moreover, in [94] Witten was able to K-theoretically interpret Sen’s construction, which allows to obtain Dp -branes with $p < 9$ as the decay product of a system of brane-antibranes of higher dimension. According to this interpretation, the group of possible charges for a Dp -brane wrapping a spin^c submanifold $Q \subset M$ is given by

$$K_{cpt}^0(N) \simeq K^0(B(N), S(N))$$

with N denoting the tubular neighbourhood identifying the normal bundle to Q in M . In particular, given a single Dp -brane wrapping Q with trivial Chan-Paton bundle, the system of $D9$ - $\overline{\text{D9}}$ -brane decaying to Q is given by

$$i_*(1) \in K^0(M)$$

where 1 denotes the class of the trivial line bundle over Q .

Applying the above machinery to a flat Dp -brane wrapping $\mathbb{R}^{1,p} \subset M^{10}$, with M^{10} the ten dimensional Minkowski spacetime, we obtain that the its charge group is given by

$$K_{cpt}^0(B(N), S(N)) \simeq K_{cpt}^0(\mathbb{R}^{1,p} \times D^{9-p}, \mathbb{R}^{1,p} \times S^{9-p}) \simeq \tilde{K}^0(S^{9-p})$$

where we have used the compact support relative K-theory, as $\mathbb{R}^{1,p}$ is not compact. As $\tilde{K}^0(S^{9-p}) = 0$ for p even, we find that the K-theoretical description agrees with, and in a certain sense justifies the statement that odd-dimensional (flat) D-branes in type IIB String theory are necessarily unstable, as they cannot carry any D-brane charge.

The discussion for type IIA is less straightforward. Indeed, the fact that Sen's construction works in even codimension let Witten propose to relate branes not to bundles on M , but to bundles on $S^1 \times M$. Namely, given a Dp -brane wrapping an odd-dimensional submanifold $Q \subset M$, one identifies Q with $pt \times Q$ in $S^1 \times M$, where pt is any point in S^1 . Applying Sen's construction, the Dp -brane wrapping Q determines an element in $K^0(S^1 \times M)$, which turns out to be trivial when restricted to $pt \times M$. By using the isomorphism

$$K^{-1}(M) \simeq \ker [K^0(S^1 \times M) \rightarrow K^0(pt \times M)]$$

it follows that type IIA brane-antibrane configurations have to be considered equivalent if they determine the same element in $K^{-1}(M)$. Of course, the construction above is not natural, as we have to make a choice of how to embed Q in $S^1 \times M$. Similarly, the group of charges for a Dp -brane wrapping Q is given by

$$K_{cpt}^{-1}(B(N), S(N))$$

which agrees with the known results for flat Dp -branes in Minkowski spacetime.

From our point of view, the appearance of the functor K^{-1} can be motivated in the following way. As we have seen in section 3.6.2, a Gysin homomorphism for a map $i : X \rightarrow Y$ such that the normal bundle admits a $spin^c$ structure *does* always exist. Hence, given a D-brane wrapping an odd codimension manifold Q equipped with a trivial Chan-Paton bundle, its D-brane charge is given by the element

$$i_*(1) \in K_{cpt}^{-1}(M)$$

The classes in $K_{cpt}^{-1}(M)$ do not represent a brane-antibrane system: instead, they can be conveniently represented as a system of D9-branes equipped with the tachyon field causing the instability of D9-branes in Type IIA String theory. We refer the reader to [55] for details on this construction.

3.8 KO-theory and Type I D-branes: torsion effects

As we mentioned in chapter 1, Type I String theory is a theory of open and closed strings, including oriented as well as unoriented worldsheets. The Fock space for a String theory of unoriented worldsheets is obtained by considering states in the Type II Fock space which are invariant under the *worldsheet parity operator* Ω , induced by reversing the orientation on the worldsheet [79, 33].

The operator Ω induces an action also on the Chan-Paton bundle to a D-brane, forcing it to be an $O(n)$ -bundle. Moreover, the Ramond-Ramond field content in Type I differs from that of Type II: indeed, the Ramond-Ramond gauge theory in Type I String theory consists of p -forms, for $p=2,6$.

Finally, Freed-Witten anomaly cancellation imposes that a D-brane can only wrap a spin manifold.

The right framework to describe D-brane charges is then KO-theory. Namely, a configuration of Type I D9-branes (E, F) modulo brane-antibrane annihilation is represented by a class x in $KO^0(M)$.

Moreover, as in Type II String theory, given a Dp -brane wrapping a spin manifold $Q \subset M$, the group of its admissible charges is given by

$$KO^0(B(N), S(N))$$

Applying this classification to a Dp -brane wrapping $\mathbb{R}^{1,p} \subset M^{10}$, we obtain

$$KO_{cpt}^0(B(N), S(N)) \simeq KO_{cpt}^0(\mathbb{R}^{1,p} \times D^{9-p}, \mathbb{R}^{1,p} \times S^{9-p}) \simeq \widetilde{KO}^0(S^{9-p})$$

where

$$\widetilde{KO}^0(S^n) \simeq \begin{cases} \mathbb{Z}, & n = 0, 4 \bmod 8 \\ \mathbb{Z}_2, & n = 1, 2 \bmod 8 \\ 0, & \text{otherwise} \end{cases}$$

We have $\widetilde{KO}^0(S^{9-p}) \simeq \mathbb{Z}$, for $p = 1, 5, 9$: for $p = 1, 5$, this agrees with the fact that the corresponding supersymmetric D-branes couple to Ramond-Ramond fields, and hence are stable. Moreover, we would expect these to be the *only* stable D-branes in Type I String theory. A new prediction of K-theory is that this is not the case. Indeed, for $p = 7, 8$, we have $\widetilde{KO}^0(S^{9-p}) \simeq \mathbb{Z}_2$: hence, D7-branes and D8-branes can carry a charge that would protect them from decaying. Moreover, since the group \mathbb{Z}_2 is a pure torsion group, this charge is not associated to any spacetime field coupling

to the D-brane, and has to be considered as a purely topological effect.

The K-theoretic description also naturally describes stable objects for $p = -1$ and $p = 0$, which are called the *D-instanton* and the *D-particle*.

In contrast to Type II String theory, constructing a system of spacetime filling brane-antibranes which has a given Dp -brane as its decay product is not very systematic, in the sense that there is no unified procedure, or homomorphism, realizing this construction. Indeed, in [94] the different stable Dp -branes are treated with different methods. This is due to the fact that the Thom isomorphism in KO-theory implies that

$$\mathrm{KO}_{cpt}^0(\mathrm{B}(\mathrm{N}), \mathrm{S}(\mathrm{N})) \simeq \mathrm{KO}_{cpt}^{-p}(\mathrm{Q}) \quad (3.8.1)$$

where Q is the wrapped manifold, and p is the rank of the normal bundle of Q in M . As we have seen, elements of $\mathrm{KO}_{cpt}^{-p}(\mathrm{Q})$ are not represented by “differences” of vector bundles on Q , hence there is no natural way to identify them as a system of D-branes. Indeed, the higher KO-groups only play a role through the isomorphism (3.8.1), which is one of the limitations of the K-theoretical description of D-branes. We will address a possible solution to this problem in the next chapter, where we will develop another description of D-branes, based on KO-homology, the dual theory to KO-theory. We refer to [76] for an alternative interpretation of the higher KO-groups, and to [75] for extensive review on the application of K-theory to String theory.

Chapter 4

KO-homology and Type I D-branes

As it was emphasized in the last sections of chapter 3, the K-theoretical description of D-branes in String theory is based on the Sen-Witten mechanism of brane-antibrane annihilation. However, we have seen that in an ordinary Maxwell theory of p-forms the sources are extended objects, and their charges are related to the homology cycles they represent. We have argued that for the Ramond-Ramond gauge theory this is not the case: anyway, intuitively we still expect the right mathematical framework to be given by some sort of homology theory that would take into account the relation between K-theory and D-branes as exposed in the previous chapter.

In Type II String theory, this simple observation suggests that a much more natural description of D-branes can be given in terms of K-homology, the homological theory associated to K-theory, as thoroughly emphasized in [77, 52, 69, 89].

More precisely, K-homology has two equivalent representations: an analytic representation, in terms of C^* -algebras and Fredholm modules, and a geometric one, constructed by Baum and Douglas in [14, 13]. In particular, the Baum-Douglas construction was extensively used in [80] to provide a rigorous geometric description of D-branes in Type II String theory in various topologically non trivial backgrounds.

In this chapter, we will present new results concerning KO-homology, the homology theory associated to KO-theory.

From the mathematical perspective, we construct a geometric realization of KO-homology, and we argue that it is indeed isomorphic to the homology theory defined via the loop spectrum of KO-theory, which we refer to as *spectral KO-homology*. We will also develop the analytic description of KO-homology, using Kasparov's formalism for real C^* -algebras, which represents a unified description of both K-theory and K-homology.

We then construct an homomorphism between geometric and analytic KO-homology,

and give a detailed and explicit proof that such homomorphism induces a natural equivalence between the two representations. We want to comment, at this point, that the equivalence between geometric and analytic K-homology has been a sort of a “folklore theorem” until recently [15], as the original work of Baum and Douglas did not contain any accurate proof of such equivalence. In particular, the work [15] also contains a proof of the equivalence between geometric and analytic KO-homology: nevertheless, the proof presented in this chapter is fundamentally different, and the whole construction is more apt for physical applications, as we will see later on. Indeed, the approach followed here has strongly been inspired by the point of view in [14] that index theory is based on the equivalence between geometric and analytic K-homology: this point of view is reinforced by the introduction of certain geometric invariants that will help us to derive some cohomological index formulas in the real case.

From the physical perspective, we introduce the notion of *wrapped D-brane* and of *wrapping charge* of a D-brane. In particular, we will illustrate how the higher K-homology groups can naturally be interpreted in terms of wrapped D-branes, and we will argue that the wrapping charge of a D-brane is a genuinely different concept in Type I String theory..

We have mentioned in chapter 2 that the interpretation of a D-brane as a submanifold of the spacetime is not very accurate, and that a distinction should be made somehow between the D-brane itself and the worldvolume it wraps. This point of view emerges here, as we will construct stable torsion D-branes wrapping a single point in the spacetime.

4.1 Dual theories and spectral KO-homology

In this section we will explain in which sense KO-homology is “associated” to KO-theory, and we will define spectral KO-homology.

First we recall some properties of cohomology theories in the category of CW-complexes, referring to [88] for a detailed exposition.

An Ω -spectrum, or *loop spectrum* for a generalized cohomology theory k^* is given by a sequence of CW-complexes $\{K_n\}_{n \in \mathbb{Z}}$ together with homotopy equivalences

$$K_n \rightarrow \Omega K_{n+1} \tag{4.1.1}$$

where Ω denotes the loop space functor, such that the functor k^n can be represented

$$k^n(X) = [X, K_n] \quad \forall n \in \mathbb{Z}$$

for any CW-complex X .

By considering the maps $\sigma_n : \Sigma(K_n) \rightarrow K_{n+1}$ adjoint to the maps in (4.1.1), we can define the unreduced generalized homology theory associated to k_* by setting

$$k_i(X) := \tilde{k}_i(X^+) := \lim_n \pi_{n+i}(X^+ \wedge K_n)$$

where the inductive limit is taken using the maps σ_i

$$\begin{aligned} \pi_{n+i}(X^+ \wedge K_n) &= [S^{n+i}, X^+ \wedge K_n]_* \xrightarrow{\text{susp}} [\Sigma(S^{n+i}), \Sigma(X^+ \wedge K_n)]_* \\ &\simeq [S^{n+i+1}, X^+ \wedge \Sigma(K_n)]_* \xrightarrow{\sigma_i} [S^{n+i+1}, X^+ \wedge K_{n+1}]_* \\ &= \pi_{n+i+1}(X^+ \wedge K_{n+1}) \end{aligned}$$

The relative homology theory can be defined as usual, and we will refer to the homology theory k_* as the *dual* theory to k^* .

In [10] it was shown that a suitable spectrum for KO-theory can be defined in the following way. For $n \geq 1$, let $\mathcal{H}_{\mathbb{R}}$ be a real \mathbb{Z}_2 -graded separable Hilbert space which is a $*$ -module for the real Clifford algebra $Cl_{n-1} = Cl(\mathbb{R}^{n-1})$. Let Fred_n be the space of all Fredholm operators on $\mathcal{H}_{\mathbb{R}}$ which are odd, Cl_{n-1} -linear and self-adjoint. Then Fred_n is the classifying space for KO^{-n} , and there are homotopy equivalences $\text{Fred}_n \rightarrow \Omega \text{Fred}_{n-1}$. For $n \leq 0$, we choose $k \in \mathbb{N}$ such that $8k + n \geq 1$ and define $\text{Fred}_n := \text{Fred}_{8k+n}$.

Then, *spectral KO-homology* can be defined by setting

$$KO_i^s(X, Y) := \lim_n \pi_{n+i}((X/Y) \wedge \text{Fred}_n) \quad (4.1.2)$$

From a general result for spectrally defined generalized homology theories, we have that

$$KO_i^s(\text{pt}) = \widetilde{KO}^0(S^i)$$

The importance of the functors KO_n^s in our context consists in the fact that any set of groups which are isomorphic to spectral KO-homology, in a suitable sense, for any space X , defines necessarily an homology theory. Indeed, this will be the case for geometric KO-homology defined later on, whose homological properties will be deduced by “comparison” with spectral KO-homology.

4.2 KKO-theory and analytic KO-homology

We will give now a detailed overview of the definition of KO-homology in terms of Kasparov’s KK-theory for real C^* -algebras [62], and describe various properties that we will need later on. Incidentally, we will eventually give an interpretation of the KK-groups for complex C^* -algebras in terms of Type II D-branes.

4.2.1 Real C^* -algebras

A *real algebra* is a ring A which is also an \mathbb{R} -vector space such that $\lambda(xy) = (\lambda x)y = x(\lambda y)$ for all $\lambda \in \mathbb{R}$ and all $x, y \in A$. A *real $*$ -algebra* is a real algebra A equipped with a linear involution $*$: $A \rightarrow A$ such that $(xy)^* = y^*x^*$ for all $x, y \in A$. A *real Banach algebra* is a real algebra A equipped with a norm $\|-\|$: $A \rightarrow \mathbb{R}$ such that $\|xy\| \leq \|x\| \|y\|$ and such that A is complete in the norm topology. If A is a unital algebra then we assume $\|1\| = 1$. A *real Banach $*$ -algebra* is a real Banach algebra which is also a real $*$ -algebra. A *real C^* -algebra* is a real Banach $*$ -algebra such that

- (i) $\|x^*x\| = \|x\|^2$ for all $x \in A$; and
- (ii) $1 + x^*x$ is invertible in \tilde{A} for all $x \in A$.

where \tilde{A} denotes the unitalization of the algebra A .

Remark 4.1. Although in the complex case invertibility of $1+x^*x$ for all $x \in A$ would follow immediately from the C^* -algebra structure, in the real case this is no longer true. For example, consider the real Banach $*$ -algebra \mathbb{C} with involution given by the identity map. Then $1 + i^*i$ is not invertible, where $i := \sqrt{-1}$. This invertibility condition is fundamental to obtaining the usual representation theorem below for C^* -algebras in terms of bounded self-adjoint operators on a real Hilbert space. However, \mathbb{C} with involution given by complex conjugation is a real C^* -algebra. Since the only \mathbb{R} -linear involutions of \mathbb{C} are the identity and complex conjugation, when we consider \mathbb{C} as a real C^* -algebra the involution will always be implicitly assumed to be complex conjugation. More generally any complex C^* -algebra, regarded as a real vector space and with the same operations, is a real C^* -algebra.

Let us now give a number of examples of real C^* -algebras, some of which we will use later on in representation theorems.

Example 4.2. Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space. Then the set of bounded linear operators $\mathcal{B}(\mathcal{H}_{\mathbb{R}})$ with the usual operations is a real C^* -algebra. Any closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H}_{\mathbb{R}})$ is also a real C^* -algebra. More generally, any closed self-adjoint subalgebra of a real C^* -algebra is always a real C^* -algebra.

Example 4.3. Let X be a locally compact Hausdorff space and $C_0(X, \mathbb{R})$ the space of real-valued continuous functions vanishing at infinity. Then $C_0(X, \mathbb{R})$ with pointwise operations, the supremum norm and involution given by the identity map is a real C^* -algebra. As in the complex case, $C_0(X, \mathbb{R})$ is unital if and only if X is compact.

Example 4.4. With X as in Example 4.3 above, let Y be a closed subspace of X and $C_0(X, Y; \mathbb{R})$ the subspace of $C_0(X, \mathbb{C})$ consisting of maps $f : X \rightarrow \mathbb{C}$ such that $f(Y) \subset \mathbb{R}$. Then with the operations inherited from $C_0(X, \mathbb{C})$, the subspace $C_0(X, Y; \mathbb{R})$ is a real C^* -algebra.

Example 4.5. Let X be a locally compact Hausdorff space with involution $\tau : X \rightarrow X$, i.e. a homeomorphism such that $\tau \circ \tau = \text{id}_X$, and consider the subset $C_0(X, \tau)$ of $C_0(X, \mathbb{C})$ consisting of maps f such that $f \circ \tau = f^* = \overline{f}$. Then $C_0(X, \tau)$, with the operations inherited from $C_0(X, \mathbb{C})$, is a real C^* -algebra. If $\tau = \text{id}_X$ then $C_0(X, \tau) = C_0(X, \mathbb{R})$. If X is compact and Y is a closed subspace of X , then there is a compact Hausdorff space Z with an involution τ such that $C(X, Y; \mathbb{R}) \simeq C(Z, \tau)$. However, the converse does not hold in general.

Example 4.6. Let \mathcal{V} be a real vector space equipped with a quadratic form q , and consider the associated real Clifford algebra $\text{Cl}(\mathcal{V}, q)$. Assume, without loss of generality, that $q(v) = \langle v, \phi(v) \rangle$ for all $v \in \mathcal{V}$ with respect to an inner product on \mathcal{V} , where the linear operator $\phi \in \mathcal{L}(\mathcal{V})$ is symmetric and orthogonal. We can then define an involution on $\text{Cl}(\mathcal{V}, q)$ by $(v_1 \cdots v_k)^* = \phi(v_k) \cdots \phi(v_1)$, i.e. if $v \in \mathcal{V}$ then $v^* = \phi(v)$. The isomorphism $\Phi : \text{Cl}(\mathcal{V} \oplus \mathcal{V}, q \oplus -q) \rightarrow \mathcal{L}(\Lambda^* \mathcal{V})$ induces a norm on $\text{Cl}(\mathcal{V} \oplus \mathcal{V}, q \oplus -q)$ by pullback of the operator norm on $\mathcal{L}(\Lambda^* \mathcal{V})$, and the inclusion $\text{Cl}(\mathcal{V}, q) \hookrightarrow \text{Cl}(\mathcal{V}, q) \hat{\otimes} \text{Cl}(\mathcal{V}, -q) \simeq \text{Cl}(\mathcal{V} \oplus \mathcal{V}, q \oplus -q)$ given by $x \mapsto x \hat{\otimes} 1$ thereby induces a norm on $\text{Cl}(\mathcal{V}, q)$. Then $\text{Cl}(\mathcal{V}, q)$ with its algebra structure, this involution and norm is a real C^* -algebra.

If A, B are real $*$ -algebras then a *real $*$ -algebra homomorphism* is a real algebra map $\phi : A \rightarrow B$, i.e. an \mathbb{R} -linear ring homomorphism, such that $\phi(x^*) = \phi(x)^*$ for all $x \in A$. The homomorphism is assumed to be unital if both algebras are unital.

If A is an algebra, we denote by $\mathbb{M}_n(A)$ the algebra of $n \times n$ matrices with entries in A . Then, we have the following general representation theorem [47]

Theorem 4.7. *Let A be a finite-dimensional real C^* -algebra. Then there exist $n_1, \dots, n_k \in \mathbb{N}$ such that $A \simeq \mathbb{M}_{n_1}(A_1) \times \cdots \times \mathbb{M}_{n_k}(A_k)$ as real C^* -algebras with $A_1, \dots, A_k \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.*

Analogously to the complex case, real C^* -algebras are always algebras of operators on some Hilbert space.

Theorem 4.8. (Ingelstam) *Let A be any real C^* -algebra. Then there exists a real Hilbert space $\mathcal{H}_{\mathbb{R}}$ such that A is isomorphic as a real C^* -algebra to a closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H}_{\mathbb{R}})$.*

Consider now a real C^* -algebra A . We denote by $A_{\mathbb{C}} := A \otimes \mathbb{C}$ the complexification of A , which is a complex algebra containing A as a real algebra. Theorem 4.2.1 assures that $A_{\mathbb{C}}$ can be given a unique C^* -algebra norm such that the natural embedding $\theta : A \rightarrow A_{\mathbb{C}}$ of A onto its complexification is an isometry.

We can define a map $J_A : A_{\mathbb{C}} \rightarrow A_{\mathbb{C}}$ by $J_A(x + iy) = x - iy$ for all $x, y \in A$. The map J_A is a conjugate linear $*$ -isomorphism of the complex C^* -algebra $A_{\mathbb{C}}$. If $\phi : A \rightarrow A$ is a continuous $*$ -homomorphism, then the map $J_A(\phi) : A_{\mathbb{C}} \rightarrow A_{\mathbb{C}}$ defined by $J_A(\phi)(x + iy) = \phi(x) + i\phi(y)$ is a continuous $*$ -homomorphism such that $J_A \circ J_A(\phi) = J_A(\phi) \circ J_A$. Conversely, if J is a conjugate linear $*$ -isomorphism of a complex C^* -algebra B , then $A = \{x \in B \mid J(x) = x\}$ is a real C^* -algebra. This implies the following result.

Proposition 4.9. *Let $\mathcal{C}_{\mathbb{R}}^*$ be the category of real C^* -algebras and continuous $*$ -algebra homomorphisms. Let $\mathcal{C}_{\mathbb{C}, \text{cl}}^*$ be the category of pairs (A, J) , where A is a complex C^* -algebra and J is a conjugate linear $*$ -isomorphism of A , and continuous $*$ -homomorphisms commuting with J . Then the assignments*

$$\begin{aligned} A &\mapsto (A_{\mathbb{C}}, J_A) \\ \phi &\mapsto J_A(\phi) \end{aligned}$$

define a functor

$$\mathcal{J} : \mathcal{C}_{\mathbb{R}}^* \longrightarrow \mathcal{C}_{\mathbb{C}, \text{cl}}^*$$

which is an equivalence of categories.

The complexification of a real C^* -algebra A is crucial in generalizing the notion of spectrum of an element. Indeed, we define the *complexified spectrum* $\text{Sp}^{\mathbb{C}}(x)$ of an element $x \in A$ as the spectrum of the element $\theta(x)$ in $A_{\mathbb{C}}$, i.e. the set of $\lambda \in \mathbb{C}$ such that $\lambda - \theta(x)$ is not invertible in $A_{\mathbb{C}}$. This definition of the spectrum assures that the functional calculus in A is well behaved. In the following, we will use the notion of positive element. An element x in a real C^* -algebra A is said to be *positive* if $x = x^*$, and $\text{Sp}^{\mathbb{C}}(x) \subseteq \mathbb{R}^+$.

As we are interested in the applications of real C^* -algebras to algebraic topology of CW-complexes, we will now specialize to the case of commutative algebras.

As with complex Banach algebras, a maximal two-sided ideal in a real Banach algebra A is closed in A . If M is a maximal two-sided ideal of a real Banach algebra A , then A/M is isomorphic to one of \mathbb{R} or \mathbb{C} as real algebras. A *character* on a real algebra A is a non-zero real algebra map $\chi : A \rightarrow \mathbb{C}$, assumed unital if A is unital. Let Ω_A be the space of characters of A . This can be given, as in the complex case, a locally compact

Hausdorff space topology such that Ω_A is homeomorphic to $\Omega_{A_{\mathbb{C}}}$. Furthermore, A is unital if and only if Ω_A is compact.

Given $x \in A$, evaluation at x gives a continuous map $\Gamma(x) : \Omega_A \rightarrow \mathbb{C}$ called the *Gel'fand transform of x* . From this we obtain the *Gel'fand transform of A* , $\Gamma : A \rightarrow C_0(\Omega_A, \mathbb{C})$, which is a continuous real algebra homomorphism of unit norm. If A is a real $*$ -algebra, then Γ is a $*$ -algebra homomorphism.

The following important results on the representation of commutative real C^* -algebras allow to unify the treatment of real commutative C^* -algebras and topological spaces with involution.

Theorem 4.10. *Let A be a commutative real C^* -algebra. Then:*

- (i) *The map $\tau : \Omega_A \rightarrow \Omega_A$ defined by $\tau(\chi) = \bar{\chi}$ is an involution; and*
- (ii) *The Gel'fand transform $\Gamma : A \rightarrow C_0(\Omega_A, \tau)$ is a real C^* -algebra isomorphism.*

Proof. (i) The map τ is a bijection. The collection of sets

$$U_{x,V} = \{\chi \in \Omega_A \mid \chi(x) \in V\}$$

for every $x \in A$ and V open in \mathbb{C} is a sub-basis for the topology of Ω_A . The complex conjugate \bar{V} of V is an open set and $\tau^{-1}(U_{x,V}) = U_{x,\bar{V}}$. Thus τ is continuous.

(ii) The map Γ is a real $*$ -algebra map with $\|\Gamma(x)\| = \|x\|$. One also has

$$\begin{aligned} \Gamma(x) \circ \tau(\chi) &= \Gamma(x)(\bar{\chi}) \\ &= \overline{\chi(x)} = \Gamma(x)^*(\chi), \end{aligned}$$

and so $\Gamma(x) \circ \tau = \Gamma(x)^*$ and $\Gamma(A) \subset C_0(\Omega_A, \tau)$. Let $\theta : A \rightarrow A_{\mathbb{C}}$ be the C^* -algebra embedding of A into its complexification. The map $\vartheta : \Omega_{A_{\mathbb{C}}} \rightarrow \Omega_A$ given by $\vartheta(f) = f \circ \theta$ is a homeomorphism and there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\Gamma} & C_0(\Omega_A, \mathbb{C}) \\ \theta \downarrow & & \downarrow \vartheta^* \\ A_{\mathbb{C}} & \xrightarrow{\Gamma} & C_0(\Omega_{A_{\mathbb{C}}}, \mathbb{C}) . \end{array}$$

Using this one then shows that $\Gamma(A) = C_0(\Omega_A, \tau)$; see [47]. □

Corollary 4.11. *Let A be a commutative real C^* -algebra with trivial involution. Then A is $*$ -isomorphic to $C_0(\Omega_A, \mathbb{R})$.*

4.2.2 Hilbert Modules

In this section we will present a generalization of the notion of Hilbert space, in which the scalar product takes value in a general real C^* -algebra.

Let A be a (not necessarily commutative) real C^* -algebra. A *pre-Hilbert module over A* is a (right) A -module \mathcal{E} equipped with an *A -valued inner product*, i.e. a bilinear map $(-, -) : \mathcal{E} \times \mathcal{E} \rightarrow A$ such that

- (i) $(x, x) \geq 0$ for all $x \in \mathcal{E}$ and $(x, x) = 0$ if and only if $x = 0$;
- (ii) $(x, y) = (y, x)^*$ for all $x, y \in \mathcal{E}$; and
- (iii) $(x, ya) = (x, y)a$ for all $x, y \in \mathcal{E}$, $a \in A$.

For $x \in \mathcal{E}$ we define $\|x\|_{\mathcal{E}} := \|(x, x)\|^{1/2}$. This defines a norm on \mathcal{E} satisfying the Cauchy-Schwartz inequality. If \mathcal{E} is complete under this norm, then it is called a *Hilbert module over A* .

As a straight generalization from the ordinary Hilbert space case, we have the following examples of Hilbert modules.

Example 4.12. A real C^* -algebra A can be given the structure of a Hilbert module over itself by defining $(a, b) = a^*b$, for any $a, b \in A$. More generally, any closed right ideal of A is a Hilbert module over A .

Example 4.13. Let \mathcal{E} consists of all sequences $(a_n)_{n \in \mathbb{N}}$, $a_n \in A$, such that

$$\sum_n \|a_n\|^2 < \infty$$

with inner product $((a_n), (b_n)) = \sum_n a_n^* b_n$.

\mathcal{E} is called the *Hilbert space over A* , and is often denoted with A^∞ .

Let \mathcal{E}, \mathcal{F} be Hilbert A -modules and $T : \mathcal{E} \rightarrow \mathcal{F}$ an A -linear map. We call a map $T^* : \mathcal{F} \rightarrow \mathcal{E}$ such that $(Tx, y)_{\mathcal{F}} = (x, T^*y)_{\mathcal{E}}$ for all $x \in \mathcal{E}$, $y \in \mathcal{F}$ the *adjoint* of T . If it exists the adjoint is unique.

In contrast to ordinary bounded operators on separable Hilbert spaces, not every A -linear map between Hilbert A -modules has an adjoint. We denote the set of all A -linear maps $T : \mathcal{E} \rightarrow \mathcal{F}$ admitting an adjoint by $\mathcal{L}(\mathcal{E}, \mathcal{F})$. Elements of $\mathcal{L}(\mathcal{E}, \mathcal{F})$ are bounded A -linear maps and $\mathcal{L}(\mathcal{E}) := \mathcal{L}(\mathcal{E}, \mathcal{E})$ is a C^* -algebra with the operator norm and involution given by the adjoint.

Notice that a submodule of a Hilbert A -module \mathcal{E} in general need not be complemented, i.e. there is generally no projection in $\mathcal{L}(\mathcal{E})$ onto the given submodule.

However, one can define some special “rank 1” operators as follows. Given $x \in \mathcal{F}$, $y \in \mathcal{E}$ we define an operator $\theta_{x,y} \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ by $\theta_{x,y}(z) = x(y, z)_\mathcal{E}$. These operators generate an $\mathcal{L}(\mathcal{E}) - \mathcal{L}(\mathcal{F})$ -bimodule whose norm closure in $\mathcal{L}(\mathcal{E}, \mathcal{F})$ is denoted $\mathcal{K}(\mathcal{E}, \mathcal{F})$. Elements of $\mathcal{K}(\mathcal{E}, \mathcal{F})$ are called *generalized compact operators*. We will use the notation $\mathcal{K}(\mathcal{E})$ for $\mathcal{K}(\mathcal{E}, \mathcal{E})$.

Example 4.14. $\mathcal{K}(A) = A$ for any real C^* -algebra A . If A is unital, we have also that $\mathcal{L}(A) = A$.

Example 4.15. If \mathcal{E} is a Hilbert A -module, then $\mathcal{L}(\mathcal{E}^n) \simeq \mathbb{M}_n(\mathbb{R}) \otimes \mathcal{L}(\mathcal{E})$, and $\mathcal{K}(\mathcal{E}^n) \simeq \mathbb{M}_n(\mathbb{R}) \otimes \mathcal{K}(\mathcal{E})$

Example 4.16. $\mathcal{K}(A^\infty) \simeq A \otimes \mathcal{K}_\mathbb{R}$, where $\mathcal{K}_\mathbb{R} := \mathcal{K}(\mathcal{H}_\mathbb{R})$.

For a real C^* -algebra A , the *multiplier algebra* of A , $\mathbf{M}(A)$, is the maximal C^* -algebra containing A as an essential ideal. Equivalently, by representing $A \subset \mathcal{L}(\mathcal{H}_\mathbb{R})$ one has

$$\mathbf{M}(A) = \{ T \in \mathcal{L}(\mathcal{H}_\mathbb{R}) \mid TS, ST \in A \text{ for all } S \in A \} .$$

The multiplier algebra $\mathbf{M}(A)$ is a C^* -algebra which is $*$ -isomorphic to the C^* -algebra of double centralizers, i.e. pairs $(T_1, T_2) \in \mathcal{L}(A) \times \mathcal{L}(A)$ such that $a T_1(b) = T_2(a) b$, $T_1(ab) = T_1(a) b$ and $T_2(ab) = a T_2(b)$ for all $a, b \in A$.

If A is unital, then $\mathbf{M}(A) = A$. Furthermore, $\mathbf{M}(\mathcal{K}_\mathbb{R}) = \mathcal{L}(\mathcal{H}_\mathbb{R})$, and $\mathbf{M}(C_0(X, \mathbb{R})) = C_b(X, \mathbb{R})$ is the C^* -algebra of real-valued bounded continuous functions on a locally compact Hausdorff space X .

We have then the following proposition, whose proof follows from the analogous result for complex Hilbert modules [18]

Proposition 4.17. *Let \mathcal{E} be a Hilbert A -module. Then there is an isomorphism*

$$\mathcal{L}(\mathcal{E}) \simeq \mathbf{M}(\mathcal{K}(\mathcal{E})) .$$

As with ordinary Hilbert spaces, one can define tensor products of Hilbert modules in the following way [18].

Let \mathcal{E}_i be a Hilbert B_i -module, for $i=1,2$, and let $\phi : B_1 \rightarrow \mathcal{L}(\mathcal{E}_2)$ be a $*$ -homomorphism. If we regard \mathcal{E}_2 as a left B_1 -module via ϕ , we can form the algebraic tensor product $\mathcal{E}_1 \otimes_{B_1} \mathcal{E}_2$, which is a right B_2 -module. Finally, we define the B_2 -valued pre-inner product on the algebraic tensor product by

$$(x_1 \otimes x_2, y_1 \otimes y_2) := (x_2, \phi((x_1, y_1)_{\mathcal{E}_1}) y_2)_{\mathcal{E}_2}$$

The completion of the algebraic tensor product with respect to this inner product, with vectors of length 0 divided out, is called the *tensor product* of \mathcal{E}_1 and \mathcal{E}_2 , and is denoted with $\mathcal{E}_1 \otimes_\phi \mathcal{E}_2$.

As for ordinary Hilbert spaces, there is a natural homomorphism from $\mathcal{L}(\mathcal{E}_1)$ to $\mathcal{L}(\mathcal{E}_1 \otimes_\phi \mathcal{E}_2)$, and we will denote the image of $T \in \mathcal{L}(\mathcal{E}_1)$ as $T \otimes 1$, or $\phi_*(T)$ when we want to emphasize the homomorphism ϕ . However, there is no homomorphism from $\mathcal{L}(\mathcal{E}_2)$ to $\mathcal{L}(\mathcal{E}_1 \otimes_\phi \mathcal{E}_2)$ in general.

Finally, if \mathcal{E} is a pre-Hilbert module over the real C^* -algebra A , we assume that the complexification $\mathcal{E} \otimes \mathbb{C}$ is a pre-Hilbert module over $A_{\mathbb{C}}$. This means that the A -valued inner product extends to a sesquilinear map. We assume that sesquilinear maps are linear in the second variable.

4.2.3 Kasparov's formalism for KKO-theory

We are now ready to define KKO-theory by using Kasparov's approach, developed in [62]. In the following we will assume that a real C^* -algebra A is separable and a real C^* -algebra B is σ -unital, i.e. B contains an element h such that $\phi(h) > 0$, for every character ϕ of B . This technical requirements will be useful in the following.

Definition 4.18. A (*Kasparov*) (A, B) -*module* is a triple (\mathcal{E}, ρ, T) , where \mathcal{E} is a countably generated \mathbb{Z}_2 -graded Hilbert B -module, $\rho : A \rightarrow \mathcal{L}(\mathcal{E})$ is an even $*$ -homomorphism and $T \in \mathcal{L}(\mathcal{E})$ such that

$$(T - T^*) \rho(a) , (T^2 - 1) \rho(a) , [T , \rho(a)] \in \mathcal{K}(\mathcal{E}) \quad (4.2.1)$$

for all $a \in A$, and T is odd with respect to the grading on \mathcal{E} . A Kasparov module (\mathcal{E}, ρ, T) is called *degenerate* if all operators in (4.2.1) are zero. Two Kasparov modules $(\mathcal{E}_i, \rho_i, T_i)$, $i = 1, 2$ are said to be *orthogonally equivalent* if there is an isometric isomorphism $U \in \mathcal{L}(\mathcal{E}_1, \mathcal{E}_2)$ such that $T_1 = U^* T_2 U$ and $\rho_1(a) = U^* \rho_2(a) U$ for all $a \in A$.

Orthogonal equivalence is an equivalence relation on the set of Kasparov modules. We denote the set of Kasparov modules by $E(A, B)$. The subset containing degenerate modules is denoted $D(A, B)$. Direct sum makes $E(A, B)$ and $D(A, B)$ into monoids.

Definition 4.19. Let $(\mathcal{E}_i, \rho_i, T_i) \in E(A, B)$ for $i = 0, 1$, $(\mathcal{E}, \rho, T) \in E(A, B \otimes C([0, 1], \mathbb{R}))$, and let $f_t : B \otimes C([0, 1], \mathbb{R}) \rightarrow B$ be the evaluation map $f_t(g) = g(t)$. Then

$(\mathcal{E}_0, \rho_0, T_0)$ and $(\mathcal{E}_1, \rho_1, T_1)$ are said to be *homotopic* and (\mathcal{E}, ρ, T) is called a *homotopy* if $(\mathcal{E} \otimes_{f_i} B, f_i \circ \rho, f_{i*}(T))$ is orthogonally equivalent to $(\mathcal{E}_i, \rho_i, T_i)$ for $i = 0, 1$.

Homotopy is an equivalence relation on $\mathbf{E}(A, B)$ and we denote the equivalence classes by $[\mathcal{E}, \rho, T]$. It is useful to consider special kinds of homotopy. If $\mathcal{E} = C([0, 1], \mathcal{E}_0)$, $\mathcal{E}_0 = \mathcal{E}_1$ and the induced maps $t \mapsto T_t$, $t \mapsto \rho_t(a)$ for all $a \in A$ are strongly $*$ -continuous, then we call (\mathcal{E}, ρ, T) a *standard homotopy*. If in addition $\rho_t = \rho$ is constant and T_t is norm continuous, then (\mathcal{E}, ρ, T) is called an *operator homotopy*. The following result holds, whose proof can be obtained by the analogous result in complex case [18].

Proposition 4.20. *Let (\mathcal{E}, ϕ, T) be an element in $\mathbf{D}(A, B)$. Then (\mathcal{E}, ϕ, T) is homotopic to the zero module.*

We can now give the definition of the Kasparov's KKO-groups.

Definition 4.21. The set of equivalence classes in $\mathbf{E}(A, B)$ with respect to homotopy of (A, B) -modules is denoted $\mathbf{KKO}(A, B)$ or $\mathbf{KKO}_0(A, B)$. For $p, q \geq 0$ we define

$$\mathbf{KKO}_{p,q}(A, B) := \mathbf{KKO}(A, B \otimes C\ell_{p,q}),$$

where $C\ell_{p,q} := C\ell(\mathbb{R}^{p,q})$ is the real Clifford algebra of the vector space \mathbb{R}^{p+q} with quadratic form of signature (p, q) .

The equivalence relation allows us to simplify the (A, B) -modules required to define $\mathbf{KKO}(A, B)$ [18]. Indeed, we need only consider modules of the form (B^∞, ρ, T) with $T = T^*$. If A is unital, we can further assume that $\|T\| \leq 1$ and $T^2 - 1 \in \mathcal{K}(B^\infty)$.

There is another equivalence relation that we can define on $\mathbf{E}(A, B)$. We say that two (A, B) -modules $(\mathcal{E}_i, \rho_i, T_i)$, $i = 0, 1$ are *stably operator homotopic*, $(\mathcal{E}_0, \rho_0, T_0) \simeq_{\text{oh}} (\mathcal{E}_1, \rho_1, T_1)$, if there exist $(\mathcal{E}'_i, \rho'_i, T'_i) \in \mathbf{D}(A, B)$ such that $(\mathcal{E}_0 \oplus \mathcal{E}'_0, \rho_0 \oplus \rho'_0, T_0 \oplus T'_0)$ and $(\mathcal{E}_1 \oplus \mathcal{E}'_1, \rho_1 \oplus \rho'_1, T_1 \oplus T'_1)$ are operator homotopic up to orthogonal equivalence. One can prove that the set of equivalence classes with respect to \simeq_{oh} coincides with the set $\mathbf{KKO}(A, B)$ defined above. For this result to hold the hypothesis that B is σ -unital is of particular importance.

The set $\mathbf{KKO}(A, B)$ is an abelian group, for any separable real C^* -algebra A and any real σ -unital C^* -algebra B . Moreover, $\mathbf{KKO}(-, -)$ is a covariant bifunctor from

the category of separable C^* -algebras into the category of abelian groups which is additive, i.e.

$$\begin{aligned} \mathrm{KKO}(A_1 \oplus A_2, B) &= \mathrm{KKO}(A_1, B) \oplus \mathrm{KKO}(A_2, B) , \\ \mathrm{KKO}(A, B_1 \oplus B_2) &= \mathrm{KKO}(A, B_1) \oplus \mathrm{KKO}(A, B_2) . \end{aligned}$$

Namely, any two $*$ -homomorphisms $f : A_2 \rightarrow A_1$ and $g : B_1 \rightarrow B_2$ induce group homomorphisms

$$\begin{aligned} f^* : \mathrm{KKO}(A_1, B) &\longrightarrow \mathrm{KKO}(A_2, B) , \\ g_* : \mathrm{KKO}(A, B_1) &\longrightarrow \mathrm{KKO}(A, B_2) \end{aligned}$$

defined by

$$\begin{aligned} f^*[\mathcal{E}, \rho, T] &= [\mathcal{E}, \rho \circ f, T] , \\ g_*[\mathcal{E}, \rho, T] &= [\mathcal{E} \otimes_g B_2, \rho \otimes 1, T \otimes 1] . \end{aligned}$$

Finally, any two homotopies $f_t : A_2 \rightarrow A_1$ and $g_t : B_1 \rightarrow B_2$ induce the same homomorphism for all $t \in [0, 1]$, i.e. $f_t^* = f_0^*$ and $g_{t*} = g_{0*}$.

4.2.4 Analytic KO-homology

As mentioned in the introduction to this chapter, Kasparov's formalism represents a unified description of both KO-theory and KO-homology. Indeed, if we denote with $\mathrm{KO}_p(B)$ the algebraic KO-theory groups of a unital real C^* -algebra B , we have the following result [18, 62].

Theorem 4.22. *Let B be a unital real C^* -algebra. Then, $\mathrm{KKO}(\mathbb{R}, B) \simeq \mathrm{KO}_0(B)$ and $\mathrm{KKO}_{p,q}(\mathbb{R}, B) \simeq \mathrm{KO}_{p-q}(B)$*

Recall that for a real unital C^* -algebra A , the algebraic K-theory group $\mathrm{KO}_0(A)$ is defined as the Grothendieck group of the monoid of unitarily equivalent projectors in $M_\infty(A)$, which is defined as the direct limit of A -valued matrix algebras $M_n(A)$ under the embedding $a \rightarrow \mathrm{diag}(a, 0)$. The higher algebraic K-theory group can be defined by $\mathrm{KO}_p(A) := \mathrm{KO}_0(C_0(\mathbb{R}^p) \otimes A)$.

If X is a compact Hausdorff space, $\mathrm{KKO}_p(\mathbb{R}, C(X, \mathbb{R})) \simeq \mathrm{KO}_p(C(X, \mathbb{R})) \simeq \mathrm{KO}^p(X)$. On the other hand, using the Gel'fand transform the contravariant functor $(X, \tau) \mapsto C(X, \tau)$ induces an equivalence of categories between the category of compact Hausdorff spaces with involution and the category of commutative real C^* -algebras. Since

$\text{KKO}_{\sharp}(-, \mathbb{R})$ is also a contravariant functor, it follows that their composition $(X, \tau) \mapsto \text{KKO}_{\sharp}(C(X, \tau), \mathbb{R})$ is a covariant functor. This motivates the following

Definition 4.23. Let (X, τ) be a compact Hausdorff space with involution. The *analytic KO-homology groups of (X, τ)* are defined by

$$\text{KO}_n^a(X, \tau) := \text{KKO}_{n,0}(C(X, \tau), \mathbb{R}) = \text{KKO}(C(X, \tau), Cl_n).$$

In the following, we will illustrate an alternative description of Kasparov's KO-homology groups for a real C^* -algebra A , referring to [53] for more details. This description is based on triples $(\mathcal{H}_{\mathbb{R}}, \rho, T)$ which are defined by the data:

- (i) $\mathcal{H}_{\mathbb{R}}$ is a separable real Hilbert space;
- (ii) $\rho : A \rightarrow \mathcal{L}(\mathcal{H}_{\mathbb{R}})$ is a $*$ -representation of A ; and
- (iii) T is a bounded linear operator on $\mathcal{H}_{\mathbb{R}}$.

These are assumed to satisfy the following conditions:

- (i) $\mathcal{H}_{\mathbb{R}}$ is equipped with a \mathbb{Z}_2 -grading such that $\rho(a)$ is even for all $a \in A$ and T is odd;
- (ii) For all $a \in A$ one has

$$(T^2 - 1) \rho(a), (T - T^*) \rho(a), T \rho(a) - \rho(a) T \in \mathcal{K}_{\mathbb{R}}; \quad (4.2.2)$$

and

- (iii) There are odd \mathbb{R} -linear operators $\varepsilon_1, \dots, \varepsilon_n$ on $\mathcal{H}_{\mathbb{R}}$ with the Cl_n algebra relations

$$\varepsilon_i = -\varepsilon_i^*, \quad \varepsilon_i^2 = -1, \quad \varepsilon_i \varepsilon_j + \varepsilon_j \varepsilon_i = 0 \quad (4.2.3)$$

for $i \neq j$ such that T and $\rho(a)$ commute with each ε_i .

We shall refer to the triple $(\mathcal{H}_{\mathbb{R}}, \rho, T)$ as an *n -graded Fredholm module*.

Let us denote by $\Gamma O^n(A)$ the set of all n -graded Fredholm modules over A . Consider the equivalence relation \sim on $\Gamma O^n(A)$ generated by the relations:

Orthogonal equivalence: $(\mathcal{H}_{\mathbb{R}}, \rho, T) \sim (\mathcal{H}'_{\mathbb{R}}, \rho', T')$ if and only if there exists an isometric degree-preserving linear operator $U : \mathcal{H}_{\mathbb{R}} \rightarrow \mathcal{H}'_{\mathbb{R}}$ such that $U \rho(a) = \rho'(a) U$ for all $a \in A$, $U T = T' U$, and $U \varepsilon_i = \varepsilon'_i U$; and

Homotopy equivalence: $(\mathcal{H}_{\mathbb{R}}, \rho, T) \sim (\mathcal{H}_{\mathbb{R}}, \rho, T')$ if and only if there exists a norm continuous function $t \mapsto T_t$ such that $(\mathcal{H}_{\mathbb{R}}, \rho, T_t)$ is a Fredholm module for all $t \in [0, 1]$ with $T_0 = T$, $T_1 = T'$.

We define the *direct sum* of two Fredholm modules $(\mathcal{H}_{\mathbb{R}}, \rho, T)$ and $(\mathcal{H}'_{\mathbb{R}}, \rho', T')$ to be the Fredholm module $(\mathcal{H}_{\mathbb{R}} \oplus \mathcal{H}'_{\mathbb{R}}, \rho \oplus \rho', T \oplus T')$.

We may now define $KO^n(A)$ as the free abelian group generated by elements in $\Gamma O^n(A)/\sim$ and quotiented by the ideal generated by the set $\{[x_0 \oplus x_1] - [x_0] - [x_1] \mid [x_0], [x_1] \in \Gamma O^n(A)/\sim\}$. In $KO^n(A)$ the *inverse* of a class represented by the module $(\mathcal{H}_{\mathbb{R}}, \rho, T)$ is given by $(\mathcal{H}_{\mathbb{R}}^{\circ}, \rho, T)$, where $\mathcal{H}_{\mathbb{R}}^{\circ}$ is the Hilbert space $\mathcal{H}_{\mathbb{R}}$ with the opposite \mathbb{Z}_2 -grading and where the operators ε_i reverse their signs. Moreover, the operator T for a triple $(\mathcal{H}_{\mathbb{R}}, \rho, T)$ representing an element in $KO^n(A)$ can be taken to be a Fredholm operator without loss of generality [18].

For a compact Hausdorff space X we define

$$KO_n^a(X) := KO^n(C(X, \mathbb{R})) = KKO(C(X, \mathbb{R}), Cl_n).$$

Moreover, for a compact pair (X, Y) , we can define the higher relative KO-homology groups as

$$KO_n^a(X, Y) := KO^n(C(X/Y, \mathbb{R})) = KKO(C(X/Y, \mathbb{R}), Cl_n).$$

The groups $KO_n^a(X, Y)$ enjoy Bott periodicity

$$KO_n^a(X, Y) \simeq KO_{n+8}^a(X, Y), \forall n \geq 0$$

which can be proven by using the periodicity of real Clifford algebras, and the isomorphisms

$$KKO(A \otimes Cl_{p,q}, B \otimes Cl_{r,s}) \simeq KKO(A \otimes Cl_{p,q} \otimes Cl_{r,s}, B) \simeq KKO(A \otimes Cl_{p-q+s-r,0}, B)$$

induced by the *intersection product*, which we will not attempt to define here.

The term ‘‘KO-homology’’ for the groups $KO_n^a(X, Y)$ is justified by the following theorem [15, 53].

Theorem 4.24. (Kasparov) *There are connecting homomorphisms*

$$\partial : KO_n^a(X, Y) \rightarrow KO_{n-1}^a(X, Y)$$

which are compatible with Bott periodicity, and give Kasparov KO-homology the structure of a \mathbb{Z}_8 -graded homology theory on the category of CW-complex pairs (X, Y) . On the subcategory of finite CW-complex pairs Kasparov’s KO-homology is isomorphic to spectral KO-homology.

In particular, Theorem 4.24 implies that

$$KO_n^a(\text{pt}) \simeq KO^{-n}(\text{pt})$$

This can indeed be proven by directly computing the groups $\text{KKO}(\mathbb{R}, Cl_n)$; see [18] for such a computation in the complex case.

4.3 Geometric KO-homology

As we have seen in the previous section, analytic KO-homology gives a representation of spectral KO-homology based on C^* -algebras and linear operators on Hilbert spaces. In this section we will develop a geometric version of KO-homology, which is analogous to the Baum-Douglas construction of K-homology [14, 13, 80]. Indeed, we will prove directly its homological properties by comparing it with other formulations of KO-homology.

Definition 4.25. Let X be a finite CW-complex. A *KO-cycle* on X is a triple (M, E, ϕ) where

- (i) M is a compact spin manifold without boundary;
- (ii) E is a real vector bundle over M ; and
- (iii) $\phi : M \rightarrow X$ is a continuous map.

There are no connectedness requirements made upon M , and hence the bundle E can have different fibre dimensions on the different connected components of M . It follows that disjoint union

$$(M_1, E_1, \phi_1) \amalg (M_2, E_2, \phi_2) := (M_1 \amalg M_2, E_1 \amalg E_2, \phi_1 \amalg \phi_2)$$

is a well-defined operation on the set of KO-cycles on X , which we will denote with $\Gamma O(X)$.

In the following we will consider some equivalence relations on the set $\Gamma O(X)$.

Definition 4.26. Two KO-cycles (M_1, E_1, ϕ_1) and (M_2, E_2, ϕ_2) on X are *isomorphic* if there exists a diffeomorphism $h : M_1 \rightarrow M_2$ such that

- (i) h preserves the spin structures;
- (ii) $h^*(E_2) \simeq E_1$ as real vector bundles; and

(iii) The diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{h} & M_2 \\ & \searrow \phi_1 & \downarrow \phi_2 \\ & & X \end{array}$$

commutes.

Definition 4.27. Two KO-cycles (M_1, E_1, ϕ_1) and (M_2, E_2, ϕ_2) on X are *spin bordant* if there exists a compact spin manifold W with boundary, a real vector bundle $E \rightarrow W$, and a continuous map $\phi : W \rightarrow X$ such that the two KO-cycles

$$(\partial W, E|_{\partial W}, \phi|_{\partial W}) \quad , \quad (M_1 \amalg (-M_2), E_1 \amalg E_2, \phi_1 \amalg \phi_2)$$

are isomorphic, where $-M_2$ denotes M_2 with the opposite class in $H^1(M; \mathbb{Z}_2)$ representing the spin structure on its tangent bundle TM_2 . The triple (W, E, ϕ) is called a *spin bordism* of KO-cycles.

We finally introduce the last equivalence relation we will need to define geometric KO-homology. Let M be a spin manifold and $F \rightarrow M$ a C^∞ real spin vector bundle with fibres of dimension $n := \dim_{\mathbb{R}} F_p \equiv 0 \pmod{8}$ for $p \in M$. Let $\mathbb{1}_M^{\mathbb{R}} := M \times \mathbb{R}$ denote the trivial real line bundle over M . Then $F \oplus \mathbb{1}_M^{\mathbb{R}}$ is a real vector bundle over M with fibres of dimension $n + 1$ and projection map λ . By choosing a C^∞ metric on it, we may define

$$\widehat{M} = \mathbb{S}(F \oplus \mathbb{1}_M^{\mathbb{R}}) \tag{4.3.1}$$

where $\mathbb{S}(F \oplus \mathbb{1}_M^{\mathbb{R}})$ denotes the sphere bundle of $F \oplus \mathbb{1}_M^{\mathbb{R}}$. The tangent bundle of $F \oplus \mathbb{1}_M^{\mathbb{R}}$ fits into an exact sequence of bundles given by

$$0 \longrightarrow \lambda^*(F \oplus \mathbb{1}_M^{\mathbb{R}}) \longrightarrow T(F \oplus \mathbb{1}_M^{\mathbb{R}}) \longrightarrow \lambda^*(TM) \longrightarrow 0 ,$$

as for a vector bundle $E \xrightarrow{\pi} M$ the vertical tangent bundle to the fibration π is isomorphic to the vector bundle $\pi^*E \rightarrow E$. Upon choosing a splitting, it follows that

$$T(F \oplus \mathbb{1}_M^{\mathbb{R}}) \simeq \lambda^*(TM) \oplus \lambda^*(F \oplus \mathbb{1}_M^{\mathbb{R}})$$

and hence the spin structures on TM and $F \oplus \mathbb{1}_M^{\mathbb{R}}$ determine a spin structure on $T(F \oplus \mathbb{1}_M^{\mathbb{R}})$. Since the sphere bundle $\mathbb{S}(F \oplus \mathbb{1}_M^{\mathbb{R}})$ is the boundary of the disk bundle $\mathbb{B}(F \oplus \mathbb{1}_M^{\mathbb{R}})$, and using the fact that we can equip $\mathbb{B}(F \oplus \mathbb{1}_M^{\mathbb{R}})$ with the spin structure induced by that on the total space of $T(F \oplus \mathbb{1}_M^{\mathbb{R}})$, it follows that \widehat{M} is a compact spin

manifold. By construction, \widehat{M} is a sphere bundle over M with n -dimensional spheres S^n as fibres. We denote the bundle projection by

$$\pi : \widehat{M} \longrightarrow M . \quad (4.3.2)$$

We may regard the total space \widehat{M} as consisting of two copies $\mathbb{B}^\pm(F)$, with opposite spin structures, of the unit ball bundle $\mathbb{B}(F)$ of F glued together by the identity map $\text{id}_{\mathbb{S}(F)}$ on its boundary so that

$$\widehat{M} = \mathbb{B}^+(F) \cup_{\mathbb{S}(F)} \mathbb{B}^-(F) . \quad (4.3.3)$$

Since $n \equiv 0 \pmod{8}$, the group $\text{Spin}(n)$ has two irreducible real half-spin representations. The spin structure on F associates to these representations real vector bundles $S_0(F)$ and $S_1(F)$ of equal rank $2^{n/2}$ over M . Their Whitney sum $S(F) = S_0(F) \oplus S_1(F)$ is a bundle of real Clifford modules over TM such that $\text{Cl}(F) \simeq \text{End } S(F)$, where $\text{Cl}(F)$ is the real Clifford algebra bundle of F . Let $\mathcal{S}^+(F)$ and $\mathcal{S}^-(F)$ be the real spinor bundles over F obtained from pullbacks to F by the bundle projection $F \rightarrow M$ of $S_0(F)$ and $S_1(F)$, respectively. Clifford multiplication induces a bundle map $F \otimes S_0(F) \rightarrow S_1(F)$ that defines a vector bundle map $\sigma : \mathcal{S}^+(F) \rightarrow \mathcal{S}^-(F)$ covering id_F which is an isomorphism outside the zero section of F . Since the ball bundle $\mathbb{B}(F)$ is a subbundle of F , we may form real spinor bundles over $\mathbb{B}^\pm(F)$ as the restriction bundles $\Delta^\pm(F) = \mathcal{S}^\pm(F)|_{\mathbb{B}^\pm(F)}$. We can then glue $\Delta^+(F)$ and $\Delta^-(F)$ along $\mathbb{S}(F) = \partial\mathbb{B}(F)$ by the Clifford multiplication map σ giving a real vector bundle over \widehat{M} defined by

$$H(F) = \Delta^+(F) \cup_\sigma \Delta^-(F) . \quad (4.3.4)$$

For each $p \in M$, the bundle $H(F)|_{\pi^{-1}(p)}$ is the real Bott generator vector bundle over the n -dimensional sphere $\pi^{-1}(p)$.

In particular, the class $[H(F)] \in \text{KO}^0(\widehat{M})$ is the image of the Thom class of F $\tau_F \in \text{KO}^0(\mathbb{B}^+(F), \mathbb{S}(F))$ under the composition of the homomorphisms

$$\text{KO}^0(\mathbb{B}^+(F), \mathbb{S}(F)) \rightarrow \text{KO}^0(\widehat{M}, \mathbb{B}^-(F)) \rightarrow \text{KO}^0(\widehat{M})$$

where the first map is given by excision, and the second map is given by restriction.

Definition 4.28. Let (M, E, ϕ) be a KO-cycle on X and F a C^∞ real spin vector bundle over M with fibres of dimension $\dim_{\mathbb{R}} F_p \equiv 0 \pmod{8}$ for $p \in M$. Then the process of obtaining the KO-cycle $(\widehat{M}, H(F) \otimes \pi^*(E), \phi \circ \pi)$ from (M, E, ϕ) is called *real vector bundle modification*.

We are now ready to define the geometric KO-homology groups of the space X .

Definition 4.29. The *geometric KO-homology group* of X is the abelian group obtained from quotienting $\Gamma\text{O}(X)$ by the equivalence relation \sim generated by the relations of

- (i) isomorphism;
- (ii) spin bordism;
- (iii) direct sum: if $E = E_1 \oplus E_2$, then $(M, E, \phi) \sim (M, E_1, \phi) \amalg (M, E_2, \phi)$; and
- (iv) real vector bundle modification.

The group operation is induced by disjoint union of KO-cycles. We denote this group by $\text{KO}_{\sharp}^t(X) := \Gamma\text{O}(X) / \sim$, and the homology class of the KO-cycle (M, E, ϕ) by $[M, E, \phi] \in \text{KO}_{\sharp}^t(X)$.

Since the equivalence relation on $\Gamma\text{O}(X)$ preserves the dimension of $M \bmod 8$ in KO-cycles (M, E, ϕ) , one can define the subgroups $\text{KO}_n^t(X)$ consisting of classes of KO-cycles (M, E, ϕ) for which all connected components M_i of M are of dimension $\dim M_i \equiv n \bmod 8$. Then

$$\text{KO}_{\sharp}^t(X) = \bigoplus_{n=0}^7 \text{KO}_n^t(X) \tag{4.3.5}$$

has a natural \mathbb{Z}_8 -grading.

The geometric construction of KO-homology is functorial. If $f : X \rightarrow Y$ is a continuous map, then the induced homomorphism

$$f_* : \text{KO}_{\sharp}^t(X) \longrightarrow \text{KO}_{\sharp}^t(Y)$$

of \mathbb{Z}_8 -graded abelian groups is given on classes of KO-cycles $[M, E, \phi] \in \text{KO}_{\sharp}^t(X)$ by

$$f_*[M, E, \phi] := [M, E, f \circ \phi] .$$

One has $(\text{id}_X)_* = \text{id}_{\text{KO}_{\sharp}^t(X)}$ and $(f \circ g)_* = f_* \circ g_*$. Since real vector bundles over M extend to real vector bundles over $M \times [0, 1]$, it follows by spin bordism that induced homomorphisms depend only on their homotopy classes. If pt denotes a one-point topological space, then the collapsing map $\zeta : X \rightarrow \text{pt}$ induces an epimorphism

$$\zeta_* : \text{KO}_{\sharp}^t(X) \longrightarrow \text{KO}_{\sharp}^t(\text{pt}) . \tag{4.3.6}$$

The *reduced* geometric KO-homology group of X is

$$\widetilde{\mathrm{KO}}_{\sharp}^t(X) := \ker \zeta_* . \quad (4.3.7)$$

Since the map (4.3.6) is an epimorphism with left inverse induced by the inclusion of a point $\iota : \mathrm{pt} \hookrightarrow X$, one has $\mathrm{KO}_{\sharp}^t(X) \simeq \mathrm{KO}_{\sharp}^t(\mathrm{pt}) \oplus \widetilde{\mathrm{KO}}_{\sharp}^t(X)$ for any space X .

As for the complex case [80], the abelian group $\mathrm{KO}_{\sharp}^t(X)$ is generated by classes $[M, E, \phi]$ where M is connected, and if $\{X_j\}_{j \in J}$ is the set of connected components of X then

$$\mathrm{KO}_{\sharp}^t(X) = \bigoplus_{j \in J} \mathrm{KO}_{\sharp}^t(X_j) .$$

Moreover, the homology class of a cycle (M, E, ϕ) on X depends only on the KO-theory class of E in $\mathrm{KO}^0(M)$, and on the homotopy class of ϕ in $[M, X]$.

4.3.1 Homological properties of KO_{\sharp}^t

In the previous section we have constructed a covariant functor KO_{\sharp}^t from the category of finite CW-complexes to the category of abelian groups, which is homotopy invariant. We will now establish that this construction actually yields a (generalized) homology theory, and in particular is the dual homology to KO-theory. The main strategy consists in “comparing” the functors KO_n^t with a realization of spectral KO-homology developed in [58], which we will denote with KO'_{\sharp} . Namely, for each pair (X, Y) we will construct a map $\mu^s : \mathrm{KO}_n^t(X, Y) \rightarrow \mathrm{KO}'_n(X, Y)$ for each $n \in \mathbb{Z}$ which defines a natural equivalence between functors on the category of topological spaces having the homotopy type of finite CW-pairs.

The set of cycles for $\mathrm{KO}'_{\sharp}(X)$ is given by triples (M, x, ϕ) as in Definition 4.25, but with $x \in \mathrm{KO}^i(M)$ being a KO-theory class over M such that $\dim M \equiv i + n \pmod{8}$. The equivalence relations are as in the previous section, apart from real vector bundle modification, which is modified from Definition 4.28 as follows. The nowhere zero section

$$\Sigma^F : M \longrightarrow F \oplus \mathbb{1}_M^{\mathbb{R}} \quad (4.3.8)$$

defined by

$$\Sigma^F(p) = 0_p \oplus 1$$

for $p \in M$ induces an embedding

$$\Sigma^F : M \hookrightarrow \widehat{M} . \quad (4.3.9)$$

Then real vector bundle modification is replaced by the relation

$$(M, x, \phi) \sim (\widehat{M}, \Sigma_1^F(x), \phi \circ \pi),$$

where the functorial homomorphism $\Sigma_1^F : KO^i(M) \rightarrow KO^{i+r}(\widehat{M})$ is the Gysin map induced by the embedding (4.3.9), with $r = \text{rank}(F)$. By construction, the normal bundle to M in \widehat{M} can be identified with F . Since $H(F)$ is the image of the Thom class of F , by the definition of Gysin morphism in section 3.6 it follows that on stable isomorphism classes of real vector bundles $[E] \in KO^0(M)$ one has

$$\Sigma_1^F [E] = [H(F) \otimes \pi^*(E)]. \quad (4.3.10)$$

To compare KO_{\sharp}^t with KO'_{\sharp} , we define $KO_{n+8k}^t(X) := KO_n^t(X)$ for all $k \in \mathbb{Z}$, $0 \leq n \leq 7$. Moreover, we give a spin bordism description of the relative geometric KO-homology groups $KO_n^t(X, Y)$ as follows. We consider the set $\Gamma O(X, Y)$ of isomorphism classes of triples (M, E, ϕ) where

- (i) M is a compact spin manifold with (possibly empty) boundary;
- (ii) E is a real vector bundle over M ; and
- (iii) $\phi : M \rightarrow X$ is a continuous map with $\phi(\partial M) \subset Y$.

The set $\Gamma O(X, Y)$ is then quotiented by relations of relative spin bordism, which is modified from Definition 4.27 by the requirement that $M_1 \amalg (-M_2) \subset \partial W$ is a regularly embedded submanifold of codimension 0 with $\phi(\partial W \setminus M_1 \amalg (-M_2)) \subset Y$, direct sum, and real vector bundle modification, which is applicable in this case since $\mathbb{S}(F \oplus \mathbb{1}_M^{\mathbb{R}})$ is a compact spin manifold with boundary $\mathbb{S}(F \oplus \mathbb{1}_M^{\mathbb{R}})|_{\partial M}$. The collection of equivalence classes is a \mathbb{Z}_8 -graded abelian group with operation induced by disjoint union of relative KO-cycles. One has $KO_i^t(X, \emptyset) = KO_i^t(X)$.

We can finally prove the following

Theorem 4.30. *The map*

$$\mu^s : KO_n^t(X, Y) \longrightarrow KO'_n(X, Y)$$

defined on classes of KO-cycles by

$$\mu^s [M, E, \phi]_t = [M, [E], \phi],$$

is an isomorphism of abelian groups which is natural with respect to continuous maps of pairs.

Proof. Taking into account the equivalence relations on $\Gamma\text{O}(X, Y)$ used to define both KO-homology groups, the map μ^s is well-defined and a group homomorphism. Let $[M, x, \phi]_t \in \text{KO}'_n(X, Y)$ with $m := \dim M$. We may assume that M is connected and x is non-zero in $\text{KO}^i(M)$. Then $m - i \equiv n \pmod{8}$. Consider the trivial spin vector bundle $F = M \times \mathbb{R}^{n+7m}$ over M . In this case the sphere bundle (4.3.1) is $\widehat{M} = M \times \mathbb{S}^{n+7m}$ and the associated Gysin homomorphism in KO-theory is a map

$$\Sigma_1^F : \text{KO}^i(M) \longrightarrow \text{KO}^{i+7m+n}(\widehat{M}).$$

Since $i + 7m + n \equiv (i + 7m + m - i) \pmod{8} \equiv 0 \pmod{8}$, one has $\text{KO}^{i+7m+n}(\widehat{M}) \simeq \text{KO}^0(\widehat{M})$. It follows that there are real vector bundles $E, H \rightarrow \widehat{M}$ such that $\Sigma_1^F(x) = [E] - [H]$, and so by real vector bundle modification one has

$$[M, x, \phi]_t = [\widehat{M}, [E], \phi \circ \pi]_t - [\widehat{M}, [H], \phi \circ \pi]_t$$

in $\text{KO}'_n(X, Y)$. Therefore $\mu'([\widehat{M}, E, \phi \circ \pi]_t - [\widehat{M}, H, \phi \circ \pi]_t) = [M, x, \phi]_t$, and we conclude that μ^s is an epimorphism.

Now suppose that $\mu^s[M_1, E_1, \phi_1]_t = \mu^s[M_2, E_2, \phi_2]_t$ are identified in $\text{KO}'_n(X, Y)$ through real vector bundle modification. Then, for instance, there is a real spin vector bundle $F \rightarrow M_1$ such that $M_2 = \widehat{M}_1$ and $[E_2] = \Sigma_1^F[E_1]$. Since

$$\Sigma_1^F[E_1] = [H(F) \otimes \pi^*(E_1)]$$

and since the class $[M_2, E_2, \phi_2]_t$ depends only on the KO-theory class $[E_2]$, it follows that the homology classes $[M_1, E_1, \phi_1]_t$ and $[M_2, E_2, \phi_2]_t$ are also identified in $\text{KO}_n^t(X, Y)$ through real vector bundle modification. As this is the only relation in $\text{KO}'_n(X, Y)$ that might identify these classes without identifying them as KO-cycles, we conclude that μ^s is a monomorphism and therefore an isomorphism. \square

Since $\text{KO}'_{\#}$ is a homological realization of the homology theory associated with KO-theory, we have thus established that geometric KO-homology is a generalized homology theory which is equivalent to spectral KO-homology. In particular, it enjoys the standard homological properties, such as the existence of a long exact sequence for any pair (X, Y) . Moreover, the connecting homomorphism

$$\partial : \text{KO}_n^t(X, Y) \longrightarrow \text{KO}_{n-1}^t(Y)$$

is given by the boundary map

$$\partial[M, E, \phi] := [\partial M, E|_{\partial M}, \phi|_{\partial M}] \quad (4.3.11)$$

on classes of KO-cycles and extended by linearity.

Other homological properties are direct translations of those of the complex case provided by [80], to which we refer for details.

4.3.2 Products and Poincaré Duality

Since we have established that geometric KO-homology is a representation of spectral KO-homology, we can define products and dualities with KO-theory.

The *cap product* is defined as the \mathbb{Z}_8 -degree preserving bilinear pairing

$$\cap : \mathrm{KO}^0(\mathbf{X}) \otimes \mathrm{KO}_\#^t(\mathbf{X}) \longrightarrow \mathrm{KO}_\#^t(\mathbf{X})$$

given for any real vector bundle $F \rightarrow X$ and KO-cycle class $[M, E, \phi] \in \mathrm{KO}_\#^t(X)$ by

$$[F] \cap [M, E, \phi] := [M, \phi^*F \otimes E, \phi] \quad (4.3.12)$$

and extended linearly. In particular, it makes $\mathrm{KO}_\#^t(X)$ into a module over the ring $\mathrm{KO}^0(X)$.

As in the complex case, this product can be extended to a bilinear form

$$\cap : \mathrm{KO}^i(X) \otimes \mathrm{KO}_j^t(X, A) \longrightarrow \mathrm{KO}_{j-i}^t(X, A) .$$

defined as

$$x \cap [M, E, \phi] := (\mu^s)^{-1}([M, x \cup \phi^*[E], \phi]) \quad (4.3.13)$$

which coincides with (4.3.12) when $x = [F]$.

If X and Y are spaces, then the *exterior product*

$$\times : \mathrm{KO}_i^t(X) \otimes \mathrm{KO}_j^t(Y) \longrightarrow \mathrm{KO}_{i+j}^t(X \times Y)$$

is given for classes of KO-cycles $[M, E, \phi] \in \mathrm{KO}_i^t(X)$ and $[N, F, \psi] \in \mathrm{KO}_j^t(Y)$ by

$$[M, E, \phi] \times [N, F, \psi] := [M \times N, E \boxtimes F, (\phi, \psi)] ,$$

where $M \times N$ has the product spin structure uniquely induced by the spin structures on M and N , and $E \boxtimes F$ is the real vector bundle over $M \times N$ with fibres $(E \boxtimes F)_{(p,q)} = E_p \otimes F_q$ for $(p, q) \in M \times N$. This product is natural with respect to continuous maps. Unfortunately, in contrast to the complex case, we don't have a version of the Künneth theorem for KO-homology. Indeed, should such a formula exist, one could use it to show that $\mathrm{KO}_\#^t(\mathrm{pt}) \otimes \mathrm{KO}_\#^t(\mathrm{pt})$ has to be a tensor product as modules over the ring $\mathrm{KO}^\#(\mathrm{pt})$. But this does not work correctly as pointed out by Atiyah [3].

Let M be a spin manifold of dimension n . The class

$$[M] := [M, \mathbb{1}_M^{\mathbb{R}}, \mathrm{id}_M] \in \mathrm{KO}_n^t(M, \partial M)$$

is called the *fundamental class*, and it induces the following *Poincaré Duality* isomorphism [58]

$$\begin{aligned} \Phi_M : \text{KO}^i(M) &\xrightarrow{\cong} \text{KO}_{n-i}^t(M, \partial M) \\ \xi &\longrightarrow \xi \cap [M] \end{aligned}$$

The fundamental class is uniquely determined.

Finally, let $V \xrightarrow{\pi} X$ be a KO-oriented vector bundle of rank r over a space X with Thom class τ_V . In analogy to K-theory, we have the homological version of Thom isomorphism

$$\mathfrak{T}_{X,V}^* : \text{KO}_{i+r}^t(\mathbb{B}(V), \mathbb{S}(V)) \xrightarrow{\cong} \text{KO}_i^t(X) . \quad (4.3.14)$$

defined as

$$\mathfrak{T}_{X,V}^*([M, E, \phi]) := \tilde{\pi}_*(\tau_V \cap [M, E, \phi]) \quad (4.3.15)$$

where $\tilde{\pi} : \mathbb{B}(V) \rightarrow X$ is the bundle projection induced by π .

We conclude this section by noticing that all the above constructions have an equivalent description in analytic KO-homology.

4.4 K-homology and Index Theorems

We have seen in the previous sections that both analytic KO-homology and geometric KO-homology are representations of spectral KO-homology. It follows straightforwardly from compositions of natural equivalences that analytic KO-homology and geometric KO-homology are naturally equivalent. It is interesting, at this point, to ask if such natural isomorphism can be induced by a map defined *at the level of the cycles*. That this is indeed the case can be viewed as the primordial formulation of the index theorem, and it is the philosophy proposed in [14, 13]. Since this point of view will be crucial in both the construction and the proof of the equivalence carried in the next section, we will briefly make the above statement more precise by illustrating the case of even complex K-homology, directing the reader to [14, 13, 53] for more information.

Let M be an even dimensional compact spin^c manifold, and let E be a complex vector bundle on M . The canonical Dirac operator on M induces the elliptic differential operator

$$\mathbb{D}_M \otimes I_E : \Gamma(\mathcal{S} \otimes E) \rightarrow \Gamma(\mathcal{S} \otimes E)$$

where \mathcal{S} denotes the spinor bundle associated to TM . After a choice of a smooth metric g on M , we can construct the Hilbert space

$$\mathcal{H}_E^M := L^2(\Gamma(\mathcal{S} \otimes E); dg^M)$$

and define the bounded Fredholm operator T_M^E as the partial isometry in the polar decomposition for $\mathcal{D}_M \otimes I_E$. Since M is even-dimensional, \mathcal{H}_E^M is \mathbb{Z}_2 -graded, and there is a $*$ -homomorphism

$$\rho_E^M : C(M; \mathbb{C}) \rightarrow \mathcal{L}(\mathcal{H}_E^M)$$

as the space of section of a vector bundle on a manifold is equipped with a module structure over the algebra of functions of the manifold itself. Hence we have a correspondence

$$(M, E, \phi) \rightarrow (\mathcal{H}_E^M, \rho_M^E \circ \phi^*, T_M^E)$$

between cycles for geometric K-homology and cycles for analytic K-homology.

Theorem 4.31. ([14]) *Let X be a finite CW-complex. Then the correspondence*

$$(M, E, \phi) \rightarrow (\mathcal{H}_E^M, \rho_M^E \circ \phi^*, T_M^E)$$

induces a natural isomorphism

$$\mu^a : K_0^t(X) \rightarrow K_0^a(X)$$

commuting with the cap product between K-theory and K-homology.

The isomorphism μ can be now used to give an elegant formulation of the Atiyah-Singer index theorem. Namely, consider a closed even-dimensional smooth manifold M , and let T^*M denote its cotangent bundle. By the results in [11], any elliptic pseudo-differential operator D between the sections of vector bundles on M can be assigned to a class in $K_{cpt}^0(T^*M)$. More precisely, let

$$D : \Gamma(E) \rightarrow \Gamma(F)$$

be an elliptic differential operator of order m , where the vector bundles E and F have rank p and q respectively. The differential operator D can be expressed in local coordinates (x_1, \dots, x_n) on M as

$$D = \sum_{|\alpha| \leq m} A^\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha}$$

where $|\alpha| := \sum_k \alpha_k$ for a n -tuple on nonnegative integers $\alpha = (\alpha_1, \dots, \alpha_n)$, and where for each α $A^\alpha(x)$ is a $q \times p$ matrix of smooth complex-valued functions on M with $A^\alpha(x) \neq 0$ for some α such that $|\alpha| = m$. The *principal symbol* of D is defined to be the section $\sigma(D)$ of the bundle $(\odot^m T^*M) \otimes \text{Hom}(E, F)$ represented by the coefficients $\{A^\alpha\}_{|\alpha|=m}$, and where \odot denotes the symmetric tensor product. Since

$\odot^m V$ is canonically isomorphic to the space of homogeneous polynomial functions of degree m on V^* , for any vector space V , it follows that for each cotangent vector $\xi \in T_x^*(M)$, the principal symbol gives a homomorphism

$$\sigma_\xi(D) : E_x \rightarrow F_x$$

For elliptic operators, by definition, such a homomorphism is invertible for each non-zero cotangent vector ξ and any point $x \in M$. The principal symbol $\sigma(D)$ can be considered to live on the cotangent bundle, i.e. it defines a bundle map

$$\sigma(D) : \pi^*E \rightarrow \pi^*F$$

where $\pi : T^*M \rightarrow M$. In particular, if D is elliptic, $\sigma(D)$ is an isomorphism away from the zero section, hence we can assign to D the class

$$i(D) := [\pi^*E, \pi^*F; \sigma(D)] \in K^0(\mathbb{B}(T^*M), \mathbb{S}(T^*M))$$

Conversely, to any class $[E, F; \mu] \in K^0(\mathbb{B}(T^*M), \mathbb{S}(T^*M))$ one can assign a pseudo-differential operator on M with total symbol μ . See [63] for details on this construction. By using the pseudo-differential operator associated to any class in $K_{cpt}^0(T^*M)$, one can define isomorphisms

$$\begin{aligned} \text{ind}_t &: K_{cpt}^0(T^*M) \rightarrow K_0^t(M) \\ \text{ind}_a &: K_{cpt}^0(T^*M) \rightarrow K_0^a(M) \end{aligned}$$

which in the case that M is a spin^c manifold are given by the composition of Thom isomorphism and Poincaré duality. The above isomorphisms can be used to state the following elegant version of the index theorem.

Theorem 4.32. (Atiyah-Singer) *Let M be a closed smooth manifold. Then the following diagram commutes*

$$\begin{array}{ccc} & K_{cpt}^0(T^*M) & \\ \text{ind}_a \swarrow & & \searrow \text{ind}_t \\ K_0^t(M) & \xrightarrow{\mu^a} & K_0^a(M) \end{array}$$

To recover the usual form of the Atiyah-Singer index theorem, we consider the following composition of commutative diagrams

$$\begin{array}{ccc} & K_{cpt}^0(T^*M) & \\ \text{ind}_t \swarrow & & \searrow \text{ind}_a \\ K_0^t(M) & \xrightarrow{\mu^a} & K_0^a(M) \\ \zeta_* \swarrow & & \searrow \zeta_* \\ & K_0(\text{pt}) \simeq \mathbb{Z} & \end{array} \tag{4.4.1}$$

where ζ is the collapsing map, and the commutativity of the bottom diagram is granted by the fact that the isomorphism μ is natural. The homomorphisms $\zeta_* \circ \text{ind}_t$, $\zeta_* \circ \text{ind}_a$ coincide with the topological and analytical index homomorphism, respectively, defined by Atiyah and Singer, and the commutativity of the two triangles implies that

$$\zeta_* \circ \text{ind}_t = \zeta_* \circ \text{ind}_a$$

which is the original formulation of the index theorem.

Theorem 4.32 motivated the authors in [14] to state that for any flavour k of K-theory, the equivalence between geometric and analytic k -homology is related to an index theorem for the given theory k . In the next section we will reinforce the above statement: we will construct a natural morphism μ^a between $KO_{\#}^t$ and $KO_{\#}^a$ defined at the level of the cycles, and use a suitable index theorem to prove that μ^a is indeed a natural equivalence.

4.5 The equivalence between $KO_{\#}^t$ and $KO_{\#}^a$

As illustrated in the previous section, our primary goal is to prove the following result.

Theorem 4.33. *There is a natural equivalence*

$$\mu^a : KO^t \xrightarrow{\cong} KO^a$$

between the topological and analytic KO-homology functors.

As we have seen, geometric and analytic KO-homology are generalized cohomology theories defined on the category of finite CW-pairs (X, Y) . For such theories, the following general result holds [30].

Theorem 4.34. *Let $h_{\#}$ and $k_{\#}$ be generalized homology theories defined on the category of finite CW-pairs, and let*

$$\phi : h_{\#} \rightarrow k_{\#}$$

a natural transformation such that

$$\phi : h_n(\text{pt}) \rightarrow k_n(\text{pt})$$

is an isomorphism for any $n \in \mathbb{Z}$. Then ϕ is a natural equivalence.

We have explained in the previous section how the fact that the natural isomorphism μ^a between geometric and analytic K-homology induces the commutativity of

the bottom triangle in (4.4.1), and consequently the Atiyah-Singer index theorem. Our strategy to prove Theorem 4.33 will be instead opposite: namely, we will construct surjective “index” homomorphisms ind_n^t and ind_n^a such that the diagram

$$\begin{array}{ccc}
 KO_n^t(\text{pt}) & \xrightarrow{\mu^a} & KO_n^a(\text{pt}) \\
 & \searrow \text{ind}_n^t & \downarrow \text{ind}_n^a \\
 & & KO^{-n}(\text{pt})
 \end{array} \tag{4.5.1}$$

commutes for every n , thanks to a suitable index theorem. Recall that

$$KO_n^{t,a}(\text{pt}) \simeq KO^{-n}(\text{pt})$$

and since the groups $KO^{-n}(\text{pt})$ are equal to either 0, \mathbb{Z} or \mathbb{Z}_2 depending on the particular value of n , the commutativity of the diagram (4.5.1) along with surjectivity of the index maps are sufficient to prove that μ^a is an isomorphism¹. Moreover, the index homomorphism will play a crucial role in the applications exploited in later sections, in particular in the construction of cycles representing the generators for $KO_{\sharp}^t(\text{pt})$.

4.5.1 The natural transformation μ^a

Let (M, E, ϕ) be a topological KO-cycle on X with $\dim M = n$. Recall that M is a compact spin manifold. We construct a corresponding class in $KO_n^a(X)$ as follows. Consider the associated vector bundle

$$\mathfrak{S}(M) := P_{\text{Spin}}(M) \times_{\lambda_n} Cl_n$$

where $Cl_n = Cl(\mathbb{R}^n)$, $\lambda_n : \text{Spin}(n) \rightarrow \text{End}(Cl_n)$ is given by left multiplication with $\text{Spin}(n) \subset Cl_n^0 \subset Cl_n$, and $P_{\text{Spin}}(M)$ is the principal $\text{Spin}(n)$ -bundle over M associated to the spin structure on the tangent bundle TM . Since $Cl_n = Cl_n^0 \oplus Cl_n^1$ is a \mathbb{Z}_2 -graded algebra, it follows that

$$\mathfrak{S}(M) = \mathfrak{S}^0(M) \oplus \mathfrak{S}^1(M) \tag{4.5.2}$$

is a \mathbb{Z}_2 -graded real vector bundle over M with respect to the $Cl(TM)$ -action. The Clifford algebra Cl_n acts by right multiplication on the fibres whilst preserving the bundle grading (4.5.2). Since $\mathfrak{S}(M)$ is a vector bundle associated to $P_{\text{Spin}}(M)$, it carries the canonical Riemannian connection associated to the spin connection, and

¹Recall that surjective group homomorphisms of \mathbb{Z} or any finite group are isomorphisms

hence it is a Dirac bundle, i.e. it admits a canonical Dirac operator which is selfadjoint on the space of L^2 -sections of $\mathfrak{S}(M)$ with respect to the given metric, and which has finite dimensional kernel. Indeed, as a vector bundle, $\mathfrak{S}(M)$ is isomorphic to the direct sum of irreducible real spinor bundles on M . Hence, after choosing a C^∞ Riemannian metric g^M on TM , we consider the canonical Dirac operator

$$\mathfrak{D}^M : C^\infty(M, \mathfrak{S}(M)) \rightarrow C^\infty(M, \mathfrak{S}(M))$$

which we will refer to as the *Atiyah-Singer* operator [10] defined locally by

$$\mathfrak{D}^M = \sum_{i=1}^n e_i \cdot \nabla_{e_i}^M, \quad (4.5.3)$$

where $\{e_i\}_{1 \leq i \leq n}$ is a local orthonormal basis of sections of the tangent bundle TM , $\nabla_{e_i}^M$ are the corresponding components of the spin connection ∇^M , and the dot denotes Clifford multiplication. The operator \mathfrak{D}^M is a Cl_n -operator [63], i.e. one has

$$\mathfrak{D}^M(\Psi \cdot \varphi) = \mathfrak{D}^M(\Psi) \cdot \varphi$$

for all $\Psi \in C^\infty(M, \mathfrak{S}(M))$ and all $\varphi \in Cl_n$, where $\cdot \varphi$ denotes right multiplication by φ . In particular, with respect to decomposition (4.5.2), the operator \mathfrak{D}^M is of the form

$$\mathfrak{D}^M = \begin{pmatrix} 0 & \mathfrak{D}_1^M \\ \mathfrak{D}_0^M & 0 \end{pmatrix}$$

where

$$\mathfrak{D}_0^M : \Gamma(\mathfrak{S}^0(M)) \rightarrow \Gamma(\mathfrak{S}^1(M))$$

is a real, elliptic first-order differential operator which commutes with the action of $Cl_n^0 \simeq Cl_{n-1}$ on $\mathfrak{S}(M)$. Since \mathfrak{D}^M commutes with the Cl_n -action, the vector space $\ker \mathfrak{D}^M$ is a finite dimensional graded Cl_n -module.

We can now construct a triple $(\mathcal{H}_E^M, \rho_E^M, T_E^M)$ comprising the following data:

- (i) The separable real Hilbert space $\mathcal{H}_E^M := L_{\mathbb{R}}^2(M, \mathfrak{S}(M) \otimes E; dg^M)$;
- (ii) The $*$ -homomorphism $\rho_E^M : C(M, \mathbb{R}) \rightarrow \mathcal{L}(\mathcal{H}_E^M)$ defined by

$$(\rho_E^M(f)(\Psi))(p) = f(p) \Psi(p)$$

for $f \in C(M, \mathbb{R})$, $\Psi \in C^\infty(M, \mathfrak{S}(M) \otimes E)$ and $p \in M$; and

- (iii) The bounded Fredholm operator

$$T_E^M := \frac{\mathfrak{D}_E^M}{\sqrt{1 + (\mathfrak{D}_E^M)^2}} \quad (4.5.4)$$

acting on \mathcal{H}_E^M , where \mathcal{D}_E^M is the Atiyah-Singer operator (4.5.3) twisted by the real vector bundle $E \rightarrow M$.

This triple satisfies the following properties [53]:

- (i) \mathcal{H}_E^M is \mathbb{Z}_2 -graded according to the splitting (4.5.2) of the Clifford bundle;
- (ii) $\rho_E^M(f)$ is an even operator on \mathcal{H}_E^M for all $f \in C(M, \mathbb{R})$;
- (iii) Since M is compact, T_E^M is an odd Fredholm operator which obeys the compactness conditions (4.2.2) with $\rho_E^M(f)$; and
- (iv) There are odd operators ε_i , $i = 1, \dots, n$ commuting with both $\rho_E^M(f)$ and T_E^M which generate a Cl_n -action on \mathcal{H}_E^M as in (4.2.3), and which are given explicitly as right multiplication by elements e_i of a basis of the vector space \mathbb{R}^n .

It follows that $(\mathcal{H}_E^M, \rho_E^M, T_E^M)$ is a well-defined n -graded Fredholm module over the real C^* -algebra $C(M, \mathbb{R})$.

We now define the map μ^a in (4.5.1) by

$$\mu^a(M, E, \phi) := \phi_*(\mathcal{H}_E^M, \rho_E^M, T_E^M) = (\mathcal{H}_E^M, \rho_E^M \circ \phi^*, T_E^M), \quad (4.5.5)$$

where $\phi^* : C(X, \mathbb{R}) \rightarrow C(M, \mathbb{R})$ is the real C^* -algebra homomorphism induced by the map ϕ . At this stage the map μ^a is only defined on KO-cycles. More precisely, we can consider the map $\mu^a : \Gamma O_n(X) \rightarrow KO_n^a(X)$ induced by the equivalence relations on the set of Fredholm modules. We need to prove, at this point, that the map μ^a gives a well defined homomorphism

$$\mu^a : KO_n^t(X) \rightarrow KO_n^a(X)$$

We will first recall some useful results concerning the above construction. Let us denote with $[\mathcal{D}_E^M]$ the class corresponding to the element $\mu^a(M, E, \text{id}_M) \in KO_n^a(M)$. We can then state the following result [53]

Theorem 4.35. *Let $M - \partial M$ be the interior of a spin manifold M of dimension n with boundary ∂M , and let E be a real vector bundle on M . Equip the boundary ∂M with the spin structure induced by that on M . Then*

$$\partial[\mathcal{D}_{E|_{M-\partial M}}^{M-\partial M}] = [\mathcal{D}_{E|\partial M}^{\partial M}]$$

where $\partial : KO_n^a(M - \partial M) \rightarrow KO_{n-1}^a(\partial M)$ is the boundary homomorphism.

Notice that in the theorem above we have used that $KO_n^a(M - \partial M) \simeq KO_n(M, \partial M)$ via excision. Moreover, one can prove that the class $[\mathcal{D}^M] := [\mathcal{D}_{\mathbb{I}}^M]$ represents the fundamental class of M in $KO_n^a(M)$, and that

$$[\mathcal{D}_E^M] = [E] \cap [\mathcal{D}^M]$$

See [24] for details.

We are now ready to prove the following

Theorem 4.36. *The map $\mu^a : \Gamma O_n(X) \rightarrow KO_n^a(X)$ induces a well defined homomorphism*

$$\mu^a : KO_n^t(X) \rightarrow KO_n^a(X)$$

for any CW-complex X , and any $n \in \mathbb{Z}$.

Proof. That the map μ^a respects the algebraic sum and independence of the direct sum relation follows straightforwardly from the fact that

$$\Gamma(\mathcal{S}(M \sqcup N) \otimes (E \sqcup F)) = \Gamma(\mathcal{S}(M) \otimes E) \oplus \Gamma(\mathcal{S}(N) \otimes F)$$

for any compact spin manifolds M, N , with vector bundles E, F over M, N respectively, and by the definition of direct sum of Fredholm modules.

To prove that the map homomorphism μ^a does not depend on the bordism relation is equivalent to proving that $\mu^a(M, E, \phi) = 0$ for every bord n -cycle (M, E, ϕ) , i.e. a cycle for which there exists an $n + 1$ -cycle (W, F, ψ) in the appropriate relative group such that

$$(M, E, \phi) = (\partial W, F|_{\partial W}, \psi|_{\partial W})$$

First, we notice that the map $\psi|_{\partial W}$ factors as $\psi|_{\partial W} = \psi \circ i$, where

$$i : \partial W \hookrightarrow W$$

denotes the inclusion of the boundary. We have that

$$\begin{aligned} \mu^a(M, E, \phi) &= (\psi|_{\partial W})_*([\mathcal{D}_{F|_{\partial W}}^{\partial W}]) \\ &= (\psi)_* \circ i_*([\mathcal{D}_{F|_{\partial W}}^{\partial W}]) \\ &= 0 \end{aligned}$$

which follows by Theorem 4.35, and the long exact sequence for the pair $(W, \partial W)$

$$\cdots \rightarrow KO_{n+1}^a(W - \partial W) \xrightarrow{\partial} KO_n^a(\partial W) \xrightarrow{i_*} KO_n^a(W) \rightarrow \cdots$$

To conclude the proof, we need to show that for any KO-cycle (M, E, ϕ) and any real spin vector bundle F of rank $8r$ over M we have

$$\mu^a(M, E, \phi) = \mu^a(\widehat{M}, H(F) \otimes \pi^*E, \phi \circ \pi)$$

By definition, we have

$$\mu^a(\widehat{M}, H(F) \otimes \pi^*E, \phi \circ \pi) := \phi_* \pi_* ([\mathcal{D}_{H(F) \otimes \pi^*E}^{\widehat{M}}])$$

By using that

$$\begin{aligned} [\mathcal{D}_{H(F) \otimes \pi^*E}^{\widehat{M}}] &= [H(F) \otimes \pi^*E] \cap [\mathcal{D}^{\widehat{M}}] \\ &= ([H(F)] \cup [\pi^*E]) \cap [\mathcal{D}^{\widehat{M}}] \\ &= \pi^*[E] \cap ([H(F)] \cap [\mathcal{D}^{\widehat{M}}]) \end{aligned}$$

we have

$$\begin{aligned} \mu^a(\widehat{M}, H(F) \otimes \pi^*E, \phi \circ \pi) &= \phi_* \pi_* (\pi^*[E] \cap ([H(F)] \cap [\mathcal{D}^{\widehat{M}}])) \\ &= \phi_* ([E] \cap \pi_* ([H(F)] \cap [\mathcal{D}^{\widehat{M}}])) \end{aligned}$$

By recalling that $[H(F)]$ is the image of the Thom class of F , and by the equation (4.3.15), the class $\pi_* ([H(F)] \cap [\mathcal{D}^{\widehat{M}}])$ is the image of $[\mathcal{D}^{\widehat{M}}]$ under the isomorphism

$$KO_{n+8r}(\widehat{M}) \rightarrow KO_n(M)$$

for the spherical fibration $\widehat{M} \xrightarrow{\pi} M$ induced by the Thom isomorphism. Since the above isomorphism maps the fundamental class of \widehat{M} to that of M , we have

$$\pi_* ([H(F)] \cap [\mathcal{D}^{\widehat{M}}]) = [\mathcal{D}^M]$$

The following equalities conclude the proof

$$\begin{aligned} \mu^a(\widehat{M}, H(F) \otimes \pi^*E, \phi \circ \pi) &= \phi_* ([E] \cap [\mathcal{D}^M]) \\ &= \phi_* ([\mathcal{D}_E^M]) \\ &= \mu^a(M, E, \phi) \end{aligned}$$

□

We refer the reader to [15] for an alternative proof of Theorem 4.36.

4.5.2 The analytic index map ind_n^a

Let $(\mathcal{H}_{\mathbb{R}}, \rho, T)$ be an n -graded Fredholm module over the real C^* -algebra $C(X, \mathbb{R})$ such that T is a Fredholm operator. Since T commutes with ε_i for $i = 1, \dots, n$, the

kernel $\ker T \subset \mathcal{H}_{\mathbb{R}}$ is a real $\mathcal{C}\ell_n$ -module with \mathbb{Z}_2 -grading induced by the grading of $\mathcal{H}_{\mathbb{R}}$. Thus we can define

$$\text{ind}_n^a(T) := [\ker T] \in \widehat{\mathfrak{M}}_n/i^*\widehat{\mathfrak{M}}_{n+1} \quad (4.5.6)$$

where $\widehat{\mathfrak{M}}_n$ is the Grothendieck group of irreducible Clifford modules defined in chapter 3.

By the Atiyah-Bott-Shapiro isomorphism, we have

$$\widehat{\mathfrak{M}}_n/i^*\widehat{\mathfrak{M}}_{n+1} \simeq KO^{-n}(\text{pt})$$

We will call (4.5.6) the *analytic* or *Clifford index* of the Fredholm operator T . An important property of this definition is the following result [63].

Theorem 4.37. *The analytic index*

$$\text{ind}_n^a : \text{Fred}_n \longrightarrow KO^{-n}(\text{pt})$$

is surjective and constant on the connected components of Fred_n .

Given two Fredholm modules $(\mathcal{H}_{\mathbb{R}}, \rho, T)$ and $(\mathcal{H}_{\mathbb{R}}, \rho, T')$ over a real C^* -algebra A , we will say that T is a *compact perturbation* of T' if $(T - T')\rho(a) \in \mathcal{K}_{\mathbb{R}}$ for all $a \in A$. We then have the following elementary result.

Lemma 4.38. *If T is a compact perturbation of T' , then the Fredholm modules $(\mathcal{H}_{\mathbb{R}}, \rho, T)$ and $(\mathcal{H}_{\mathbb{R}}, \rho, T')$ are operator homotopic over A .*

Proof. Consider the path $T_t = (1 - t)T + tT'$ for $t \in [0, 1]$. Then the map $t \mapsto T_t$ is norm continuous. We will show that for any $t \in [0, 1]$, the triple $(\mathcal{H}_{\mathbb{R}}, \rho, T_t)$ is a Fredholm module over A , i.e. that the operator T_t satisfies

$$(T_t^2 - 1)\rho(a), (T_t - T_t^*)\rho(a), T_t\rho(a) - \rho(a)T_t \in \mathcal{K}_{\mathbb{R}} \quad (4.5.7)$$

for all $a \in A$. The last two inclusions in (4.5.7) are easily proven because the path T_t is “linear” in the operators T and T' . To establish the first one, for any $t \in [0, 1]$ and $a \in A$ we compute

$$(T_t^2 - 1)\rho(a) = \left[(T^2 - 1) + t^2(T - T')^2 - t(T^2 - 1) - t(T - T')^2 + t(T'^2 - 1) \right] \rho(a). \quad (4.5.8)$$

By using the fact that $(\mathcal{H}_{\mathbb{R}}, \rho, T)$ and $(\mathcal{H}_{\mathbb{R}}, \rho, T')$ are Fredholm modules, that T is a compact perturbation of T' , and that $\mathcal{K}_{\mathbb{R}}$ is an ideal in $\mathcal{L}(\mathcal{H}_{\mathbb{R}})$, one easily verifies that the right-hand side of (4.5.8) is a compact operator. This implies that $(\mathcal{H}_{\mathbb{R}}, \rho, T_t)$ is a well-defined family of Fredholm modules over A . \square

By Lemma 4.38, we can choose the operator T in the class $[\mathcal{H}_{\mathbb{R}}, \rho, T]$ to be selfadjoint without loss of generality. Indeed, given a Fredholm module we simply replace the operator T with $\tilde{T} := \frac{1}{2}(T + T^*)$. Moreover, such a choice of \tilde{T} is compatible with operator homotopy. Hence in the following we will always assume that the operator T is selfadjoint and Fredholm.

Proposition 4.39. *The induced map*

$$\text{ind}_n^a : KO_n^a(X) \longrightarrow KO^{-n}(\text{pt})$$

given on classes of n -graded Fredholm modules by

$$\text{ind}_n^a[\mathcal{H}_{\mathbb{R}}, \rho, T] = [\ker T]$$

is a well-defined surjective homomorphism for any $n \in \mathbb{N}$.

Proof. We first show that to the direct sum of two Fredholm modules $(\mathcal{H}_{\mathbb{R}}, \rho, T)$ and $(\mathcal{H}'_{\mathbb{R}}, \rho', T')$ over $A = C(X, \mathbb{R})$, the map ind_n^a associates the class $[\ker T] + [\ker T'] \in \widehat{\mathfrak{M}}_n / i^* \widehat{\mathfrak{M}}_{n+1} \simeq KO^{-n}(\text{pt})$. The kernel

$$\ker(T \oplus T') = \ker(T) \oplus \ker(T')$$

is a real graded Cl_n -module. By the definition of the group $\widehat{\mathfrak{M}}_n$ and of its quotient by $i^* \widehat{\mathfrak{M}}_{n+1}$, one thus has $\text{ind}_n^a(T \oplus T') = [\ker T] + [\ker T']$ and so the map ind_n^a respects the algebraic structure on $\Gamma O_n(A)$.

Consider now two Fredholm modules $(\mathcal{H}_{\mathbb{R}}, \rho, T)$ and $(\mathcal{H}'_{\mathbb{R}}, \rho', T')$ which are orthogonally equivalent. Then there exists an even isometry $U : \mathcal{H}_{\mathbb{R}} \rightarrow \mathcal{H}'_{\mathbb{R}}$ such that

$$T' = U T U^* , \quad \varepsilon'_i = U \varepsilon_i U^* .$$

This implies that $\ker T' = U(\ker T)$, and that the graded Cl_n representations given respectively by ε'_i and ε_i are equivalent. In particular, they represent the same class in $\widehat{\mathfrak{M}}_n / i^* \widehat{\mathfrak{M}}_{n+1}$.

Finally, consider two homotopic n -graded Fredholm modules $(\mathcal{H}_{\mathbb{R}}, \rho, T)$, $(\mathcal{H}_{\mathbb{R}}, \rho, T')$ over A . There exists by definition a continuous path $t \rightarrow T_t$ connecting T and T' in Fred_n . Hence, by Theorem 4.37 $\text{ind}_n^a(T) = \text{ind}_n^a(T')$. \square

4.5.3 The topological index map ind_n^t

Given a KO-cycle (M, E, ϕ) on X with M an n -dimensional compact spin manifold, we can assign to it the *associated Atiyah-Milnor-Singer (AMS) invariant* [63] defined by

$$\widehat{\mathcal{A}}_E(M) = \beta \circ f_!([E]) \in KO^{-n}(\text{pt}) \tag{4.5.9}$$

where f_i is the Gysin homomorphism for the smooth embedding

$$f : M \hookrightarrow \mathbb{R}^{n+8k} \tag{4.5.10}$$

for some $k \in \mathbb{N}$ sufficiently large, and β is induced by the Bott periodicity isomorphism

$$\beta : KO_{\text{cpt}}(\mathbb{R}^{n+8k}) \rightarrow KO^{-n}(\text{pt})$$

This definition does not depend on the embedding (4.5.10) nor on the integer k .

We define

$$\text{ind}_n^t(M, E, \phi) := \widehat{\mathcal{A}}_E(M) . \tag{4.5.11}$$

Proposition 4.40. *The map*

$$\text{ind}_n^t : KO_n^t(X) \longrightarrow KO^{-n}(\text{pt})$$

induced by (4.5.11) is a well-defined surjective homomorphism for any $n \in \mathbb{N}$.

Proof. We first prove that the map ind_n^t respects the algebraic structure on the abelian group $KO_n^t(X)$. Given two n -dimensional compact spin manifolds M_1 and M_2 , let $M = M_1 \amalg M_2$. Embed M in the Euclidean space \mathbb{R}^{n+8k} for some k sufficiently large as in (4.5.10). Recall now that the Gysin homomorphism of f is the composition of Thom isomorphism with respect to the normal bundle of M in \mathbb{R}^{n+8k} with the map “extending by zero”, and that the Thom isomorphism is the isomorphism induced by the Thom class. Moreover, notice that the normal bundle ν to the embedding of M is given by $\nu_1 \amalg \nu_2$, where ν_1 and ν_2 are respectively the normal bundles to the embeddings of M_1 and M_2 induced by the embedding of M . The Thom class of ν is given by

$$\begin{aligned} \tau_\nu &:= [\varpi^* \mathcal{S}^+(\nu), \varpi^* \mathcal{S}^-(\nu); \sigma] \\ &= [\varpi_1^* \mathcal{S}^+(\nu_1) \amalg \varpi_2^* \mathcal{S}^+(\nu_2), \varpi_1^* \mathcal{S}^-(\nu_1) \amalg \varpi_2^* \mathcal{S}^-(\nu_2); \sigma_1 \amalg \sigma_2] \\ &= \tau_{\nu_1} + \tau_{\nu_2} \in KO^0(B(\nu), S(\nu)) \simeq KO^0(B(\nu_1), S(\nu_1)) \oplus KO^0(B(\nu_2), S(\nu_2)) . \end{aligned}$$

where $\varpi = \varpi_1 \amalg \varpi_2 : \nu_1 \amalg \nu_2 \rightarrow M_1 \amalg M_2$ is the normal bundle projection. Let E_1 and E_2 be real vector bundles over M_1 and M_2 , respectively, and let $E = E_1 \amalg E_2$. Then in $KO^0(B(\nu), S(\nu))$ one has

$$\begin{aligned} \tau_\nu(E) &= \tau_\nu \smile [\varpi^* E] \\ &= \tau_\nu \smile [\varpi_1^* E_1 \amalg \varpi_2^* E_2] \\ &= \tau_{\nu_1} \smile [\varpi_1^* E_1] + \tau_{\nu_2} \smile [\varpi_2^* E_2] = \tau_{\nu_1}(E_1) + \tau_{\nu_2}(E_2) . \end{aligned}$$

Since the map extending by zero is a homomorphism, one then finds

$$\begin{aligned} \text{ind}_n^t((M_1, E_1, \phi_1) \amalg (M_2, E_2, \phi_2)) &:= \widehat{\mathcal{A}}_E(M) \\ &= \widehat{\mathcal{A}}_{E_1}(M_1) + \widehat{\mathcal{A}}_{E_2}(M_2) \\ &= \text{ind}_n^t(M_1, E_1, \phi_1) + \text{ind}_n^t(M_2, E_2, \phi_2) \in KO^{-n}(\text{pt}) , \end{aligned}$$

showing that ind_n^t respects the algebraic sum of cycles.

Next we have to check that the map ind_n^t is independent of the choice of representative of a homology class in $KO_n^t(X)$. The independence of the direct sum relation follows from the discussion above, while spin bordism independence is guaranteed by the property that the AMS invariant $\widehat{\mathcal{A}}_E(M)$ is a spin cobordism invariant [63]. Finally, we have to verify that the map ind_n^t does not depend on real vector bundle modification. Let M be a smooth n -dimensional compact spin manifold and let $E \rightarrow M$ be a smooth real vector bundle. Let F be a real spin vector bundle over M with fibres of real dimension $8l$ for some $l \in \mathbb{N}$. Consider the corresponding sphere bundle (4.3.1) with projection (4.3.2). Real vector bundle modification of a KO-cycle (M, E, ϕ) on X induced by F produces the KO-cycle $(\widehat{M}, \widehat{E}, \phi \circ \pi)$, where $\widehat{E} = H(F) \otimes \pi^*(E)$ is the real vector bundle over \widehat{M} such that

$$[\widehat{E}] = \Sigma_!^F[E]$$

with $[E] \in KO^0(M)$, $[\widehat{E}] \in KO^0(\widehat{M})$, and Σ^F defined as in (4.3.9). We may compute the AMS invariant for the pair $(\widehat{M}, \widehat{E})$ by choosing an embedding

$$\widehat{f} : \widehat{M} \hookrightarrow \mathbb{R}^{n+8k+8l}$$

so that

$$\begin{aligned} \widehat{\mathcal{A}}_{\widehat{E}}(\widehat{M}) &= \beta \circ \widehat{f}_!([\widehat{E}]) \\ &= \beta \circ \widehat{f}_! \circ \Sigma_!^F[E] = \beta \circ (\widehat{f} \circ \Sigma^F)_![E] , \end{aligned}$$

where in the last equality we have used functoriality of the Gysin map. Notice that

$$\widehat{f} \circ \Sigma^F : M \hookrightarrow \mathbb{R}^{n+8k+8l} =: \mathbb{R}^{n+8m}$$

is an embedding of M into a ‘‘large enough’’ Euclidean space. Since $\widehat{\mathcal{A}}_E(M)$ is independent of the embedding and the integer m , we have

$$\widehat{\mathcal{A}}_{\widehat{E}}(\widehat{M}) = \widehat{\mathcal{A}}_E(M)$$

as required. □

4.5.4 The Isomorphism Theorem

We can now assemble the constructions of the previous sections to finally establish our main result. Notice first of all that since $\ker \mathcal{D}_E^M \simeq \ker T_E^M$, one has

$$\text{ind}_n^a \circ \mu^a(M, E, \phi) = \text{ind}_n^a(\mathcal{D}_E^M) \tag{4.5.12}$$

for any KO-cycle (M, E, ϕ) on X with $\dim M = n$. At this point we can use an important result from spin geometry called the *Cl_n -index theorem* [63].

Theorem 4.41. *Let M be a compact spin manifold of dimension n and let E be a real vector bundle over M . Let*

$$\mathcal{D}_E^M : C^\infty(M, \mathcal{S}(M) \otimes E) \longrightarrow C^\infty(M, \mathcal{S}(M) \otimes E)$$

be the Cl_n -linear Atiyah-Singer operator with coefficients in E . Then

$$\text{ind}_n^a(\mathcal{D}_E^M) = \widehat{\mathcal{A}}_E(M) .$$

The proof of Theorem 4.33 is now completed once we establish the following result.

Proposition 4.42. *The map*

$$\mu^a : KO_n^t(\text{pt}) \longrightarrow KO_n^a(\text{pt})$$

is an isomorphism for any $n \in \mathbb{N}$.

Proof. As noticed at the beginning of this section, it suffices to establish the commutativity of the diagram (4.5.1), i.e. that

$$\text{ind}_n^t = \text{ind}_n^a \circ \mu^a .$$

Let $[M, E, \phi]$ be the class of a KO-cycle over pt with $\dim M = n$. Using Theorem 4.41 and (4.5.12) we have

$$\begin{aligned} \text{ind}_n^t[M, E, \phi] &:= \widehat{\mathcal{A}}_E(M) \\ &= \text{ind}_n^a(\mathcal{D}_E^M) = \text{ind}_n^a \circ \mu^a[M, E, \phi] \end{aligned}$$

as required. □

4.6 The Real Chern Character

In this section we will describe the natural complexification map from geometric KO-homology to geometric K-homology and use it to define the Chern character homomorphism in topological KO-homology. We describe various properties of this homomorphism, most notably its intimate connection with the AMS invariant which was the crux of the isomorphism of the previous section.

Let X be a compact topological space. Consider the topological, generalized homology groups $K_{\sharp}^t(X)$ and $KO_{\sharp}^t(X)$, along with the corresponding K-theory and KO-theory groups. The complexification of a real vector bundle over X is a complex vector bundle over X which is isomorphic to its own conjugate vector bundle. The complexification map is compatible with stable isomorphism of real and complex vector bundles, and thus defines a homomorphism from stable equivalence classes of real vector bundles to stable equivalence classes of complex vector bundles. It thereby induces a natural transformation of cohomology theories

$$(\otimes \mathbb{C})^* : KO^*(X) \longrightarrow K^*(X)$$

induced by

$$[E] - [F] \longmapsto [E_{\mathbb{C}}] - [F_{\mathbb{C}}]$$

where $E_{\mathbb{C}} := E \otimes \mathbb{C}$ is the complexification of the real vector bundle $E \rightarrow X$.

We can also define a complexification morphism relating the homology theories

$$(\otimes \mathbb{C})_* : KO_{\sharp}^t(X) \longrightarrow K_{\sharp}^t(X) \tag{4.6.1}$$

by

$$[M, E, \phi] \otimes \mathbb{C} := [M, E_{\mathbb{C}}, \phi]$$

and extended by linearity, where on the right-hand side we regard M endowed with the spin^c structure induced by its spin structure as a KO-cycle. One can easily see that

$$[M, E, \phi] \otimes \mathbb{C} = \phi_*([E_{\mathbb{C}}] \frown [M]_{\mathbb{K}}) \tag{4.6.2}$$

where $[M]_{\mathbb{K}} \in K_{\sharp}(M)$ denotes the K-theory fundamental class of M . Thus in the case when X is a compact spin manifold, the homomorphism $(\otimes \mathbb{C})_*$ is just the Poincaré dual of $(\otimes \mathbb{C})^*$. This is clearly a natural transformation of homology theories.

A related natural transformation between cohomology theories is the realification morphism

$$(_ \mathbb{R})^* : K^{\sharp}(X) \longrightarrow KO^{\sharp}(X)$$

induced by assigning to a complex vector bundle over X the underlying real vector bundle over X . Because a spin^c manifold is not necessarily spin, we cannot implement this transformation in the homological setting in general. Rather, we must assume that X is a compact spin manifold. In this case the K -homology group $K_{\sharp}^t(X)$ has generators $[X \times S^n, E_i, \text{pr}_1] - [X \times S^n, F_i, \text{pr}_1]$, $0 \leq n \leq 7$, where $\text{pr}_1 : X \times S^n \rightarrow X$ is the projection onto the first factor [80]. We can then define the morphism

$$(_ \mathbb{R})_* : K_{\sharp}^t(X) \longrightarrow KO_{\sharp}^t(X)$$

by

$$([X \times S^n, E_i, \text{pr}_1] - [X \times S^n, F_i, \text{pr}_1]) _ \mathbb{R} := [X \times S^n, E_i _ \mathbb{R}, \text{pr}_1] - [X \times S^n, F_i _ \mathbb{R}, \text{pr}_1]$$

and extending by linearity. Since this definition depends on a choice of generators for $K_{\sharp}^t(X)$, the transformation is not natural. As for the complexification morphism, the morphism $(_ \mathbb{R})_*$ thus defined is Poincaré dual to $(_ \mathbb{R})^*$. It follows that the composition $(_ \mathbb{R})_* \circ (\otimes \mathbb{C})_*$ is multiplication by 2.

We can use the natural transformation provided by the complexification homomorphism (4.6.1) to define a real homological Chern character

$$\text{ch}_{\bullet}^{\mathbb{R}} : KO_{\sharp}^t(X) \longrightarrow H_{\sharp}(X, \mathbb{Q}) \tag{4.6.3}$$

by

$$\text{ch}_{\bullet}^{\mathbb{R}}(\xi) = \text{ch}_{\bullet}(\xi \otimes \mathbb{C})$$

for $\xi \in KO_{\sharp}^t(X)$, where on the right-hand side we use the K -homology Chern character $\text{ch}_{\bullet} : K_{\sharp}^t(X) \rightarrow H_{\sharp}(X, \mathbb{Q})$, which can be defined as

$$\text{ch}_{\bullet}(M, E, \phi) = \phi_*((\text{ch}(E) \cup e^{d(TM)/2} \hat{A}(TM)) \cap [M])$$

where the cohomology class $d(TM)$ is defined in Appendix C.

The real Chern character (4.6.3) is a natural transformation of homology theories, and it preserves the cap product, i.e. the diagram

$$\begin{array}{ccc} KO_t^*(X) \otimes KO_{\sharp}^t(X) & \xrightarrow{\cap} & KO_{\sharp}^t(X) \\ \text{ch}_{\bullet} \otimes \text{ch}_{\bullet} \downarrow & & \downarrow \text{ch}_{\bullet} \\ H_t^*(X; \mathbb{Q}) \otimes H_{\sharp}^t(X; \mathbb{Q}) & \xrightarrow{\cap} & H_{\sharp}^t(X; \mathbb{Q}) \end{array}$$

commutes for any space X . This property is indeed guaranteed by the fact that $\text{ch}_{\bullet}^{\mathbb{R}}$ is the natural transformation in KO -homology induced by the real Chern character in KO -theory. See [58] for details. Finally, tensoring with \mathbb{Q} gives a map

$$\text{ch}_{\bullet}^{\mathbb{R}} \otimes \text{id}_{\mathbb{Q}} = (\text{ch}_{\bullet} \otimes \text{id}_{\mathbb{Q}}) \circ ((\otimes \mathbb{C})_* \otimes \text{id}_{\mathbb{Q}}) : KO_{\sharp}^t(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H_{\sharp}(X, \mathbb{Q})$$

In contrast with the complex case, this map is *not* an isomorphism.

4.7 Cohomological Index formulas

We will now explore the relation between the homological real Chern character and the topological index defined in (4.5.11). In particular, we will be able to give cohomological $C\ell_n$ -index formulas. We first show that up to Poincaré duality the topological index is the homological morphism induced by the collapsing map. Recall that up to isomorphism, the AMS invariant is given by

$$\widehat{\mathcal{A}}_E(M) = \tilde{\zeta}_!^{\text{KO}}[E]$$

where M is a compact spin manifold of dimension n , E is a real vector bundle over M , $\tilde{\zeta} : M \rightarrow \text{pt}$ is the collapsing map on M , and $\tilde{\zeta}_!^{\text{KO}}$ is the corresponding Gysin homomorphism² on KO-theory. In this case we have

$$\tilde{\zeta}_*^{\text{KO}} = \Phi_{\text{pt}} \circ \tilde{\zeta}_!^{\text{KO}} \circ \Phi_M^{-1}$$

where $\tilde{\zeta}_*^{\text{KO}}$ is the induced morphism on $\text{KO}_\#^t(M)$, and Φ_{pt} and Φ_M are the Poincaré duality isomorphisms on pt and M , respectively. It then follows that

$$\begin{aligned} \Phi_{\text{pt}} \circ \text{ind}_n^t(M, E, \phi) &= \Phi_{\text{pt}} \circ \tilde{\zeta}_!^{\text{KO}}[E] \\ &= \Phi_{\text{pt}} \circ \tilde{\zeta}_!^{\text{KO}} \circ \Phi_M^{-1}(M, E, \text{id}_M) \\ &= \tilde{\zeta}_*^{\text{KO}}[M, E, \text{id}_M] \\ &= [M, E, \tilde{\zeta}] = \zeta_*^{\text{KO}}[M, E, \phi] \end{aligned} \quad (4.7.1)$$

where $\zeta : X \rightarrow \text{pt}$ is the collapsing map on X with $\tilde{\zeta} = \zeta \circ \phi$.

We will next describe how the real Chern character can be used to give a characteristic class description of the map ind_n^t in the torsion-free cases. Consider first the case $n \equiv 4 \pmod{8}$. We begin by showing that there is a commutative diagram

$$\begin{array}{ccc} \text{KO}_4^t(X) & \xrightarrow{\zeta_*^{\text{KO}}} & \text{KO}_4^t(\text{pt}) \\ & \searrow \zeta_*^{\text{H} \circ \text{ch}_\bullet^{\mathbb{R}}} & \downarrow \text{ch}_\bullet^{\mathbb{R}} \\ & & \text{H}_0(\text{pt}, \mathbb{Q}) \end{array} \quad (4.7.2)$$

where ζ_*^{H} is the induced morphism on homology. Recall that $\text{ch}_\bullet^{\mathbb{R}} = \text{ch}_\bullet \circ (\otimes \mathbb{C})_*$, where $(\otimes \mathbb{C})_*$ is the complexification map (4.6.1), hence it can be expressed as

$$\text{ch}_\bullet^{\mathbb{R}}(M, E, \phi) = \phi_*((\text{ch}(E_{\mathbb{C}}) \cup \hat{A}(TM)) \cap [M])$$

²We are using the Gysin homomorphism definition for continuous and proper maps, which are not necessarily embeddings. See [61] for details.

Then one has

$$\begin{aligned}
 \zeta_*^H \circ \text{ch}_\bullet^{\mathbb{R}}(M, E, \phi) &= \zeta_*^H \circ \phi_*((\text{ch}^\bullet(E_{\mathbb{C}}) \cup \widehat{A}(\text{TM})) \cap [M]) \\
 &= (\zeta \circ \phi)_*((\text{ch}^\bullet(E_{\mathbb{C}}) \cup \widehat{A}(\text{TM})) \cap [M]) \\
 &= \text{ch}_\bullet^{\mathbb{R}}(M, E, \tilde{\zeta}) = \text{ch}_\bullet^{\mathbb{R}} \circ \zeta_*^{\text{KO}}[M, E, \phi].
 \end{aligned}$$

Now we use the fact that the map $\text{ch}_\bullet^{\mathbb{R}} : \text{KO}_4^t(\text{pt}) \rightarrow H_0(\text{pt}, \mathbb{Q})$ sends $\mathbb{Z} \rightarrow 2\mathbb{Z} \subset \mathbb{Q}$ [61]. On its image, the homomorphism $\text{ch}_\bullet^{\mathbb{R}}$ is thus invertible and its inverse is given as division by 2. This can be proven as follows. We have

$$\begin{aligned}
 \zeta_*^H \circ \text{ch}_\bullet^{\mathbb{R}}(M, E, \phi) &= \zeta_*^H \circ \Phi_M(\text{ch}^\bullet(E_{\mathbb{C}}) \cup \widehat{A}(\text{TM})) \\
 &= \Phi_{\text{pt}} \circ \zeta_{\text{pt}}^H(\text{ch}^\bullet(E_{\mathbb{C}}) \cup \widehat{A}(\text{TM})) \\
 &= \langle \text{ch}^\bullet(E_{\mathbb{C}}) \cup \widehat{A}(\text{TM}), [M] \rangle, \tag{4.7.3}
 \end{aligned}$$

where $\langle -, - \rangle : H^{\sharp}(M, \mathbb{Q}) \times H_{\sharp}(M, \mathbb{Q}) \rightarrow \mathbb{Q}$ is the canonical dual pairing between cohomology and homology. In (4.7.3) we have used the fact that Φ_{pt} is the identity on $H_0(\text{pt}, \mathbb{Q}) \simeq \mathbb{Q}$, and the proof of the last equality uses the Atiyah-Hirzebruch version of the Grothendieck-Riemann-Roch theorem [61]. Recall that for a spin manifold M of dimension $4k + 8$, one has

$$\langle \text{ch}^\bullet(E_{\mathbb{C}}) \cup \widehat{A}(\text{TM}), [M] \rangle \in 2\mathbb{Z}$$

After using the isomorphism $\text{KO}_4(\text{pt}) \simeq \mathbb{Z}$, we thus deduce that

$$\zeta_*^{\text{KO}}[M, E, \phi] = \frac{1}{2} \langle \text{ch}^\bullet(E_{\mathbb{C}}) \cup \widehat{A}(\text{TM}), [M] \rangle$$

and from (4.7.1) we arrive finally at

$$\text{ind}_n^t(M, E, \phi) = \frac{1}{2} \langle \text{ch}^\bullet(E_{\mathbb{C}}) \cup \widehat{A}(\text{TM}), [M] \rangle$$

When $n \equiv 0 \pmod{8}$, one obtains a similar result but now without the factor $\frac{1}{2}$, since in that case $\text{ch}_\bullet^{\mathbb{R}} : \text{KO}_0^t(\text{pt}) \rightarrow H_0(\text{pt}, \mathbb{Q})$ is the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ [61]. In the remaining non-trivial cases $n \equiv 1, 2 \pmod{8}$ the homological Chern character is of no use, as $\text{KO}^{-n}(\text{pt})$ is the pure torsion group \mathbb{Z}_2 , and there is no cohomological formula for the AMS invariant in these instances. However, by using Theorem 4.41 one still has an interesting mod 2 index formula for the topological index in these cases as well [63]. We can summarize our homological derivations of these index formulas as follows.

Theorem 4.43. *Let $[M, E, \phi] \in \text{KO}_n^t(X)$, and let \mathcal{D}_E^M be the Atiyah-Singer operator on M with coefficients in E . Let $\mathcal{H}_E^M := \ker \mathcal{D}_E^M$ denote the vector space of real harmonic*

E -valued spinors on M , where \mathcal{D}_E^M is the Atiyah-Singer operator for the irreducible real spinor bundle on M . Then one has the Cl_n -index formulas

$$\text{ind}_n^t(M, E, \phi) = \begin{cases} \langle \text{ch}^\bullet(E_{\mathbb{C}}) \cup \widehat{A}(TM), [M] \rangle, & n \equiv 0 \pmod{8}, \\ \dim_{\mathbb{C}} \mathcal{H}_E^M \pmod{2}, & n \equiv 1 \pmod{8}, \\ \dim_{\mathbb{H}} \mathcal{H}_E^M \pmod{2}, & n \equiv 2 \pmod{8}, \\ \frac{1}{2} \langle \text{ch}^\bullet(E_{\mathbb{C}}) \cup \widehat{A}(TM), [M] \rangle, & n \equiv 4 \pmod{8}, \\ 0, & \text{otherwise.} \end{cases}$$

4.8 D-branes and K-homology

In this section we will show how geometric K-homology can be used to describe D-branes in type II and type I String theory in a topological nontrivial spacetime. We will introduce the notion of *wrapped D-brane* on a given submanifold of spacetime, we will define the group of charges of wrapped D-branes, and construct explicit examples of wrapped D-branes which have torsion charge.

Recall by section 3.7 that the group of topological charges of a Dp -brane realized as a spin^c submanifold $W \subset X$ is given by

$$K_{\text{cpt}}^0(\nu_W) \simeq K^0(B(N(W)), S(N(W)))$$

as proposed by Witten. According to the Sen-Witten construction, the classes in $K_{\text{cpt}}^0(\nu_W)$ are interpreted as systems of $D9 - D\bar{9}$ branes which are unstable, and will decay onto the worldvolume W , which correspond to the zero loci of the appropriate tachyon field. In particular, this process happens in spacetime, and it depends on how the worldvolume is embedded in it. On the other hand, the role played by the Chan-Paton vector bundle on the Dp -brane is not manifest in this classification. However, there is a natural way of classifying the Dp -branes on W by means which manifestly takes into account the Chan-Paton bundle contribution. Indeed, from the Dp -brane data, we can naturally construct the Baum-Douglas cycle (W, E, id) , where E denotes the Chan-Paton bundle, and declare that its charge is given by the class $[W, E, \text{id}] \in K_{p+1}(W)$. As the group $K_{p+1}(W)$ contains no information about the embedding of the worldvolume W in X , we can intuitively think the charge $[W, E, \text{id}]$ takes into account how the D-brane wraps the submanifold W . Notice that this is analogous to the charge classification of an extended object in an abelian gauge theory via the homology cycle of its worldvolume. Finally, the equivalence relation

that defines the group $K_{p+1}(W)$ are very natural from the physical point of view. See [80] for details. At this point, we notice that by definition the elements of $K_{p+1}^t(W)$ are given by (differences of) classes $[M, E, \phi]$ where M is a $p+1$ -dimensional manifold. However, it is *not* always possible to choose the map ϕ in $[M, W]$ in such a way that ϕ is a diffeomorphism. This motivates the following

Definition 4.44. Let X be a type II String theory spacetime, described by a 10-dimensional spin manifold, and let $W \subset X$ be a spin^c submanifold. A Dp -brane *wrapping* the worldvolume W is defined as the K-cycle (M, E, ϕ) , where $\dim M = p+1$, and $\phi(M) \subset W$. We will call E the Chan-Paton bundle associated to the wrapped Dp -brane, and we will say that the Dp -branes *fills* W if $\phi(M) = W$. The charge of the wrapped Dp -brane is given by the class $[M, E, \phi]$ in the group $K_{p+1}^t(W)$.

Notice that in the above definition we have relaxed the condition that $\dim W = p+1$, as we are not requiring that the wrapping preserves the dimension of the D-brane. This is an attempt to take into account, at least at the topological level, the well-known fact that D-branes are not always representable as submanifolds equipped with vector bundles, since they are boundary conditions for a superconformal field theory, and that a distinction should be made between the wrapping D-brane, in this case identified with a K-cycle representing a particular type of boundary conditions, and the worldvolume it wraps. Notice also that the group of charges of wrapped Dp -branes does *not* depend on how the manifold W is embedded into the spacetime, and hence it seems to represent a genuine worldvolume concept. In particular, as mentioned above, the wrapped D-brane definition is very natural in the ordinary case of a D-brane realized as a submanifold W of spacetime equipped with a Chan-Paton bundle E , as it only depends on how the vector bundle is defined on the submanifold, and not on the procedure used to “extend” it to the spacetime. Finally, in the case of ordinary D-branes wrapping W with $\dim W = p+1$, the group $K_{p+1}(W)$ coincides with the group of charges of type IIB Dp -branes that can be obtained via the Sen-Witten construction, i.e. via brane-antibrane decay. This can be shown as follows. Since the normal bundle $\nu_W \rightarrow W$ is a spin^c vector bundle, we can use the Thom isomorphism in K-theory to establish that

$$K_{\text{cpt}}^0(\nu_W) \simeq K^0(W)$$

As W is a spin^c submanifold of the spacetime, we can use Poincaré Duality to get

$$K^0(W) \simeq K_{p+1}^t(W)$$

where $p+1 = \dim W$. This suggests that for ordinary type II Dp -branes the wrapping charge is completely determined by the decay of the tachyon field. It is natural at this point to extend the notion of wrapped D-brane and of wrapping charge to type I String theory. In this case, though, the two notions of charge do not coincide, as we will show in the following.

Recall that in type I String theory the group of topological charges of a Dp -brane realized as a spin submanifold $W \subset X$ is given by

$$KO_{\text{cpt}}^0(\nu_W) \simeq KO^0(B(N(W)), S(N(W))) \quad (4.8.1)$$

By using the Thom isomorphism in KO-theory, we have that

$$KO_{\text{cpt}}^0(\nu_W) \simeq KO^{p-9}(W)$$

Finally, by Poincaré Duality, we get

$$KO^{p-9}(W) \simeq KO_{10}(W)$$

The group $KO_{10}(W)$ is in general not isomorphic to the group $KO_{p+1}(W)$, and explicitly depends on the dimension of the spacetime. We can physically interpret the elements of $KO_{10}(W)$ as equally charged systems of wrapping $D9 - \bar{D}9$ -branes decaying on the submanifold W , and via the inclusion $i : W \hookrightarrow X$ they can be related to the $D9$ -branes used in the Sen-Witten construction. This is not surprising, as the decay mechanism is somehow at the heart of the spacetime D-brane charge classification, and it reinforces the statement that the group (4.8.1) encodes spacetime properties of the Dp -brane. After introducing the K-theoretical description of Ramond-Ramond fields in the next chapter, we will argue that wrapped D-branes can in principle couple to Ramond-Ramond fields.

We comment at this point that the definition of wrapped branes and of wrapped charge as presented in this section is very natural from the mathematical point of view, but is still heuristic in nature. Indeed, stronger evidences for wrapped D-branes and their coupling to Ramond-Ramond fields should come from the boundary conformal field theory describing type I String theory in topologically nontrivial settings, which has not yet been investigated in full generality.

4.9 The group $KO_{\sharp}^t(\text{pt})$ and torsion branes

In this section we will find explicit generators for the group $KO_{\sharp}^t(\text{pt})$, which, as explained in the previous section, we can interpret as stable D-branes wrapping a point

in spacetime. The stability of such D-branes is related to the fact that their charge is the “lightest” in the group of possible charges, hence the conservation of charge does not allow these D-branes to decay. To this end, we first give sufficient combinatorial criteria on the rational homology of X which ensure that torsion-free D-branes can wrap non-trivial spin cycles of the spacetime X , which is an adaptation of an analogous result in [80]

Theorem 4.45. *Let X be a compact connected finite CW-complex of dimension n whose rational homology can be presented as*

$$H_{\sharp}(X, \mathbb{Q}) = \bigoplus_{p=0}^n \bigoplus_{i=1}^{m_p} [M_i^p] \mathbb{Q} ,$$

where M_i^p is a p -dimensional compact connected spin submanifold of X without boundary and with orientation cycle $[M_i^p]$ given by the spin structure. Then the KO -homology lattice $\Lambda_{KO_{\sharp}^t(X)} := KO_{\sharp}^t(X) / \text{tor}_{KO_{\sharp}^t(X)}$ contains a set of linearly independent elements given by the classes of KO -cycles

$$[M_i^p, \mathbb{1}_{M_i^p}^{\mathbb{R}}, \iota_i^p] , \quad 0 \leq p \leq n , \quad 1 \leq i \leq m_p .$$

Proof. By [80] the cycles $[M_i^p, \mathbb{1}_{M_i^p}^{\mathbb{C}}, \iota_i^p]$ form a rational basis for the lattice $\Lambda_{K_{\sharp}^t(X)} := K_{\sharp}^t(X) / \text{tor}_{K_{\sharp}^t(X)}$ in K -homology. The conclusion follows from the fact that

$$[M_i^p, \mathbb{1}_{M_i^p}^{\mathbb{R}}, \iota_i^p] \otimes \mathbb{C} = [M_i^p, \mathbb{1}_{M_i^p}^{\mathbb{C}}, \iota_i^p] ,$$

i.e. that the elements $\text{ch}_{\bullet}^{\mathbb{R}}(M_i^p, \mathbb{1}_{M_i^p}^{\mathbb{R}}, \iota_i^p)$ form a set of generators of $H_{\sharp}(X, \mathbb{Q})$. \square

Recall now, that we have $KO_n^t(\text{pt}) \simeq \mathbb{Z}$ for $n = 0, 4$, $KO_n^t(\text{pt}) \simeq \mathbb{Z}_2$ for $n = 1, 2$, and 0 otherwise. For $n = 0, 4$ Theorem 4.45 and the Chern character assure that the classes $[\text{pt}, \mathbb{1}_{\text{pt}}^{\mathbb{R}}, \text{id}_{\text{pt}}]$ and $[S^4, \mathbb{1}_{S^4}^{\mathbb{R}}, \zeta]$ are generators of the groups $KO_0^t(\text{pt}) \simeq \mathbb{Z}$ and $KO_4^t(\text{pt}) \simeq \mathbb{Z}$, respectively. Let us now consider the group $KO_1^t(\text{pt})$. Consider the circle S^1 and assign to it a Riemannian metric. Since there is only one unit tangent vector at any point of S^1 , one has $P_{SO}(S^1) \simeq S^1$, where $P_{SO}(S^1)$ is the orthonormal frame bundle of S^1 . A spin structure on S^1 is thus given by a double covering

$$P_{\text{Spin}}(S^1) \longrightarrow S^1$$

and by the fibration

$$\begin{array}{ccc} \mathbb{Z}_2 & \longrightarrow & P_{\text{Spin}}(S^1) . \\ & & \downarrow \\ & & S^1 \end{array}$$

There are only two double coverings of the circle, one disconnected and the other connected, given respectively by

$$S^1 \times \mathbb{Z}_2 \longrightarrow S^1, \quad S_M^1 \longrightarrow S^1$$

where S_M^1 is the total space of the principal \mathbb{Z}_2 -bundle associated to the Möbius strip. We will call these two spin structures the “interesting” and the “uninteresting” spin structures, respectively.

Corresponding to these two spin structures (labelled ‘i’ and ‘u’, respectively), we construct classes in $KO_1^{\dagger}(\text{pt})$ given by $[S_i^1, \mathbb{1}_{S^1}^{\mathbb{R}}, \zeta]$ and $[S_u^1, \mathbb{1}_{S^1}^{\mathbb{R}}, \zeta]$ where $\zeta : S^1 \rightarrow \text{pt}$ is as usual the collapsing map. We will now compute the topological indices in detail, finding the AMS invariants [63]

$$\widehat{\mathcal{A}}_{\mathbb{1}_{S^1}^{\mathbb{R}}}(S_i^1) = 1, \quad \widehat{\mathcal{A}}_{\mathbb{1}_{S^1}^{\mathbb{R}}}(S_u^1) = 0$$

in $KO^{-1}(\text{pt}) \simeq \mathbb{Z}_2$. Hence the two classes above represent the elements of $KO_1^{\dagger}(\text{pt}) \simeq \mathbb{Z}_2$. In particular, $[S_i^1, \mathbb{1}_{S^1}^{\mathbb{R}}, \zeta]$ is a generator, analogous to the non-BPS Type I D-particle that arises from tachyon condensation.

Let us first consider the circle with the interesting spin structure. Since $Cl_1 \simeq \mathbb{C}$, one has $\mathfrak{S}(S^1) := P_{\text{Spin}}(S^1) \times_{\mathbb{Z}_2} Cl_1 \simeq S^1 \times \mathbb{C}$. By decomposing $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$, one has the identifications $\mathfrak{S}^0(S^1) = S^1 \times \mathbb{R}$ and $\mathfrak{S}^1(S^1) = S^1 \times i\mathbb{R}$. As the Clifford bundle is trivial, its space of sections is given by $C^{\infty}(S^1, \mathfrak{S}(S^1)) = C^{\infty}(S^1, \mathbb{C})$. By coordinatizing the circle S^1 with arc length s , the Atiyah-Singer operator can be expressed as

$$\mathcal{D}^{S^1} = i \frac{d}{ds} \tag{4.9.1}$$

where $e_1 = i$ is a generator of the Clifford algebra Cl_1 . To compute the topological index $\widehat{\mathcal{A}}_{\mathbb{1}_{S^1}^{\mathbb{R}}}(S_i^1)$, we use the Cl_1 -index Theorem 4.41 and hence determine the vector space $\ker \mathcal{D}^{S^1}$, or equivalently the chiral subspace $\ker(\mathcal{D}^{S^1})^0$. Since $C^{\infty}(S^1, \mathfrak{S}^0) = C^{\infty}(S^1, \mathbb{R})$, the kernel of the chiral Atiyah-Singer operator $(\mathcal{D}^{S^1})^0 : C^{\infty}(S^1, \mathfrak{S}^0) \rightarrow C^{\infty}(S^1, \mathfrak{S}^1)$ is given by the space of real-valued constant functions on S^1 . The dimension of this vector space, as a module over $Cl_1^0 \simeq \mathbb{R}$, is 1 and hence

$$\text{ind}_1^{\dagger}(S_i^1, \mathbb{1}_{S^1}^{\mathbb{R}}, \zeta) = [\ker(\mathcal{D}^{S^1})^0] = 1$$

in $\mathfrak{M}_0/i^*\mathfrak{M}_1 \simeq KO^{-1}(\text{pt}) \simeq \mathbb{Z}_2$. (Note that here we are using *ungraded* Clifford modules.)

We now turn to the uninteresting spin structure on S^1 . This time the bundle $\mathfrak{S}(S^1)$ is the (infinite complex) Möbius bundle. It can be described by a trivialization made

of three charts U_1 , U_2 and U_3 with \mathbb{Z}_2 -valued transition functions $g_{12} = 1$, $g_{23} = 1$ and $g_{31} = -1$. In this case, the vector space $\ker(\mathcal{D}^{S^1})^0$ consists of locally constant real-valued functions ψ_i defined on U_i which satisfy $\psi_j = g_{ji} \psi_i$ on the intersections $U_i \cap U_j \neq \emptyset$. Because of the non-trivial transition function g_{31} , there are no non-zero solutions ψ to the equation $(\mathcal{D}^{S^1})^0 \psi = 0$. The kernel $\ker(\mathcal{D}^{S^1})^0$ is thus trivial, and so

$$\mathrm{ind}_1^t(S^1, \mathbb{1}_{S^1}^{\mathbb{R}}, \zeta) = 0 .$$

Let us now consider the structure of the group $\mathrm{KO}_2^t(\mathrm{pt})$. Analogously to the construction above, one can equip the torus $T^2 = S^1 \times S^1$ with an “interesting” spin structure and show that

$$\widehat{\mathcal{A}}_{\mathbb{1}_{T^2}^{\mathbb{R}}}(T^2) = 1 ,$$

and also that

$$\widehat{\mathcal{A}}_{\mathbb{1}_{S^2}^{\mathbb{R}}}(S^2) = 0$$

in $\mathrm{KO}^{-2}(\mathrm{pt}) \simeq \mathbb{Z}_2$. It follows that the classes $[T^2, \mathbb{1}_{T^2}^{\mathbb{R}}, \zeta]$ and $[S^2, \mathbb{1}_{S^2}^{\mathbb{R}}, \zeta]$ represent the elements of the group $\mathrm{KO}_2^t(\mathrm{pt}) \simeq \mathbb{Z}_2$. In particular, $[T^2, \mathbb{1}_{T^2}^{\mathbb{R}}, \zeta]$ is a generator, and it is analogous to the Type I non-BPS D-instanton which is usually constructed as the Ω -projection of the Type IIB D(-1) brane-antibrane system. We will now give some details of these results.

Equip T^2 with the flat metric $d\theta_1 \otimes d\theta_1 + d\theta_2 \otimes d\theta_2$, where (θ_1, θ_2) are angular coordinates on $S^1 \times S^1$. Since T^2 is a Lie group, its tangent bundle is trivializable, and hence the oriented orthonormal frame bundle is canonically given by $P_{\mathrm{SO}}(T^2) = T^2 \times S^1$. Consider the spin structure on T^2 given by

$$P_{\mathrm{Spin}}(T^2) = T^2 \times S^1 \xrightarrow{\mathrm{id}_{T^2} \times z^2} T^2 \times S^1 .$$

Since $\mathrm{Cl}_2^0 \simeq \mathbb{H}$ and $\mathrm{Cl}_2^1 \simeq \mathbb{C}$, the corresponding Clifford bundles are $\mathcal{C}(T^2) = T^2 \times \mathbb{H}$ and $\mathcal{C}^0(T^2) = T^2 \times \mathbb{C}$. In the riemannian coordinates (θ_1, θ_2) , the Atiyah-Singer operator can be expressed as

$$\mathcal{D}^{T^2} = \sigma_1 \frac{\partial}{\partial \theta_1} + \sigma_2 \frac{\partial}{\partial \theta_2}$$

where the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

represent the generators e_1, e_2 of Cl_2 , acting by left multiplication. The chiral operator $(\mathcal{D}^{T^2})^0$ is locally the Cauchy-Riemann operator, and hence its kernel consists

of holomorphic sections of the chiral Clifford bundle $\mathfrak{S}^0(\mathbb{T}^2)$. These are simply the complex-valued constant functions on \mathbb{T}^2 , as the torus is a compact complex manifold. As a module over $\mathcal{C}\ell_2^0$, this vector space is one-dimensional and so

$$\text{ind}_2^t(\mathbb{T}^2, \mathbb{1}_{\mathbb{T}^2}^{\mathbb{R}}, \zeta) = [\ker(\mathfrak{D}^{\mathbb{T}^2})^0] = 1$$

in $\mathfrak{M}_1/i^*\mathfrak{M}_2 \simeq KO^{-2}(\text{pt}) \simeq \mathbb{Z}_2$.

Consider now the two-sphere S^2 as a riemannian manifold. It is not difficult to see that

$$P_{\text{SO}}(S^2) = \text{SO}(3) \longrightarrow \text{SO}(3)/\text{SO}(2) \simeq S^2$$

is the oriented orthonormal frame bundle over S^2 . The (unique) spin structure on S^2 is thus given by

$$\begin{array}{ccc} P_{\text{Spin}}(S^2) \simeq \text{SU}(2) & \xrightarrow{h} & P_{\text{SO}}(S^2) \simeq \text{SO}(3) \\ & \searrow \text{U}(1) & \downarrow \text{SO}(2) \\ & & S^2 \end{array}$$

with $h : \text{SU}(2) \rightarrow \text{SO}(3)$ the usual double covering, and by

$$\begin{array}{ccc} \text{U}(1) & \longrightarrow & P_{\text{Spin}}(S^2) \\ & & \downarrow \\ & & S^2 \end{array}$$

which is the Hopf fibration of S^2 . Recall that the group $\text{Spin}(2) \simeq \text{U}(1) \simeq \text{SO}(2)$ acts on $\mathcal{C}\ell_2 \simeq \mathbb{H}$ as multiplication by

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in [0, 2\pi).$$

If one gives the sphere S^2 the structure of the complex projective line $\mathbb{C}\mathbb{P}^1$, then there are isomorphisms $\mathfrak{S}^0(S^2) = P_{\text{Spin}}(S^2) \times_{\text{U}(1)} \mathbb{C} \simeq T^{1,0}\mathbb{C}\mathbb{P}^1$ since the bundle $\mathfrak{S}^0(S^2)$ has the same transition functions as the Hopf fibration. In other words, $\mathfrak{S}^0(S^2)$ is isomorphic to the canonical line bundle $L_{\mathbb{C}}$ over $\mathbb{C}\mathbb{P}^1$. The vector space $\ker(\mathfrak{D}^{S^2})^0$ thus consists of the holomorphic sections of $L_{\mathbb{C}}$. The only such section on $\mathbb{C}\mathbb{P}^1$ is the zero section [72], and we finally find

$$\text{ind}_2^t(S^2, \mathbb{1}_{S^2}^{\mathbb{R}}, \zeta) = [\ker(\mathfrak{D}^{S^2})^0] = 0$$

in $\mathfrak{M}_1/i^*\mathfrak{M}_2 \simeq \mathbb{Z}_2$.

Remark As we have seen above, the problem of finding generators of the geometric

KO-homology groups of a space X becomes increasingly involved at a very rapid rate. Even in the case of spherical D-branes, we have not been able to find a nice explicit solution. Nevertheless, at least in these cases we can find a formal solution as follows, which also illustrates the generic problems at hand.

Suppose that we want to construct generating branes for the group $\mathrm{KO}_k^t(S^n)$ for some $n > 0$. Poincaré duality gives the map

$$\mathrm{KO}^{n-k}(S^n) \longrightarrow \mathrm{KO}_k^t(S^n), \quad \xi \longmapsto \xi \cap [S^n, \mathbb{1}_{S^n}^{\mathbb{R}}, \mathrm{id}_{S^n}]. \quad (4.9.2)$$

As Poincaré duality is a group isomorphism, picking a generator in $\mathrm{KO}^{n-k}(S^n)$ will give a generator in $\mathrm{KO}_k^t(S^n)$. But the problem is that the class ξ is not a (virtual or stable) vector bundle over S^n in the cases of interest $k < n$. To this end, we rewrite the cap product in (4.9.2) by using the suspension isomorphism Σ and the desuspension Σ^{-1} to get

$$\xi \cap [S^n, \mathbb{1}_{S^n}^{\mathbb{R}}, \mathrm{id}_{S^n}] = \Sigma^{-1} \left(\Sigma(\xi) \cap \Sigma[S^n, \mathbb{1}_{S^n}^{\mathbb{R}}, \mathrm{id}_{S^n}] \right).$$

As we are interested only in generators, we can substitute $\Sigma(\xi)$ with the generators of the KO-theory group $\mathrm{KO}^0(\Sigma^{n-k}S^n) = \widetilde{\mathrm{KO}}^0(S^{2n-k})$. The generators of the latter groups are given by [61] the canonical line bundle $L_{\mathbb{F}}$ over the projective line $\mathbb{F}\mathbb{P}^1$, with \mathbb{F} the reals \mathbb{R} for $k = 2n - 1$, the complex numbers \mathbb{C} for $k = 2n - 2$, the quaternions \mathbb{H} for $k = 2n - 4$ and the octonions \mathbb{O} for $k = 2n - 8$.

Chapter 5

Abelian Gauge Theories and Differential Cohomology

We have seen in the previous chapters that D-brane charges require the use of K-theory and K-homology to be properly described. Moreover, since D-branes are electric and magnetic sources for Ramond-Ramond fields, we expect that some form of K-theory will play a relevant role also in the description of these generalized gauge fields. In this chapter we introduce some basic notions of *generalized differential cohomology theories*, which we will see constitute a powerful mathematical machinery to describe abelian gauge theories of differential forms, and in particular the theory of Ramond-Ramond fields. We first motivate the use of this formalism in the case of ordinary electromagnetism, following [40, 44].

5.1 An example: the electromagnetic case

In ordinary electromagnetism formulated on the four dimensional Minkowski space-time $M^4 = \mathbb{R}_t \times \mathbb{R}^3$, the Maxwell equations are given by

$$\begin{aligned}dF &= 0 \\ d \star F &= j_e\end{aligned}\tag{5.1.1}$$

where $F \in \Omega^2(M; \mathbb{R})$, and where $j_e \in \Omega^3(M; \mathbb{R})$ is the electric current distribution. Since M^4 is a contractible space¹, equations (5.1.1) and Poincaré's Lemma imply that there exists a form $A \in \Omega^1(M; \mathbb{R})$, called the *vector potential*, such that

$$F = dA\tag{5.1.2}$$

¹More precisely, we only use that every 2-sphere is the boundary of a 3-ball.

As we have seen in section 2.5, the total electric charge of the distribution can be identified with the class $[j_e|_{\mathbb{R}^3}]$ in $H_{\text{cpt}}^3(\mathbb{R}^3; \mathbb{R}) \simeq \mathbb{R}$.

We can modify the equations (5.1.1) by introducing a magnetic current $j_m \in \Omega^3(M; \mathbb{R})$, and allowing the equations

$$\begin{aligned} dF &= j_m \\ d \star F &= j_e \end{aligned} \tag{5.1.3}$$

The magnetic charge of the system Q_m is given by the class $[j_m|_{\mathbb{R}^3}]$ in $H_{\text{cpt}}^3(\mathbb{R}^3; \mathbb{R})$. The equations (5.1.3) have changed the “global properties” of F : indeed, it is no longer a closed form, and equation (5.1.2) no longer holds. If we denote with W_e and W_m the support of the forms j_e and j_m respectively, and supposing $W_e \cap W_m = \emptyset$, we can consider the following equations defined on $M^4 - W_m$

$$\begin{aligned} dF &= 0 \\ d \star F &= j_e \end{aligned} \tag{5.1.4}$$

Notice at this point that equations (5.1.4) do not imply equation (5.1.2), since the space $M^4 - W_m$ is in general not contractible. However, for any contractible open set $\mathcal{U}_\alpha \subset M^4 - W_m$ we have

$$F|_{\mathcal{U}_\alpha} = dA_\alpha$$

and on overlaps $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ we have

$$A_\alpha - A_\beta = dg_{\alpha\beta} \quad g_{\alpha\beta} \in C^\infty(M; \mathbb{R})$$

Finally, on triple overlaps $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$

$$g_{\alpha\beta} + g_{\beta\gamma} + g_{\gamma\alpha} = c_{\alpha\beta\gamma}$$

where $c_{\alpha\beta\gamma}$ is a real constant over the entire triple overlap. The fact that in the presence of magnetic charges the vector potential is not globally defined requires that the coupling term

$$\int_{M^4} A \wedge j_e = \int_{W_e} A$$

be carefully defined, as W_e could intersect more patches \mathcal{U}_α . However, one can show that the classical action evaluated over a given worldline W_e is ambiguous up to a constant $c_{\alpha\beta\gamma}$ [2], and this does not affect the classical equations of motion. Indeed, equations (5.1.4) only depend on the fieldstrength F , which is a globally defined two form. At the quantum level, instead, this classical ambiguity leads to inconsistencies, unless some restrictions are imposed on the collection of all $\{c_{\alpha\beta\gamma}\}$. For example, in

the path integral quantization, an ambiguous phase factor of $\exp\{i c_{\alpha\beta\gamma}\}$ is potentially present at each non-empty triple intersection of patches: the only way to avoid this ambiguity is to require that $c_{\alpha\beta\gamma} = 2\pi n_{\alpha\beta\gamma}$, where $n_{\alpha\beta\gamma}$ are *integer* numbers. Moreover, the constants $c_{\alpha\beta\gamma}$ can be directly related to the total magnetic flux, hence the total magnetic charge, implying that the two form F restricted to $M^4 - W_m$ has *integral periods*. This is the so called *Dirac quantization condition*, and was proposed by Dirac in [34], albeit in a different way. Mathematically, the Dirac quantization condition states that the class $[F]$ in $H^2(M^4 - W_m; \mathbb{R})$ lies in the image of the map

$$H^2(M^4 - W_m; \mathbb{Z}) \rightarrow H^2(M^4 - W_m; \mathbb{R})$$

induced in cohomology by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$.

Notice at this point that the above argument applies even to the case of Maxwell equations on a spacetime $M = \mathbb{R}_t \times N$ in the absence of any magnetic current, provided that $H^2(N; \mathbb{R}) \neq 0$. Also in this case, the Dirac quantization condition requires that the fieldstrength F has integer periods. Differential geometry provides a beautiful solution to the necessity of having a local vector potential A in the quantum theory, without forgetting the obstructions that prevent its global existence, and including the Dirac quantization condition. Indeed, one regards the fieldstrength F as the curvature of a connection A defined on a principal $U(1)$ -bundle $\pi : \mathcal{L} \rightarrow M$ with first Chern class $c_1(\mathcal{L}) = [c_{\alpha\beta\gamma}] \in H^2(M; \mathbb{Z})$. The relevant space² of fields for the quantum theory is then given by the space of all principal $U(1)$ -bundles with connection over M . Notice that this space extends the space of classical solutions of Maxwell equations by the flat connections, which may contribute nontrivially to the quantum theory.

As we have seen through the argument above, the Dirac quantization of charges is a required condition in order to have a well defined quantum theory. We have also seen that the theory of principal bundles can be used to geometrically encode this condition. However, this framework cannot be applied to the case of generalized electromagnetic theories introduced in section 2.5: indeed, since the fieldstrength F is given by a p -form, it cannot be realized as the curvature of some connection. One can then resort to a local description of these gauge fields, as done for the B-field in section 1.5. This approach has several disadvantages, though: it is usually difficult in this framework to determine the space of gauge equivalent field configurations, over which the path integral should be performed, and in particular it is difficult to introduce a

²It is actually a *groupoid*, where the morphisms are given by connection preserving gauge transformations.

coupling of these fields with their sources, as one needs a proper notion of pullback, integration, etc. In the following section, we will introduce the proper mathematical formalism to treat these fields, whose recent development has been greatly motivated by the very problems mentioned above.

5.2 Differential Cohomology

In the past 30 years, Differential Cohomology has appeared in the mathematical literature as the theory of differential characters, Deligne cocycles, sparks, and more recently differential functions [28, 23, 51, 54]. From the mathematical point of view, it has provided a refinement of the theory of characteristic classes and characteristic forms, which, in the appropriate contexts, gives rise to obstruction to conformal immersion of Riemannian manifolds in euclidean spaces; more recently, in combination with differential K-theory, it has been used to construct an index theorem for flat bundles. As mentioned above, from the physical point of view it constitutes a powerful formalism to describe gauge theories of p -forms whose Dirac quantization condition is dictated by integer cohomology. In the following we will focus on two particular descriptions of Differential Cohomology, which use Cheeger-Simons characters and Deligne cocycles, respectively. We will illustrate the relation between differential cohomology and electromagnetism with Dirac quantization of charges, and we will give a definition of *generalized abelian gauge theories* in terms of these objects.

5.2.1 Cheeger-Simons characters

Let M denote a smooth manifold, and let $\mathcal{C}_k \supset \mathcal{Z}_k \supset \mathcal{B}_k$ denote the groups of smooth singular chains, cycles, and boundaries with $\partial : \mathcal{C}_k \rightarrow \mathcal{C}_{k-1}$ and $\delta : \mathcal{C}^k \rightarrow \mathcal{C}^{k+1}$ the usual boundary and coboundary operators, respectively. Let $\Omega_{\mathbb{Z}}^k(M) \subset \Omega^k(M)$ denote the lattice of closed k -forms with integral periods. Notice that if we denote with r the map induced on cohomology by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$, then a k -form ω has integral periods if and only if $[\omega] = r(u)$, for some $u \in H^k(M; \mathbb{Z})$. Given $\omega \in \Omega^k(M)$, we have the map

$$\Omega^k(M) \rightarrow \mathcal{C}^k(M; \mathbb{R}/\mathbb{Z}) \tag{5.2.1}$$

which assigns to ω the \mathbb{R}/\mathbb{Z} -valued cochain defined as

$$\tilde{\omega}(\sigma) := \int_{\sigma} \omega \bmod \mathbb{Z}, \quad \forall \sigma \in \mathcal{C}_k(M)$$

As the integral of a generic differential form over the set of all cycles never takes values only in \mathbb{Z} , the map (5.2.1) is an injection, and we denote the image of $\Omega^k(M)$ in $\mathcal{C}^k(M; \mathbb{R}/\mathbb{Z})$ with $\tilde{\Omega}^k(M)$. We have then the following

Definition 5.1. The *n*-th Cheeger-Simons group of a smooth manifold M is the group

$$\check{H}^n(M) := \left\{ f \in \text{Hom}(\mathcal{Z}_{n-1}, \mathbb{R}/\mathbb{Z}) : f \circ \partial \in \tilde{\Omega}^n(M) \right\}$$

We set $\check{H}^0(M) = \mathbb{Z}$. The elements of $\check{H}^*(M) := \bigoplus_n \check{H}^n(M)$ are called *differential characters*.

A smooth map $\phi : M \rightarrow N$ naturally induces a homomorphism

$$\phi^* : \check{H}^n(N) \rightarrow \check{H}^n(M)$$

Hence, \check{H}^n is a contravariant functor from the category of smooth manifolds to the category of abelian groups.

A main result is the following [28]

Proposition 5.2. *There exist surjective maps*

$$\begin{aligned} \check{H}^n(M) &\xrightarrow{F} \Omega_{\mathbb{Z}}^n(M) \\ \check{H}^n(M) &\xrightarrow{c} H^n(M; \mathbb{Z}) \end{aligned}$$

for any smooth manifold M and any $n \in \mathbb{N}$.

The map F is called the fieldstrength map, and the map c is called the characteristic class map.

Proof. Let $f \in \check{H}^n(M)$. Consider an element $T' \in \text{Hom}(\mathcal{Z}_{n-1}, \mathbb{R})$ such that

$$\tilde{T}' = f$$

where the tilde means mod \mathbb{Z} . Consider now the following (split) exact sequence

$$0 \rightarrow \mathcal{Z}_{n-1} \xrightarrow{i} \mathcal{C}_{n-1} \xrightarrow{\partial} \mathcal{B}_{n-2} \rightarrow 0$$

Since $\text{Hom}(\cdot, \mathbb{R})$ is an exact functor, the following exact sequence holds

$$0 \rightarrow \text{Hom}(\mathcal{B}_{n-2}, \mathbb{R}) \xrightarrow{\delta} \text{Hom}(\mathcal{C}_{n-1}, \mathbb{R}) \xrightarrow{i^*} \text{Hom}(\mathcal{Z}_{n-1}, \mathbb{R}) \rightarrow 0 \quad (5.2.2)$$

where the map i^* is given by restriction. Hence, there exist $T \in \text{Hom}(\mathcal{C}_{n-1}, \mathbb{R})$ such that $T|_{\mathcal{Z}_{n-1}} = T'$, and consequently

$$\tilde{T}|_{\mathcal{Z}_{n-1}} = f$$

Since by construction $\delta T := T \circ \partial$, and $\widetilde{\delta T} = \delta \widetilde{T}$, we have that

$$\delta \widetilde{T} = f \circ \partial$$

By assumption $\delta \widetilde{T}$ is an element of $\widetilde{\Omega}^n(M)$. Any such element can be written as $\omega - c$, where ω is a n -form regarded as a real cochain by integration, and $c \in \mathcal{C}^n(M, \mathbb{Z})$. Since $\delta T = \omega - c$, we have

$$0 = \delta^2 T = \delta \omega - \delta c = d\omega - \delta c \quad (5.2.3)$$

where d denotes the deRham differential. Since the map (5.2.1) is an injection, equation (5.2.3) implies that $d\omega = 0$ and $\delta c = 0$. Finally, since $\delta T = \omega - c$ we have that $[\omega] = r([c])$, with $[c] \in H^n(M; \mathbb{Z})$, which implies that $\omega \in \Omega_{\mathbb{Z}}^n(M)$. The elements ω and $[c]$ do not depend on the *lift* T . Indeed, let T' be another real cochain such that $\widetilde{T}'|_{\mathcal{Z}_{n-1}} = f$. Then $\widetilde{T - T'}|_{\mathcal{Z}_{n-1}} = 0$, and the sequence (5.2.2) implies that

$$T' = T + \delta\alpha + \beta \quad (5.2.4)$$

with $\alpha \in \mathcal{C}^{n-2}(M; \mathbb{R})$ and $\beta \in \mathcal{C}^{n-1}(M; \mathbb{Z})$. By the argument above there exist ω' and c' such that $\delta T' = \omega' - c'$. Hence we have

$$\omega' - c' = \delta T' = \delta T + \delta\beta = \omega - c + \delta\beta$$

which implies

$$\omega' - \omega = c' - c + \delta\beta \in \text{Cl}(V; q)$$

By the same reason as before, we have $\omega' = \omega$ and $[c'] = [c]$.

We define $F(f) = \omega$ and $c(f) = [c]$, where ω and c are obtained as above. Moreover, we will refer to the form ω as the *fieldstrength* and to the class $[c]$ as *the characteristic class*; the use of this terminology will be clear later.

To prove that the maps F and c are surjective, notice that any $\omega \in \Omega_{\mathbb{Z}}^n(M)$ determines a $u \in H^n(M; \mathbb{Z})$ such that $[\omega] = r(u)$, and conversely for any u we can find such a ω . Let $[c] = u$. Then $\omega - c$ is exact as a real cochain, and there exists T such that $\delta T = \omega - c$. Then $\widetilde{T}|_{\mathcal{Z}_{n-1}}$ defines a homomorphism $f : \mathcal{Z}_{n-1} \rightarrow \mathbb{R}/\mathbb{Z}$ with the property that $f \circ \partial \in \widetilde{\Omega}^n(M)$. \square

We will now investigate the kernels of the fieldstrength and characteristic class maps.

Let $f \in \check{H}^n(M)$ satisfy $F(f) = 0$. Then the real cochain T satisfies $\delta T = -c$. Hence

$\delta\tilde{T} = 0$, and thus \tilde{T} is an \mathbb{R}/\mathbb{Z} -valued cocycle. If we consider another lift T' , equation (5.2.4) implies that $\tilde{T}' = \tilde{T} + \delta\tilde{\alpha}$, and hence $[T'] = [T] \in H^{n-1}(M; \mathbb{R}/\mathbb{Z})$. So an element f with $F(f) = 0$ determines an \mathbb{R}/\mathbb{Z} cohomology class. Conversely, given an \mathbb{R}/\mathbb{Z} class represented by a cocycle s , we have that $s|_{z_{n-1}}$ defines a differential character f , and such a definition is independent of the choice of the representing cocycle s .

Finally, let $f \in \check{H}^n(M)$ satisfy $c(f) = 0$. Then $\delta T = \omega - c$, with $c = \delta e$, for some $e \in \mathcal{C}^{n-1}(M; \mathbb{Z})$. Hence, $\delta(T - e) = \omega$. Since $[\omega] = r([c])$, by the deRham theorem we have that $\omega = d\theta$, for some $\theta \in \Omega^{n-1}(M)$. Since $\delta(T - e - \theta) = 0$, we have that $T - e - \theta = z$, for some cocycle $z \in \mathcal{Z}^{n-1}(M; \mathbb{R})$. Again by the deRham theorem, there exists a closed form $\phi \in \Omega^{n-1}(M)$ such that $\phi|_{z_{n-1}} = z|_{z_{n-1}}$, which implies that

$$f = \tilde{T}|_{z_{n-1}} = \theta + \phi + e = \tilde{\theta} + \phi$$

Hence f is in the image of the map $\omega \rightarrow \tilde{\omega}|_{z_{n-1}}$, with $\omega \in \Omega^{n-1}(M)$, whose kernel is given by $\Omega_{\mathbb{Z}}^{n-1}(M)$.

We have then proved the following [28]

Proposition 5.3. *There are natural exact sequences*

$$\begin{aligned} 0 \rightarrow H^{n-1}(M; \mathbb{R}/\mathbb{Z}) \rightarrow \check{H}^n(M) \xrightarrow{F} \Omega_{\mathbb{Z}}^n(M) \rightarrow 0 \\ 0 \rightarrow \Omega^{k-1}(M)/\Omega_{\mathbb{Z}}^{k-1}(M) \rightarrow \check{H}^n(M) \xrightarrow{c} H^n(M; \mathbb{Z}) \rightarrow 0 \end{aligned}$$

Another useful exact sequence can be obtained as follows. Consider the group

$$A^n(M) := \{(\omega, u) \in \Omega_{\mathbb{Z}}^n(M) \times H^n(M; \mathbb{Z}) : [\omega] = r(u)\}$$

We have then a surjective map

$$\check{H}^n(M) \xrightarrow{(F,c)} A^n(M)$$

The kernel of this map is given by elements $f \in \check{H}^n(M)$ such that $F(f) = 0$ and $c(f) = 0$. By the same argument as before, we have that $\delta(T - e) = 0$, hence the element f determines a class in $H^{n-1}(M; \mathbb{R})$ represented by the closed form ϕ that satisfies $T|_{z_{n-1}} = \phi + e$. As ϕ is a closed form, $\tilde{T}|_{z_{n-1}}$ is determined only by the class $[\phi] \in H^{n-1}(M; \mathbb{R})$. Conversely, any class $[\phi]$ determines a differential character $f = \tilde{\phi}|_{z_{n-1}}$, and the kernel of such an assignment is given precisely by $r(H^{n-1}(M; \mathbb{Z}))$. We have then showed that the following exact sequence holds

$$0 \rightarrow H^{n-1}(M; \mathbb{R})/r(H^{n-1}(M; \mathbb{Z})) \rightarrow \check{H}^n(M) \xrightarrow{(F,c)} A^n(M) \rightarrow 0 \quad (5.2.5)$$

The above sequences are very important, as they are usually the only computational technique available since the groups $\check{H}^n(M)$ are usually infinite dimensional.

Notice that the group $A^*(M) := \bigoplus_n A^n(M)$ carries an obvious ring structure given by

$$(\omega, u) \cdot (\phi, v) := (\omega \wedge \phi, u \cup v)$$

We expect then that the group $\check{H}^*(M)$ carries an analogous product. This is indeed the case. However, the ring product of differential characters is more subtle to define: this is due to the fact that given two forms ω_1 and ω_2 of degrees l and p , respectively, in general we have

$$\omega_1 \wedge \omega_2 \neq \omega_1 \cup \omega_2$$

where on the right hand side ω_1 and ω_2 are realized as real cochains. However, one has that

$$\delta E(\omega_1, \omega_2) = \omega_1 \wedge \omega_2 - \omega_1 \cup \omega_2$$

for some $E(\omega_1, \omega_2) \in \mathcal{C}^{l+p-1}(M; \mathbb{R})$. The technical difficulty consists exactly in constructing the cochain $E(\omega_1, \omega_2)$ in a canonical way, and we refer the reader to [28] for details. A definition of the ring product can then be given in the following way. Let $f \in \check{H}^l(M)$ and $g \in \check{H}^p(M)$, and choose the lifts $T_f \in \mathcal{C}^{l-1}(M; \mathbb{R})$ and $T_g \in \mathcal{C}^{p-1}(M; \mathbb{R})$ for f and g respectively. Then we can define [28]

$$f \star g := \widetilde{T_f \cup \omega_f} - (-1)^{l-1} \widetilde{\omega_f \cup T_g} - \widetilde{T_f \cup \delta T_g} + E(\omega_f, \omega_g)|_{\mathcal{Z}_{l+p-1}}$$

and it can be shown that $f \star g$ is independent of the choice of T_f and T_g . Moreover, the product \star is associative, graded commutative, and it is such that the fieldstrength and characteristic class maps are ring homomorphisms. Even if the product \star is fairly complicated in general, there are special cases in which it greatly simplifies. Indeed, if $g \in \ker F \subset \check{H}^p(M)$, then $f \star g$ is the image of $(-1)^l c(f) \cup g$, for $f \in \check{H}^l(M)$, while if $g \in \ker c \subset \check{H}^p(M)$, $f \star g$ is the image of $(-1)^l F(f) \wedge g$.

In the following we will give some examples.

Example 5.4. Let $M = S^1$ and consider the group $\check{H}^1(S^1)$. Since $\mathcal{Z}_0(S^1)$ is freely generated by the points of S^1 , we have that $\check{H}^1(S^1) \simeq C^\infty(S^1, S^1)$, where we have realized \mathbb{R}/\mathbb{Z} as S^1 . Consider then a smooth function

$$f : S^1 \rightarrow S^1$$

Any such function f can be expressed as

$$f(p) = e^{2\pi i \Theta(f(p))}$$

where $\Theta = \theta \circ f$, with θ the usual local angular coordinate on the circle. Then a lift T of f is given by

$$T = \frac{1}{2\pi i} \log f = \Theta(p) + \eta(p)$$

where η is a locally constant \mathbb{Z} -valued function. By assumption, we know that on any curve γ on S^1

$$\delta\tilde{T}(\gamma) = e^{2\pi i(\Theta(b) - \Theta(a))}$$

where a and b are the endpoints of the curve. The right hand side of the above equation can be written as

$$\exp \left\{ 2\pi i \int_{\gamma} d\Theta \right\}$$

Notice that $d\Theta$ is not an exact 1-form, but it is integral, as $d\Theta = f^*d\theta$. Hence we have shown that

$$d \left(\frac{1}{2\pi i} \log f \right)$$

is the fieldstrength of f . Since the group $H^1(S^1; \mathbb{Z})$ contains no torsion, the fieldstrength determines the characteristic class: hence, we have

$$[c] = f^*[d\theta]$$

The same argument can be applied to the group $\check{H}^1(M)$ for any smooth manifold M .

Example 5.5. For any smooth manifold M with $n = \dim M$ we have

$$\begin{aligned} \check{H}^{n+1}(M) &\simeq H^n(M; \mathbb{R}/\mathbb{Z}) \\ \check{H}^k(M) &= 0, \quad k > n + 1 \end{aligned}$$

This can be shown by using the exact sequences introduced above.

Example 5.6. Let $M = \text{pt}$. Then

$$\check{H}^n(M) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{R}/\mathbb{Z}, & n = 1 \\ 0, & \text{otherwise} \end{cases}$$

Example 5.7. Finally, we present an example coming from differential geometry, which is in a certain sense the ‘‘canonical’’ one.

Consider a complex line bundle $\mathcal{L} \rightarrow M$ with connection ∇ , and denote with F_{∇} its curvature form. Since $\frac{1}{2\pi} F_{\nabla}$ represents the real first Chern class of \mathcal{L} , we have that $F_{\nabla} \in \Omega_{\mathbb{Z}}^2(M)$. Let γ be a closed curve, and define

$$f(\gamma) := \text{hol}_{\nabla}(\gamma) \in \mathbb{R}/\mathbb{Z}$$

where hol_∇ denotes the holonomy of the connection ∇ . We can extend the homomorphism f to the whole of $\mathcal{Z}_1(M)$ as follows. Let $z \in \mathcal{Z}_1(M)$. Represent z as $z = \sum_i n_i \gamma_i + \partial y$, where γ_i is a closed curve, and $y \in \mathcal{C}_2(M)$. Then we can define

$$f(z) := \prod_i f(\gamma_i)^{n_i} + \frac{1}{2\pi} \tilde{F}_\nabla(y)$$

The homomorphism f above is well defined, and independent of the presentation of z , since F_∇ has integral periods. Since $f \circ \partial = \frac{1}{2\pi} F_\nabla$, we have that $f \in \check{H}^2(M)$. Moreover, one can check that

$$\begin{aligned} F(f) &= \frac{1}{2\pi} \tilde{F}_\nabla \\ c(f) &= c_1(\mathcal{L}) \end{aligned}$$

Notice that the character f contains more information than F_∇ and $c_1(\mathcal{L})$ together, since both may vanish when f does not, e.g. when $M = S^1$.

It is immediate to realize in the above example that if we perform a connection preserving gauge transformation on \mathcal{L} , the homomorphism f does not change, hence it is a gauge invariant quantity. Conversely, given an element $f \in \check{H}^2(M)$ we can construct up to isomorphism a line bundle \mathcal{L} classified by $c(f)$, equipped with a connection ∇ defined by requiring that

$$\text{hol}_\nabla(\gamma) := f(\gamma)$$

for any closed curve γ . Hence, for any smooth manifold M the group $\check{H}^2(M)$ is equivalent to the space of all gauge equivalent complex line bundles with connection over M : the group structure on the latter space is induced by tensor product of line bundles. We have then realized that the group $\check{H}^2(M)$ is the set of orbits of the group of gauge transformations acting on the space of connections on *all* line bundles over M : noncanonically, this space can be expressed as

$$\bigcup_{c_1 \in H^2(M; \mathbb{Z})} \mathcal{A}(\mathcal{L}_{c_1}) / \mathcal{G}$$

where $\mathcal{A}(\mathcal{L}_{c_1})$ is the affine space of connections on the line bundle \mathcal{L}_{c_1} classified by the class c_1 , and \mathcal{G} is the group of gauge transformations. As seen in section 5.1, this is precisely the space of all possible gauge inequivalent solutions to Maxwell equations on M , taking into account the Dirac quantization condition. Realizing this space as the group $\check{H}^2(M)$ naturally suggests the following

Definition 5.8. A *generalized abelian gauge theory* on a smooth manifold M is a field theory whose space of gauge inequivalent configurations is given by $\check{H}^n(M)$, for some $n \in \mathbb{N}$.

The generalized electromagnetism discussed in section 2.5.1, once provided with a suitable Dirac quantization condition, is a generalized abelian gauge theory. Because of the relation between the group $\check{H}^n(M)$ and gauge theories of n -forms, one usually refers to $H^{n-1}(M; \mathbb{R}/\mathbb{Z})$ as the group of *flat fields*, while $\Omega^{n-1}(M)/\Omega_{\mathbb{Z}}^{n-1}(M)$ is called the group of *topologically trivial fields*. Consequently, we can see that the group $H^{n-1}(M; \mathbb{R})/r(H^{n-1}(M; \mathbb{Z}))$ classifies the (gauge equivalence classes of) *flat and topologically trivial fields*. Notice that if we classify inequivalent field configurations only by curvature and characteristic class, we are not taking into account the effect of flat and topologically trivial fields, which may be nonvanishing, according to the topology of M .

Since the electromagnetic field can be described with complex line bundles and connections thereon, this suggests that elements of $\check{H}^n(M)$ should represent isomorphism classes of “higher” line bundles with “higher” connections. This is indeed the case: for example, the group $\check{H}^3(M)$ is given by equivalence classes of bundle gerbes with connections. However, to be able to talk about the gauge fields and higher line bundles whose equivalent classes are the elements in $\check{H}^n(M)$, we need a *model* representing such a group. Recall indeed, that the group $\check{H}^2(M)$ can be obtained as the set of isomorphism classes for the groupoid of connections on M . To give such a model is the aim of the next section.

5.2.2 Deligne cohomology

In this section we will describe a cochain model for the Cheeger-Simons groups. This is done by introducing *Deligne cohomology*, which is defined via the cohomology of a certain complex. The Deligne cohomology groups are isomorphic to the Cheeger-Simons groups: however, we will not prove this result, and will focus instead on a detailed construction of the cochain model, showing how it can naturally describe the gauge fields in a generalized abelian gauge theory in topologically nontrivial backgrounds. In the following we will describe the *smooth* Deligne cohomology³.

In a nutshell, the differential complex used to define Deligne cohomology is a “modification” of the Čech-de Rham complex: we will recall some basics about the Čech-de Rham complex in order to clearly show the nature of such a modification.

Let M be a paracompact smooth manifold with $n = \dim M$, and consider a good cover

³Deligne cohomology originated in the context of algebraic geometry.

$\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$, where the index set I may be infinite. Denote

$$U_{\alpha_0 \alpha_1 \dots \alpha_p} := U_{\alpha_0} \cap U_{\alpha_1} \cap \dots \cap U_{\alpha_p}$$

Let $\Omega^p(M)$ be the set of real valued p -forms, and denote with $\Omega^0(M)$ the space of smooth real functions $C^\infty(M; \mathbb{R})$. Since each $U_{\alpha_0 \alpha_1 \dots \alpha_p}$ is contractible, Poincaré's Lemma implies that for any sequence $\alpha_0, \alpha_1, \dots, \alpha_p$ we have an exact sequence

$$0 \rightarrow C_{lc}^\infty(U_{\alpha_0 \alpha_1 \dots \alpha_p}; \mathbb{R}) \rightarrow \Omega^0(U_{\alpha_0 \alpha_1 \dots \alpha_p}) \xrightarrow{d} \Omega^1(U_{\alpha_0 \alpha_1 \dots \alpha_p}) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(U_{\alpha_0 \alpha_1 \dots \alpha_p}) \rightarrow 0$$

where $C_{lc}^\infty(M; \mathbb{R})$ denotes the space of smooth locally constant real valued functions. Define the group of p -cochains of the cover \mathfrak{U} with values in the q -forms as

$$\check{C}^p(\mathfrak{U}; \Omega^q) := \prod_{\alpha_0 < \alpha_1 < \dots < \alpha_p} \Omega^q(U_{\alpha_0 \alpha_1 \dots \alpha_p})$$

Notice that the inclusion

$$U_{\alpha_0 \dots \alpha_p} \hookrightarrow U_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_p}$$

induces a map

$$\phi_{\alpha_i} : \Omega^q(U_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_p}) \rightarrow \Omega^q(U_{\alpha_0 \dots \alpha_p})$$

which is given by restriction. We can then define the homomorphism

$$\delta : \check{C}^p(\mathfrak{U}; \Omega^q) \rightarrow \check{C}^{p+1}(\mathfrak{U}; \Omega^q)$$

which on $\omega \in \check{C}^p(\mathfrak{U}; \Omega^q)$ is defined as

$$\delta \omega := \sum_{i=0}^{p+1} (-1)^i \phi_{\alpha_i}(\omega)$$

The homomorphism δ satisfies $\delta^2 = 0$, hence it can be used to define a cohomology theory. However, we have the following Generalized Mayer-Vietoris exact sequence [21]

$$0 \rightarrow \Omega^q(M) \rightarrow \prod \Omega^q(U_{\alpha_0 \alpha_1}) \xrightarrow{\delta} \prod \Omega^q(U_{\alpha_0 \alpha_1 \alpha_3}) \xrightarrow{\delta} \dots$$

for any $q \geq 0$, which tells us that the δ -cohomology of the complex $\check{C}^*(\mathfrak{U}; \Omega^q)$ vanishes identically. Notice that for $q = 0$ this is related to the fact that $H^p(M; \underline{\mathbb{R}})$, the p -th cohomology group of M valued in the sheaf of smooth real valued functions, is 0 for all $p \geq 0$, since $\underline{\mathbb{R}}$ is a fine sheaf⁴.

Since the de Rham differential

$$d : \check{C}^p(\mathfrak{U}; \Omega^q) \rightarrow \check{C}^p(\mathfrak{U}; \Omega^{q+1})$$

⁴A fine sheaf is loosely speaking a sheaf with a “partition of unity”.

commutes with the coboundary homomorphism δ , we can form the following double complex, called the *Čech-de Rham* complex

$$\check{C}^*(M; \Omega^*) := \bigoplus_{p,q \geq 0} \check{C}^p(\mathfrak{U}; \Omega^q)$$

with coboundary operator

$$D = \delta + (-1)^{\deg(\cdot)} d$$

where for $\omega \in \check{C}^p(\mathfrak{U}; \Omega^q)$, $\deg(\omega) := p$.

As usual for differential double complexes, we can consider the diagonal subcomplex

$$K^n := \bigoplus_{p+q=n} \check{C}^p(\mathfrak{U}; \Omega^q)$$

By “tic-tac-toeing”, i.e by chasing diagrams, one can prove that

$$H_{\text{dR}}^*(M) \simeq H^*(K^*; D) \simeq \check{H}_{\text{Ch}}^*(M; \mathbb{R})$$

where $\check{H}_{\text{Ch}}^*(M; \mathbb{R})$ denote Čech cohomology. The above isomorphism can be seen as a consequence of the fact that *all* the “rows” and “columns” of the Čech-de Rham complex are exact⁵.

The basic idea behind Deligne cohomology consists in constructing a complex analog to the Čech-de Rham one, but based on the following exact sequence

$$\Omega_{\text{U}}^0(U_{\alpha_0 \alpha_1 \dots \alpha_p}) \xrightarrow{d \log} \Omega^1(U_{\alpha_0 \alpha_1 \dots \alpha_p}) \xrightarrow{d} \dots \xrightarrow{d} \Omega^l(U_{\alpha_0 \alpha_1 \dots \alpha_p})$$

where $\Omega_{\text{U}}^0(V) := C^\infty(V; \text{U}(1))$ for any open set V , and $0 \leq l \leq n$. In other words, we truncate the complex of forms on the right for some l , and we substitute in the 0 degree real valued functions with circle valued ones. We can then define a modified Čech-de Rham complex as

$$\check{C}^*[l](M; \Omega^*) := \bigoplus_{p \geq 0, 0 \leq q \leq l} \check{C}^p(\mathfrak{U}; \Omega^q)$$

and the associated single complex

$$K[l]^n := \bigoplus_{p+q=n} \check{C}^p(\mathfrak{U}; \Omega^q)$$

The Čech-de Rham coboundary D is unchanged, and we will refer to an element $\omega \in K[l]^*$ with $D\omega = 0$ as a *Deligne cochain*. For a given l , the Deligne cohomology is given by

$$\mathbb{H}^*(M; \mathcal{D}^l) := H^*(K[l]^*; D)$$

⁵Indeed, $H_{\text{dR}}^*(M)$ and $\check{H}_{\text{Ch}}^*(M; \mathbb{R})$ are the cohomology groups of the augmented column and row, respectively.

As for Čech cohomology, the result does not depend on the particular choice of good cover.

Notice that it is no longer true that all the rows of the double complex $\check{C}^*[l](M; \Omega^*)$ are exact. Indeed, for $q = 0$ the sequence

$$\prod \Omega_U^0(U_{\alpha_0}) \xrightarrow{\delta} \prod \Omega_U^0(U_{\alpha_0\alpha_1}) \xrightarrow{\delta} \prod \Omega_U^0(U_{\alpha_0\alpha_1\alpha_2}) \xrightarrow{\delta} \cdots$$

is in general not exact, and the δ -cohomology is given by $\check{H}_{\text{Ch}}^*(M; \underline{U(1)})$, the sheaf cohomology with value in the circle functions. Moreover, one can prove that

$$\check{H}_{\text{Ch}}^*(M; \underline{U(1)}) \simeq \check{H}_{\text{Ch}}^{*+1}(\mathfrak{U}; \mathbb{Z})$$

for any good cover \mathfrak{U} . This innocuous modification implies interesting properties for the groups $\mathbb{H}^n(M; \mathcal{D}^l)$. Indeed, consider the case $n < l$. Then a Deligne class is determined by an n -tuple

$$(\omega_0^n, \omega_1^{n-1}, \dots, \omega_n^0), \quad \omega_j^i \in \prod \Omega^i(U_{\alpha_0\alpha_1\dots\alpha_j})$$

satisfying the equations

$$\begin{aligned} d\omega_0^n &= 0 \\ \delta\omega_0^n - d\omega_1^{n-1} &= 0 \\ &\vdots \\ \delta\omega_{n-1}^1 + (-1)^n d \log \omega_n^0 &= 0 \\ \delta\omega_n^0 &= 0 \end{aligned}$$

Since the first equation implies that $\omega_0^n = d\alpha_0^{n-1}$, for some α_0^{n-1} , one can “descend” through the equations, showing that $\omega_n^0 = f + \delta\alpha^0$, where $f \in \prod C_{lc}^\infty(U_{\alpha_0\alpha_1\dots\alpha_n}; U(1))$. Moreover, by the last equation $\delta f = 0$, hence f represents a class in $\check{H}_{\text{Ch}}^n(M; \mathbb{R}/\mathbb{Z})$. Conversely, any class $[f]$ in $\check{H}_{\text{Ch}}^n(M; \mathbb{R}/\mathbb{Z})$ determines a Deligne class up to D -exact terms. We have then

$$\mathbb{H}^n(M; \mathcal{D}^l) \simeq \check{H}_{\text{Ch}}^n(M; \mathbb{R}/\mathbb{Z}), \quad n < l$$

Consider now the case $n > l$. In this case a Deligne class is represented by an l -tuple

$$(\omega_{n-l}^l, \omega_{n-(l-1)}^{l-1}, \dots, \omega_n^0)$$

satisfying the equations

$$\begin{aligned} \delta\omega_{n-l}^l + (-1)^{n-(l-1)} d\omega_{n-(l-1)}^{l-1} &= 0 \\ \delta\omega_{n-l-1}^{l-1} + (-1)^{n-(l-2)} d\omega_{n-(l-2)}^{l-2} &= 0 \\ &\vdots \\ \delta\omega_{n-1}^1 + (-1)^n d \log \omega_n^0 &= 0 \\ \delta\omega_n^0 &= 0 \end{aligned}$$

Notice that in this case the element ω_{n-l}^l is not in general d-closed. Then ω_n^0 represents a class in $\check{H}_{\text{Ch}}^n(\text{M}; \underline{\text{U}}(1)) \simeq \check{H}_{\text{Ch}}^{n+1}(\text{M}; \mathbb{Z})$. Conversely, given a class $[f]$ in $\check{H}_{\text{Ch}}^n(\text{M}; \underline{\text{U}}(1))$ one can reconstruct the Deligne class up to D -exact terms. This is possible thanks to the fact that any element ω_j^i for $j \geq 1$ which is δ -closed, is also δ -exact. Hence we have that

$$\mathbb{H}^n(\text{M}; \mathcal{D}^l) \simeq \check{H}_{\text{Ch}}^{n+1}(\text{M}; \mathbb{Z}), \quad n > l$$

The case $n = l$ is the most interesting one. Indeed, in this case the arguments above cannot be applied. Instead, we have the following [23]

Theorem 5.9. *Let M be a smooth manifold. Then*

$$\mathbb{H}^n(\text{M}; \mathcal{D}^n) \simeq \check{H}^{n+1}(\text{M})$$

for any $n \geq 0$.

The complex above defined to compute Deligne cohomology constitutes then a local description of differential characters, which instead are defined in an intrinsic and global way. One can easily define the fieldstrength and the characteristic class for the groups $\mathbb{H}^n(\text{M}; \mathcal{D}^n)$ in the following way. A Deligne class $\xi \in \mathbb{H}^n(\text{M}; \mathcal{D}^n)$ is represented as

$$\xi = [(\omega_0^n, \omega_1^{n-1}, \dots, \omega_n^0)]$$

We can then define the fieldstrength of ξ as the $n + 1$ -form

$$F(\xi) := d\omega_0^n$$

This is a globally defined $n+1$ -form, since

$$\delta d\omega_0^n = d\delta\omega_0^n = d^2\omega_1^{n-1} = 0$$

The characteristic class of ξ is instead defined as

$$c(\xi) := [\omega_n^0] \in \mathbb{H}^n(\text{M}; \mathbb{Z})$$

By chasing diagrams one can prove that

$$[F(\xi)] = r(c(\xi))$$

The groups $\mathbb{H}^n(\text{M}; \mathcal{D}^n)$ satisfy the same exact sequences as the Cheeger-Simons groups. Moreover, the relation between the Cheeger-Simons groups and Deligne cohomology clarifies the very use of the term ‘‘cohomology’’. In other words, the groups $\check{H}^{n+1}(\text{M})$

can be realized as the cohomology of a differential complex, but this does *not* mean that the collection of functors $\check{H}^*(\cdot)$ constitute a cohomology theory on the category of manifolds in the sense of the Eilenberg-Steenrod axioms. First, notice that the various groups $\check{H}^{n+1}(M)$ are the cohomologies of *different* complexes, for different n . Moreover, they do not satisfy important cohomological properties like homotopy invariance. This is clear, since the groups $\check{H}^{n+1}(M)$ are extensions of the groups of closed forms, which are not homotopy invariant.

As we have seen in section 5.2.1, the first few Cheeger-Simons groups have a geometric interpretation. Indeed, for $n = 0$ we have shown that

$$\check{H}^1(M) \simeq C^\infty(M, S^1)$$

A Deligne class for $\mathbb{H}^0(M; \mathcal{D}^0)$ is determined by an element

$$f \in \prod C^\infty(U_{\alpha_0}; U(1))$$

satisfying $\delta f = 0$, which implies that f is a globally defined function from M to $U(1)$. For $n = 1$, the group $\mathbb{H}^1(M; \mathcal{D}^1)$ is given by isomorphism classes of line bundles with connection. An element in $\mathbb{H}^1(M; \mathcal{D}^1)$ can be represented as a pair

$$(\omega_0^1, \omega_1^0)$$

satisfying the equations

$$\begin{aligned} \delta\omega_0^1 - d \log \omega_1^0 &= 0 \\ \delta\omega_1^0 &= 0 \end{aligned}$$

We can then see that ω_0^1 gives local representatives $\{A_\alpha\}$ of a connection on the line bundle with transition functions $g_{\alpha\beta}$ given by ω_1^0 . The representatives in the same Deligne class differ by D -exact terms

$$(\tilde{\omega}_0^1, \tilde{\omega}_1^0) = (\omega_0^1, \omega_1^0) + D\xi_0^0 = (\omega_0^1 + d \log \xi_0^0, \omega_1^0 + \delta\xi_0^0)$$

which correspond to the gauge transformations

$$\begin{aligned} \tilde{A}_\alpha &= A_\alpha + d \log f_\alpha \\ \tilde{g}_{\alpha\beta} &= f_\beta^{-1} g_{\alpha\beta} f_\alpha \end{aligned}$$

for some circle valued function f_α . Notice that the gauge transformations above correctly include a “change” in the transition functions of the line bundle.

The case $n = 2$ is in a sense the first interesting case for the application to abelian

gauge theories of p -forms. Indeed, a Deligne class in $\mathbb{H}^2(M; \mathcal{D}^2)$ can be represented by a triple

$$(\omega_0^2, \omega_1^1, \omega_2^0)$$

satisfying the equations

$$\begin{aligned} \delta\omega_0^2 - d\omega_1^1 &= 0 \\ \delta\omega_1^1 + d \log \omega_2^0 &= 0 \\ \delta\omega_2^0 &= 0 \end{aligned}$$

The triple $(\omega_0^2, \omega_1^1, \omega_2^0)$ satisfies the equations (1.5.2)-(1.5.4) we used in section 1.5 to define the B-field, apart from the fact that the functions ω_2^0 are circle valued, rather than real valued. This is due to the fact that Deligne classes in $\mathbb{H}^2(M; \mathcal{D}^2)$ represent gauge equivalence classes of B-fields with a Dirac quantization condition as dictated by the group $H^3(M; \mathbb{Z})$. Having realized the B-field as a Deligne cochain, we also have determined the correct notion of gauge transformation, which consists simply in adding terms of the form $D\alpha$, with $\alpha = (\alpha_0^1, \alpha_1^0)$.

Using the Deligne complex as a model for the groups $\check{H}^{n+1}(M)$ clarifies the sense of Definition 5.8, since, as we have shown above, the groups $\mathbb{H}^p(M; \mathcal{D}^p)$ have a natural description in terms of gauge equivalence classes of p -form gauge fields. From the mathematical point of view, it is very natural to interpret the elements of $\mathbb{H}^p(M; \mathcal{D}^p)$ as *higher circle bundles with connections* [39]. In contrast to ordinary circle bundles, these objects are *not* fiber bundles over a base, since there is no total space which makes sense as a manifold. Nevertheless, usual operations like pullback can be performed. More importantly, a theory of integration can be constructed for such objects, i.e one can define the notion of the holonomy of a Deligne class around a suitable cycle. This is of major importance for physical applications, as the holonomy usually describes the coupling term of the gauge field to the relative current. In String theory, in particular, this is needed to make sense of the term

$$\int_{\Sigma} f^*B$$

describing the coupling of a fundamental string with the B-field. In this sense, the holonomy of a Deligne class ξ is exactly the differential character assigned to ξ by the isomorphism in Theorem 5.9.

For more details on Deligne cohomology we refer the reader to [23, 27, 39].

5.3 Ramond-Ramond fields and charge quantization

As we have seen in the previous sections, differential cohomology, in the Cheeger-Simons or the Deligne approach, represents a suitable formalism to describe fields in generalized abelian gauge theories, and we have also seen how the formalism automatically incorporates the Dirac quantization condition. It is natural then to try to describe within this formalism the gauge theory of Ramond-Ramond fields. However, because of the fact that D-branes, which are sources for Ramond-Ramond fields, have charges taking values in K-theory, it turns out that Ramond-Ramond fields themselves, in the absence of branes, are classified by K-theory. In the following we will present the main arguments developed in [74] which support this statement, and the fact that the Dirac quantization condition for Ramond-Ramond fields should be expressed in a K-theoretic language. Some of the arguments we will present here are of a heuristic nature, and should be understood as an educated guess for the use of differential K-theory, which will be introduced in the next section.

Let us first consider the case of ordinary generalized electromagnetism on a d -dimensional spacetime $M = \mathbb{R} \times Y$, where Y is noncompact. As discussed in section 2.5.1, in presence of a magnetic source j_m we have the equation

$$dG = j_m \tag{5.3.1}$$

where the fieldstrength G is an n -form.

Recall that the (total) magnetic charge is given by the class $[i_t^* j_m] \in H_{\text{cpt}}^{n+1}(Y; \mathbb{R})$. However, to enforce equation (5.3.1), the class $[i_t^* j_m]$ must be in the kernel of the natural map

$$i : H_{\text{cpt}}^{n+1}(Y; \mathbb{R}) \rightarrow H^{n+1}(Y; \mathbb{R})$$

which “forgets” the compact support condition. Let us define N as the boundary of Y : the term boundary is used in a loose way, e.g. when $Y = \mathbb{R}^{d-1}$, the boundary is considered to be the sphere S^{d-2} “at infinity”. We have then that

$$H_{\text{cpt}}^{n+1}(Y; \mathbb{R}) \simeq H^{n+1}(Y, N; \mathbb{R})$$

By using the long exact cohomology sequence for the pair (Y, N) we have

$$\dots \rightarrow H^n(Y; \mathbb{R}) \xrightarrow{j} H^n(N; \mathbb{R}) \rightarrow H^{n+1}(Y, N; \mathbb{R}) \xrightarrow{i} H^{n+1}(Y; \mathbb{R}) \rightarrow \dots$$

where j is given by restriction. The sequence above can be “broken”, and we have

$$\ker i \simeq H^n(N; \mathbb{R})/j(H^n(Y; \mathbb{R}))$$

The above isomorphism leads to the following interpretation: the total charge of the source j_m can be detected by classes of fields on the boundary N , i.e. at infinity, that are not restrictions of fields defined on Y . Moreover, the group $H^n(Y; \mathbb{R})$ classifies field configurations G which do not contribute to the total charge. In other words, it classifies gauge fields in the absence of sources.

The same reasoning can now be applied to type II String theory, where the sources are classified by the integral K-theory group $K(Y)$. Indeed, in type IIB String theory, for D-branes with compact support in space, the brane charge can be realized as an element in $K_{\text{cpt}}^0(Y)$. To ensure that the equations of motion for the Ramond-Ramond field have a solution, the brane charge should take values in the kernel of the map

$$i : K_{\text{cpt}}^0(Y) \rightarrow K^0(Y)$$

As before, the long exact sequence

$$\cdots \rightarrow K^{-1}(Y) \xrightarrow{j} K^{-1}(N) \rightarrow K^0(Y, N) \xrightarrow{i} K^0(Y) \rightarrow \cdots$$

implies that

$$\ker i \simeq K^{-1}(N)/j(K^{-1}(Y))$$

In analogy with the cohomological case, we can interpret the above isomorphism as the fact that in type IIB, gauge equivalence classes of Ramond-Ramond fields in the absence of D-branes are classified topologically by the group $K^{-1}(Y)$. By the same argument, we find that in type IIA, Ramond-Ramond fields are topologically classified by the group $K^0(Y)$.

Having established a relation between Ramond-Ramond fields and K-theory, we still must determine the relation between Ramond-Ramond fields and cohomology. Indeed, to be able to write equations of motion for such fields, we need to specify the de Rham cohomology class associated to an element $x \in K(Y)$ that determines (the class of) a Ramond-Ramond fieldstrength G . Let us then first consider the type IIA case. Recall that in type IIA the total Ramond-Ramond potential is of the form

$$C = C_1 + C_3 + \cdots$$

where C_i is a locally defined i -form on $M = \mathbb{R} \times Y$. Let us introduce a collection of 8-branes and $\bar{8}$ -branes with worldvolume $p \times Y$, with p a point in \mathbb{R} , and with arbitrary Chan-Paton bundles (E, F) over Y . These D-branes are in a sense instantonic configurations. The (electric) coupling term is given by

$$\int_{p \times Y} C \wedge \frac{1}{\sqrt{\hat{A}(TY)}} (\text{ch}(E) - \text{ch}(F)) \tag{5.3.2}$$

where we have used $\sqrt{\hat{A}(TM)} = \sqrt{\hat{A}(TY)}$. The Ramond-Ramond equations of motion can be formally written as

$$d \star G = \delta(p) \frac{1}{\sqrt{\hat{A}(TY)}} (\text{ch}(E) - \text{ch}(F))$$

where $\delta(p)$ is a Dirac delta distribution supported on $p \times Y$. By integrating both sides, we have the relation

$$\star G_R - \star G_L = \frac{1}{\sqrt{\hat{A}(TY)}} (\text{ch}(E) - \text{ch}(F)) \quad (5.3.3)$$

where G_R and G_L denote the value of the field G on spatial slices on the ‘‘right’’ and on the ‘‘left’’ of the D-brane, where the notion of left and right is given by the orientation of M and Y . Now, by imposing the selfduality constraint on the total Ramond-Ramond fieldstrength G , we have the following relation

$$G_R - G_L = \frac{1}{\sqrt{\hat{A}(TY)}} (\text{ch}(E) - \text{ch}(F)) \quad (5.3.4)$$

Notice that this at best a heuristic conclusion, since requiring the Ramond-Ramond field to be selfdual implies that the coupling (5.3.2) is *not* defined. Moreover, notice that the right hand side of (5.3.3) and (5.3.4) are not selfdual, while the left hand side is supposed to be. A way of circumnavigating the problem is to use selfduality to eliminate half the fields in G . For G_0, G_2, G_4 one would introduce the magnetic coupling as arising from the equation

$$dG = \delta(p) \frac{1}{\sqrt{\hat{A}(TY)}} (\text{ch}(E) - \text{ch}(F)) \quad (5.3.5)$$

and introduce the magnetic coupling for G_6, G_8, G_{10} via the electric coupling (5.3.2) [74].

Apart from these difficulties, equation (5.3.4) gives a good educated guess for the cohomology class of a Ramond-Ramond field. Indeed, let $b, a \in K^0(Y)$ classify the Ramond-Ramond field on the right and on the left of the collection of D-branes, respectively. Then we have the following equation in de Rham cohomology

$$[G_R(b)] - [G_L(a)] = [\sqrt{\hat{A}(TY)}]^{-1} \cup [\text{ch}(x)] \quad (5.3.6)$$

where $x = [E] - [F]$. Now, by considering the limit $p \rightarrow -\infty$, and requiring that for $a = 0$, the associated de Rham cohomology class vanishes, equation (5.3.6) determines the class of G on all of the spacetime M as

$$[G(x)] = [\sqrt{\hat{A}(TY)}]^{-1} \cup [\text{ch}(x)] \quad (5.3.7)$$

where we have used that the fact that the Chern character is an isomorphism over the reals. In other words, in this particular configuration, the element $x \in K^0(Y)$ classifies the class of Ramond-Ramond fields whose fieldstrength is a solution of the equation (5.3.5). Moreover, the fieldstrength form G associated to x must satisfy the condition (5.3.7). An analogous argument can be formulated for type IIB String theory, by using the Chern character definition as in section 3.6.2. The condition (5.3.7) is the Dirac quantization condition for Ramond-Ramond fieldstrengths in type II String theory. It is a “quantization” condition since the Chern character

$$\text{ch} : K^{0,-1}(Y) \rightarrow H^{\text{ev,odd}}(Y; \mathbb{R})$$

maps the K-theory group to a lattice in real cohomology, since the group $K^{0,-1}(Y)$ is a \mathbb{Z} -module.

At this point, one assumes that the K-theoretical classification of Ramond-Ramond fields in the absence of branes, and the condition (5.3.7), are valid for general space-times M not of the form $\mathbb{R} \times Y$, and not only in the situation used to motivate it.

We are then left with the following mathematical problem: assign to any manifold M abelian groups that can naturally be interpreted as classes of gauge inequivalent fields carrying a topological charge with values in the group $K^{0,-1}(M)$, and such that the associated fieldstrength satisfies the condition (5.3.7). We see that the differential cohomology defined in the previous section is not the suitable formalism to solve this problem, since objects in $\check{H}^p(M)$ or $\mathbb{H}^p(M; \mathcal{D}^p)$ carry a topological charge in integer cohomology. We then intuitively need a “generalized” version of differential cohomology, which is the subject of the next section.

5.4 Generalized differential cohomology

As we have seen in the previous section, the K-theoretical description of Ramond-Ramond fields requires a new framework which generalizes the differential cohomology formalism. We need indeed a theory of some sort which is able to describe objects with local degrees of freedom, and at same time takes into account global properties of such objects. A hint on how to define such a theory is given by a closer inspection of the Dirac quantization condition, as appeared in the examples before, along the lines of the arguments proposed in [40, 41]. Essentially, given a gauge theory of fields \check{A} with fieldstrength $\omega \in \Omega^*(M)$, to impose a Dirac quantization condition is tantamount to requiring that

$$[\omega] \in \Lambda \subset H_{\text{dR}}^*(M; \mathbb{R})$$

where Λ is a lattice in $H_{\text{dR}}^*(M; \mathbb{R})$. For instance, in the case of ordinary electromagnetism, the lattice Λ_{H} is given by the image of the map

$$i : H^*(M; \mathbb{Z}) \hookrightarrow H_{\text{dR}}^*(M; \mathbb{R})$$

while for Ramond-Ramond fields in type II String theory, the lattice Λ_{K} is given by the image of the map

$$\text{ch} : K^{0,1}(M) \rightarrow H_{\text{dR}}^{\text{ev,odd}}(M; \mathbb{R})$$

up to a “scale” factor.

Notice that both the above maps induce an *isomorphism* when tensored over the real field. It is clear that the lattice Λ is greatly affected by the chosen (generalized) cohomology theory which topologically classifies the fields in the given gauge theory, and by the map realizing the free part of the relevant cohomology groups in real de Rham cohomology. In general, as in the above cases, the choice of the cohomology theory for the given gauge field is suggested by physical properties of the system. In principle, though, there is no argument to exclude a given generalized cohomology theory Γ^* . The above arguments suggest the following mathematical idea, which constitute the starting point in [54] for the construction of generalized differential cohomology theories. Let Γ^* be an arbitrary multiplicative⁶ generalized cohomology theory on the category of smooth manifolds. Denote with

$$\pi_{-*}\Gamma := \Gamma^*(\text{pt})$$

the coefficient ring of the point. For example, for $\Gamma^* = H^*$, we have

$$\pi_{-*}H = \mathbb{Z}$$

while for $\Gamma^* = K^*$ we have

$$\pi_{-*}K = \mathbb{Z}[[u^{-1}, u]]$$

where u^{-1} is an element of degree -2, and corresponds to the Bott generator. For any generalized cohomology theory Γ^* there exists a *canonical* map

$$\varphi : \Gamma^*(X) \rightarrow H(X; \mathbb{R} \otimes \pi_{-*}\Gamma)^*$$

which induces an isomorphism when tensored over the reals for any topological space X [54]. In the above expression, the grading on the cohomology groups is such that

$$H(X; \mathbb{R} \otimes \pi_{-*}\Gamma)^n := \bigoplus_{p+q=n} H^p(X; \mathbb{R} \otimes \pi_{-q}\Gamma)$$

⁶This condition can be relaxed.

In the case $\Gamma^* = H^*$, the map φ coincides with i , while for $\Gamma^* = K^*$, the map φ is given by the Chern character. Notice indeed that

$$H(X; \mathbb{R} \otimes \pi_{-*}K)^0 \simeq H^{ev}(X; \mathbb{R}), \quad H(X; \mathbb{R} \otimes \pi_{-*}K)^1 \simeq H^{odd}(X; \mathbb{R})$$

We are then interested in assigning to any smooth manifold M an abelian group which “completes the square”

$$\begin{array}{ccc} ? & \longrightarrow & \Omega_{cl}(M; \mathbb{R} \otimes \pi_{-*}\Gamma)^* \\ \downarrow & & \downarrow \\ \Gamma^*(M) & \xrightarrow{\varphi} & H(M; \mathbb{R} \otimes \pi_{-*}\Gamma)^* \end{array} \quad (5.4.1)$$

where the map

$$\Omega_{cl}(M; \mathbb{R} \otimes \pi_{-*}\Gamma)^* \rightarrow H(M; \mathbb{R} \otimes \pi_{-*}\Gamma)^*$$

is given by assigning to an element $\omega \in \Omega_{cl}(M; \mathbb{R} \otimes \pi_{-*}\Gamma)^*$ its de Rham class in $H(M; \mathbb{R} \otimes \pi_{-*}\Gamma)^*$. A first guess to complete the square would be to consider the fibered product of $\Gamma^*(M)$ and $\Omega_{cl}(M; \mathbb{R} \otimes \pi_{-*}\Gamma)^*$ over $H(M; \mathbb{R} \otimes \pi_{-*}\Gamma)^*$, i.e. the group

$$A_{\Gamma}^* := \{(\omega, u) \in \Omega_{cl}(M; \mathbb{R} \otimes \pi_{-*}\Gamma)^* \times \Gamma^*(M) : [\omega] = \varphi(u)\}$$

However, we know that these groups, for each degree, are only a first approximation to our desired generalized differential cohomology theory: indeed, we know that for $\Gamma^* = H^*$, the Cheeger-Simons groups are an *extension* of A_{Γ}^* . In other words, by considering only the fibered product we are losing information about the group of flat and topologically trivial fields, which in general may be nonvanishing. Mathematically this can be understood as follows: for the two cohomology classes $[\omega]$ and $[v]$ to be equal, the cocycle representatives must satisfy the equation

$$\omega - \delta h = v$$

for some cycle h . The cycle h realizes the homotopy between the two representatives ω and v , and the information about it is lost if we only consider cohomology classes. In the case in which Γ^* is obtained as the cohomology of a differential complex, the square (5.4.1) could be completed by a *homotopy refinement*. The difficulty in doing this is in the fact that the functor $\Omega_{cl}(\cdot; \mathbb{R} \otimes \pi_{-*}\Gamma)^*$ is *not* a cohomology functor on the category of manifolds. In the case in which $\Gamma^* = H^*$ the problem can be solved by regarding for each $q \geq 0$ the space $\Omega_{cl}^q(M; \mathbb{R})$ as the 0-th cohomology of the complex

$$\Omega^q(M; \mathbb{R}) \xrightarrow{d} \Omega^{q+1}(M; \mathbb{R}) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M; \mathbb{R}) \rightarrow 0$$

where $n = \dim M$. By using standard results, then, one can define for a given q the complex [54]

$$\check{C}^p(q) := \begin{cases} C^p(M; \mathbb{Z}) \times C^{p-1}(M; \mathbb{R}) \times \Omega^p(M; \mathbb{R}) & n \geq q \\ C^p(M; \mathbb{Z}) \times C^{p-1}(M; \mathbb{R}) & n < q \end{cases} \quad (5.4.2)$$

with differential

$$d(c, h, \omega) := (\delta c, \omega - c - \delta h, d\omega)$$

and

$$d(c, h) := \begin{cases} (\delta c, -c - \delta h, 0) & n = q - 1 \\ (\delta c, -c - \delta h) & \text{otherwise} \end{cases}$$

Notice that a triple (c, h, ω) is a cocycle if it satisfies the equations

$$\begin{aligned} \delta c &= 0 \\ d\omega &= 0 \\ \omega - c - \delta h &= 0 \end{aligned} \quad (5.4.3)$$

The last of the above equations implies that $[\omega] = i([c])$: this means that the cohomology groups

$$\check{H}(q)^*(M) := H^*(\check{C}^*(q); d)$$

fit the square (5.4.1). The groups $\check{H}(q)^*(M)$ are called the *Cheeger-Simons cohomology groups* and as one can expect we have

$$\check{H}(q)^p(M) \simeq \mathbb{H}^p(M; \mathcal{D}^q)$$

The fieldstrength map assigns to the class $[(c, h, \omega)] \in \check{H}(q)^q(M)$ the closed form ω , while the characteristic class map assigns the class $[c]$. Moreover, the groups $\check{H}(q)^q(M)$ satisfies the same exact sequences as the Cheeger-Simons groups.

Unfortunately, the approach followed in the above paragraph to construct the theory completing the square (5.4.1) for ordinary cohomology cannot be used in general for generalized cohomology theories as K-theory, since these are not obtained as the cohomology of a certain differential complex. Nevertheless, a homotopy refinement for a given theory Γ can be obtained by substituting cocycles with maps to the classifying space $B\Gamma$, and defining an analog of conditions (5.4.3). By using this strategy, Hopkins and Singer showed in [54] that to *any* arbitrary generalized cohomology theory one can assign a theory, that they call a *generalized differential cohomology theory*, which fits the square (5.4.1) and satisfies analogous properties to the Cheeger-Simons or Deligne cohomology groups. We will focus in the following on a particular definition

of differential K-theory, which will be the basis in the following chapter for a suitable generalization to the equivariant setting; we refer the reader to [54] for an extensive and detailed introduction to this beautiful and exciting part of modern mathematics. Let Fred denote the space of Fredholm operators on an infinite dimensional Hilbert space: recall that Fred is a classifying space for the group $K^0(X)$ for a given topological space, i.e.

$$K^0(X) \simeq [X, \text{Fred}]$$

where the isomorphism above is given by considering the index bundle

$$[\text{Ker } F_f] - [\text{Coker } F_f] \tag{5.4.4}$$

where

$$\begin{aligned} \text{Ker } F_f &:= \bigcup_x \text{Ker } f(x) \\ \text{Coker } F_f &:= \bigcup_x \text{Coker } f(x) \end{aligned}$$

for a given map $f : X \rightarrow \text{Fred}$. Of course, the expression (5.4.4) is naive, since the dimension of $\text{Ker } f(x)$ and $\text{Coker } f(x)$ can change while x varies, and needs to be stabilized, as shown in [4], for instance.

Let u denote a cocycle in

$$\mathcal{Z}(\text{Fred}; \mathbb{R} \otimes \pi_{-*}K)^0 = \bigoplus_n \mathcal{Z}^{2n}(\text{Fred}; \mathbb{R})$$

representing the universal Chern character, i.e. such that if $f : X \rightarrow \text{Fred}$ classifies a vector bundle E , f^*u represents $\text{ch}(E)$. For a manifold M , an element in the *differential K-theory* group $\check{K}^0(M)$ is represented by a triple

$$(c, h, \omega)$$

where $c : M \rightarrow \text{Fred}$, $h \in \mathcal{C}^{ev-1}(M; \mathbb{R})$, and $\omega \in \Omega^{ev-1}(M; \mathbb{R})$ such that

$$\delta h = \omega - c^*u \tag{5.4.5}$$

Moreover, the triples above defined must satisfy the following equivalence relation. Two triples (c^0, h^0, ω^0) and (c^1, h^1, ω^1) are said to be equivalent if there exists a triple (c, h, ω) on $M \times [0, 1]$, with ω constant along $[0, 1]$, such that

$$\begin{aligned} (c, h, \omega)|_0 &= (c^0, h^0, \omega^0) \\ (c, h, \omega)|_1 &= (c^1, h^1, \omega^1) \end{aligned} \tag{5.4.6}$$

Notice that equation (5.4.5) enforces the condition

$$[\omega] = \text{ch}([c])$$

The relations (5.4.6) can be rephrased [40] in a way that will be useful later on. Indeed, two triple (c^0, h^0, ω^0) and (c^1, h^1, ω^1) are equivalent if there exists a map

$$f : M \times [0, 1] \rightarrow \text{Fred}$$

and an element $\sigma \in \Omega^{ev-2}(M; \mathbb{R})$ such that

$$\begin{aligned} f|_0 &= c^0 \\ f|_1 &= c^1 \\ \omega^1 &= \omega^0 \\ h^1 &= h^0 + \pi_* f^* u + d\sigma \end{aligned}$$

where $\pi : M \times [0, 1] \rightarrow M$ is the projection onto the first factor, and π_* denotes integration along the fibre, i.e. pairing with the fundamental class of $[0, 1]$. The last of the above equations is obtained by imposing the condition (5.4.5).

To define the higher differential K-groups, recall that the iterated loop spaces $\Omega^i \text{Fred}$ classify the functors K^{-i} . Let u^{-i} be the cocycle

$$u^{-i} \in \mathcal{Z}^{2n-i}(\Omega^i \text{Fred}; \mathbb{R})$$

defined as

$$u^{-i} := \Pi_* ev^* u$$

where

$$ev : S^i \times \Omega^i \text{Fred} \rightarrow \text{Fred}$$

denotes the evaluation map, and Π_* denotes the integration over the fiber of the projection map $\Pi : S^i \times \Omega^i \text{Fred} \rightarrow \Omega^i \text{Fred}$. Hence, elements of the higher differential groups $\check{K}^{-i}(M)$ are represented by triples

$$(c, h, \omega)$$

where $c : M \rightarrow \Omega^i \text{Fred}$, $h \in \mathcal{C}^{ev-i}(M; \mathbb{R})$, and $\omega \in \Omega^{ev-i}(M; \mathbb{R})$ such that

$$\delta h = \omega - c^* u^{-i}$$

and satisfying the equivalence relations (5.4.6).

The differential K-groups satisfy the following exact sequences

$$\begin{aligned} 0 &\rightarrow K^{-i-1}(M; \mathbb{R}/\mathbb{Z}) \rightarrow \check{K}^{-i}(M) \rightarrow \Omega_K^{ev-i}(M) \rightarrow 0 \\ 0 &\rightarrow \Omega^{ev-i-1}(M)/\Omega_K^{ev-i-1}(M) \rightarrow \check{K}^{-i}(M) \rightarrow K^{-i}(M) \rightarrow 0 \\ 0 &\rightarrow H^{ev-i-1}(M)/\text{ch}(K^{-i-1}(M)) \rightarrow \check{K}^{-i}(M) \rightarrow A_K^{-i}(M) \rightarrow 0 \end{aligned} \quad (5.4.7)$$

where $\Omega_{\mathbb{K}}^{ev-i}(M)$ denotes the elements in $\Omega^{ev-i}(M)$ whose cohomology class is in the image of the Chern character. We will give a proof of the existence of the above exact sequences in the context of equivariant K-theory in the next chapter. Finally, the homotopy equivalence $\Omega^n \text{Fred} \simeq \Omega^{n+2} \text{Fred}$ induces Bott periodicity on the differential K-groups, and in particular it allows to define them for positive degrees as

$$\check{K}^n(M) := \check{K}^{n-2N}(M) \quad n - 2N < 0$$

This definition of differential K-theory will be at the core of a generalization we will propose in the next chapter. However, we should point out that there are different models for differential K-theory, such as the one introduced in [26], where axioms for generalized differential cohomology theories are given. We have seen that ordinary differential cohomology admits different models, which lead to isomorphic theories, characterized by the fact that they satisfy the same exact sequences. As shown in [86], these exact sequences in a sense completely characterize ordinary differential cohomology. As far as the author knows, a similar result does not exist for generalized differential cohomology theories.

Chapter 6

Ramond-Ramond fields and Orbifold differential K-theory

As we have seen in the previous chapter, Ramond-Ramond fields in type II String theory require the use of differential K-theory to be properly described. We have also explained that this is due to the fact that the Dirac quantization condition for Ramond-Ramond fieldstrength is dictated by K-theory, rather than integral cohomology. Moreover, the lattice in which charges take their values has been suggested by the form of the coupling between D-branes and Ramond-Ramond field.

In this chapter we will generalize the above arguments to the case of superstring theory defined on an orbifold. To be precise, we will only consider the case of presentable orbifolds of the form $[M/G]$, where M is a spin manifold, and G is a finite group acting by spin structure preserving diffeomorphisms. In this case, as proposed by Witten in [94], D-branes are classified by the equivariant K-theory of the “covering” space M . It is natural, then, to ask that the Dirac quantization condition for the total fieldstrength of Ramond-Ramond fields on the orbifold $[M/G]$ is given by a Chern character homomorphism on the equivariant K-theory $K_G^*(M)$, which induces an isomorphism when tensored over \mathbb{R} . This is supported by the fact that we expect to obtain type II String theory features when $G = \{e\}$. A Chern character with the above properties has been constructed in [67], and makes use of Bredon cohomology, an equivariant cohomology theory defined on the category of G -CW complexes. The important feature of this equivariant cohomology theory is that it naturally takes into account some important features of String theory on orbifolds, namely the presence of twisted sectors. We then pass to propose a definition of differential K-theory suitable for good orbifolds. Indeed, the existence theorem for generalized differential cohomology theories developed in [54] cannot be applied to equivariant cohomology functors

on the category of G-manifolds. More precisely, we define abelian groups that behave as a natural generalization of the ordinary differential K-theory groups, in the sense that they agree in the case of a trivial group and they satisfy analogous exact sequences. We will “test” our definition in the case of linear abelian orbifolds, and give a proposal for the group of (equivalence classes of) flat fields. We will also make a brief digression to describe D-branes using geometric equivariant K-homology, showing that the use of K-cycle is well-suited to the description of *fractional* D-branes and their topological charges computed using equivariant Dirac operator theory. By using the Chern character, we will define electric Ramond-Ramond couplings to D-branes on good orbifolds, and compare with previous examples in the literature. Finally, we will support our definition of orbifold differential K-theory, or to be precise a complex version of it, by generalizing Moore and Witten argument as introduced in the previous chapter to the equivariant setting, and by expressing the Ramond-Ramond equations of motion by writing an equivariant version of the Ramond-Ramond current in term of the equivariant Chern character and an equivariant version of the Riemann-Roch theorem.

6.1 G-CW complexes and equivariant cohomology theories

In this section we will recall some basic notions about (generalized) equivariant cohomology theories. In the following, X denotes a topological space and G a finite group, unless otherwise stated. In the following, a (left) action $G \times X \rightarrow X$ of G on X will be denoted $(g, x) \mapsto g \cdot x$, and we will call X a *G-space*. The stabilizer or isotropy group of a point $x \in X$ is denoted $G_x = \{g \in G \mid g \cdot x = x\}$. Recall that a continuous map $f : X \rightarrow Y$ of G -spaces is a *G-map* if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$.

Definition 6.1. A *G-equivariant CW-decomposition* of a G -space X consists of a filtration X_n , $n \in \mathbb{N}_0$ such that

$$X = \bigcup_{n \in \mathbb{N}_0} X_n$$

and X_n is obtained from X_{n-1} by “attaching” equivariant cells via the following procedure. Define

$$X_0 = \coprod_{j \in J_0} G/K_j ,$$

with K_j a collection of subgroups of G and the standard (left) G -action on any coset

space G/K_j . For $n \geq 1$ set

$$X_n = \left(X_{n-1} \amalg \coprod_{j \in J_n} (D_j^n \times G/K_j) \right) / \sim \quad (6.1.1)$$

where the equivalence relation \sim is generated by G-equivariant “attaching maps”

$$\phi_j^n : S_j^{n-1} \times G/K_j \longrightarrow X_{n-1} . \quad (6.1.2)$$

One requires that X carries the colimit topology with respect to (X_n) , i.e., $B \subset X$ is closed if and only if $B \cap X_n$ is closed in X_n for all $n \in \mathbb{N}_0$. We call the image of $D_j^n \times G/K_j$ (resp. $\overset{\circ}{D}_j^n \times G/K_j$) a *closed* (resp. *open*) n -cell of orbit type G/K_j . As usual, we call the subspace X_n the n -skeleton of X . If $X = X_n$ and $X \neq X_{n-1}$, then n is called the (*cellular*) *dimension* of X and X is said to be of *finite type*. A G-space with a G-equivariant CW-decomposition is called a *G-complex*.

When $G = e$ is the trivial group, a G-complex is just an ordinary CW-complex. In general, if X is a G-complex then the orbit space X/G is an ordinary CW-complex. Conversely, there is an intimate relation between G-complexes and ordinary CW-complexes whenever G is a discrete group. Let X be a G-space which is an ordinary CW-complex. We say that G *acts cellularly* on X if

- 1) For each $g \in G$ and each open cell E of X , the left translation $g \cdot E$ is again an open cell of X ; and
- 2) If $g \cdot E = E$, then the induced map $E \rightarrow E$, $x \mapsto g \cdot x$ is the identity.

Then we have the following [92]

Proposition 6.2. *Let X be a CW-complex with a cellular action of a discrete group G . Then X is a G-complex with n -skeleton X_n .*

In the case that X is a smooth manifold, we require the G-action on X to be smooth and there is an analogous result. Recall that the applicability of algebraic topology to manifolds relies on the fact that any manifold comes equipped with a canonical CW-decomposition. In the case in which a group acts on the manifold one has the following result due to Illman [57, 57].

Theorem 6.3. *If G is a compact Lie group or a finite group acting on a smooth compact manifold X , then X is triangulable as a finite G-complex.*

The collection of G-complexes with G-maps as morphisms form a category. We are interested in equivariant cohomology theories defined on this category (or on subcategories thereof).

We will now briefly spell out the main ingredients involved in building an equivariant cohomology theory on the category of finite G-complexes, leaving the details to the comprehensive treatments of [92] and [66], and focusing instead on some explicit examples. Fix a group G and a commutative ring R. A *G-cohomology theory* E_G^* with values in R-modules is a collection of contravariant functors E_G^n from the category of G-CW pairs to the category of R-modules indexed by $n \in \mathbb{Z}$ together with natural transformations

$$\delta_G^n(X, A) : E_G^n(X, A) \longrightarrow E_G^{n+1}(X) := E_G^{n+1}(X, \emptyset)$$

for all $n \in \mathbb{Z}$ satisfying the axioms of G-homotopy invariance, long exact sequence of a pair, excision, and disjoint union. The theory is called *ordinary* if for any orbit G/H one has $E_G^q(G/H) = 0$ for all $q \neq 0$. These axioms are formulated in an analogous way to that of ordinary cohomology. The new ingredients in an equivariant cohomology theory (which we have not yet defined) are the *induction structures*, which we shall now describe.

Let $\alpha : H \rightarrow G$ be a group homomorphism, and let X be an H-space. Define the *induction of X with respect to α* to be the G-space $\text{ind}_\alpha X$ given by

$$\text{ind}_\alpha X := G \times_\alpha X .$$

This is the quotient of the product $G \times X$ by the H-action $h \cdot (g, x) := (g\alpha(h^{-1}), h \cdot x)$, with the G-action on $\text{ind}_\alpha X$ given by $g' \cdot [g, x] = [g'g, x]$. If $H < G$ is a subgroup, and α is the subgroup inclusion, the induced G-space is denoted $G \times_H X$.

An *equivariant cohomology theory* $E_{(-)}^*$ with values in R-modules consists of a collection of G-cohomology theories E_G^* with values in R-modules for each group G such that for any group homomorphism $\alpha : H \rightarrow G$ and any H-CW pair (X, A) with $\ker(\alpha)$ acting freely on X, there are for each $n \in \mathbb{Z}$ natural isomorphisms

$$\text{ind}_\alpha : E_G^n(\text{ind}_\alpha(X, A)) \xrightarrow{\cong} E_H^n(X, A) \tag{6.1.3}$$

satisfying

- (a) Compatibility with the coboundary homomorphisms:

$$\delta_H^n \circ \text{ind}_\alpha = \text{ind}_\alpha \circ \delta_G^n ;$$

- (b) Functoriality: If $\beta : G \rightarrow K$ is another group homomorphism such that $\ker(\beta \circ \alpha)$ acts freely on X , then for every $n \in \mathbb{Z}$ one has

$$\text{ind}_{\beta \circ \alpha} = \text{ind}_{\alpha} \circ \text{ind}_{\beta} \circ E_K^n(f_1)$$

where

$$\begin{aligned} f_1 : \text{ind}_{\beta}(\text{ind}_{\alpha}(X, A)) &\xrightarrow{\cong} \text{ind}_{\beta \circ \alpha}(X, A) \\ (k, g, x) &\longmapsto (k \beta(g), x) \end{aligned}$$

is a K -homeomorphism and $E_K^n(f_1)$ is the morphism on K -cohomology induced by f_1 ; and

- (c) Compatibility with conjugation: For $g, g' \in G$ define $\text{Ad}_g(g') = g g' g^{-1}$. Then the homomorphism ind_{Ad_g} coincides with $E_G^n(f_2)$, where

$$\begin{aligned} f_2 : (X, A) &\xrightarrow{\cong} \text{ind}_{\text{Ad}_g}(X, A) \\ x &\longmapsto (e, g^{-1} \cdot x) \end{aligned}$$

is a G -homeomorphism, where throughout e denotes the identity element in the group G .

Thus the induction structures connect the various G -cohomologies and keep track of the equivariance. They will be very important in the construction of the equivariant Chern character for equivariant K -theory in a later section, even if we are only interested in a fixed group G .

Example 6.4 (*Borel cohomology*). Let H^* be a cohomology theory for CW-pairs (for example, singular cohomology). Define

$$H_G^n(X, A) := H^n(EG \times_G (X, A))$$

where EG is the total space of the classifying principal G -bundle $EG \rightarrow BG$ which is contractible and carries a free G -action. This is called (*equivariant*) *Borel cohomology*, and is the most commonly used form of equivariant cohomology in the physics literature. Note that H_G^* is well-defined because the quotient $EG \times_G X$ is unique up to the homotopy type of X/G . The ordinary G -cohomology structures on H_G^* are inherited from the cohomology structures on H^* . The induction structures for H_G^* are constructed as follows. Let $\alpha : H \rightarrow G$ be a group homomorphism and X an H -space. Define

$$\begin{aligned} b : EH \times_H X &\longrightarrow EG \times_G G \times_{\alpha} X \\ (\varepsilon, x) &\longmapsto (E\alpha(\varepsilon), e, x) \end{aligned}$$

where $\varepsilon \in EH$, $x \in X$ and $E\alpha : EH \rightarrow EG$ is the α -equivariant map induced by α . The induction map ind_α is then given by pullback

$$\text{ind}_\alpha := b^* : H_G^n(\text{ind}_\alpha X) = H^n(EG \times_G G \times_\alpha X) \longrightarrow H^n(EH \times_H X) = H_H^n(X) .$$

If $\ker(\alpha)$ acts freely on X , then the map b is a homotopy equivalence and hence the map ind_α is an isomorphism.

Example 6.5 (*Equivariant K-theory*). In [82], equivariant topological K-theory is defined for any G -complex X as the abelian group completion of the semigroup $\text{Vect}_G^{\mathbb{C}}(X)$ of complex G -vector bundles over X , with G a compact Lie group. Recall that for a G -space X , a complex G -vector bundle is given by a G -space E , and a G -map $\pi : E \rightarrow X$ such that E is the total space of a complex vector bundle, and such that for any $g \in G$ and any $x \in X$, the map $g : E_x \rightarrow E_{gx}$ is an homomorphism. The compactness property of G assures that the Grothendieck functor K_G^* satisfies the G -homotopy invariance¹. The higher groups are defined via iterated suspension, similarly to ordinary K-theory, and Bott periodicity holds.

To define the induction structures, recall that if X is an H -space and $\alpha : H \rightarrow G$ is a group homomorphism, then the map

$$\begin{aligned} \varphi : X &\longrightarrow G \times_\alpha X \\ x &\longmapsto (e, x) \end{aligned}$$

is an α -equivariant map which embeds X as the subspace $H \times_\alpha X$ of $G \times_\alpha X$, and which induces via pullback of vector bundles the homomorphism

$$\varphi^* : K_G^*(G \times_\alpha X) \longrightarrow K_H^*(X) .$$

This map defines the induction structure. It is invertible when $\ker(\alpha)$ acts freely on X , with inverse the “extension” map $E \mapsto G \times_H E$ for any H -vector bundle E over X . This can be proven by using the following [82]

Theorem 6.6. *Let G be a compact group, and let N be a normal subgroup acting freely on X . Then*

$$\text{pr}^* : K_{G/N}^*(X/N) \xrightarrow{\cong} K_G^*(X) \tag{6.1.4}$$

where $\text{pr} : X \rightarrow X/N$ is the usual projection.

¹This is due to the fact that pullbacks of a G -bundle via G -homotopic maps are isomorphic only if G is compact.

By noticing that

$$X/N \simeq (G/N) \times_G X$$

and if we define denote with N the kernel of $\alpha : H \rightarrow G$, then

$$K_G^*(G \times_\alpha X) \simeq K_{H/N}^*((H/N) \times_\alpha X) \simeq K_{H/N}^*(X/N) \simeq K_H^*(X)$$

since N acts freely on X by hypothesis.

In the case in which $X = \text{pt}$, a G -vector bundle is just a G -module, hence we have

$$K_G^0(\text{pt}) \simeq R(G), \quad K_G^{-1}(\text{pt}) \simeq 0 \tag{6.1.5}$$

where $R(G)$ is the *representation ring* of G , i.e. the ring generated over \mathbb{Z} by the irreducible representations of G .

We can use the above results to show that

$$K_G^0(G/H) \simeq K_G^0(G \times_H \text{pt}) \simeq K_H^0(\text{pt}) \simeq R(H), \quad K_G^{-1}(G/H) \simeq 0$$

for $H < G$.

In the case in which the group G acts freely on the space X , theorem 6.6 implies

$$K^*(X/G) \simeq K_G^*(X)$$

On the other extreme, when G acts trivially on X , the following isomorphism holds [82]

$$K_G^*(X) \simeq K^*(X) \otimes R(G) \tag{6.1.6}$$

6.2 The equivariant Chern character

As we mentioned in the beginning of this chapter, we expect that the Dirac quantization condition for the total Ramond-Ramond fieldstrength on a good orbifold is dictated by some homomorphism on equivariant K-theory which is a generalization of the ordinary Chern character. One might naively think that the correct target theory for the equivariant Chern character would naturally be Borel cohomology, as defined in example 6.4, with real coefficients. This is not the case, as emphasised in particular by a *completion theorem* of Atiyah and Segal, which we briefly summarize. Any conjugacy classes of an element $\gamma \in G$ induces a homomorphism

$$\nu_\gamma : R(G) \rightarrow \mathbb{C}$$

given by $\nu_\gamma(\rho) := \chi_\rho(\gamma)$, where χ_ρ is the character associated to the representation ρ , which is constant on conjugacy classes. The kernel of such a homomorphism is a

prime ideal² in the ring $R(G)$. Let us denote with I_G the prime ideal associated to element $e \in G$. Then we have [8]

Theorem 6.7. *For a G -space X , with G a compact Lie group, the Borel cohomology $H_G^*(X; \mathbb{Q})$ is isomorphic to the completion of the $R(G)$ -module $K_G^*(X) \otimes \mathbb{Q}$ with respect to the ideal I_G .*

The completion of $K_G^*(X; \mathbb{Q}) := K_G^*(X) \otimes \mathbb{Q}$ is defined as the tensor product $K_G^*(X; \mathbb{Q}) \otimes_{R(G)} \widehat{R}(G)$, where $\widehat{R}(G)$ is given by the limit of the quotients $R(G)/I_G^n \cdot R(G)$ for n going to infinity.

The above theorem suggests then that Borel cohomology is not the correct target theory for a Chern character inducing an isomorphism over \mathbb{R} . If we think of $R(G)$ as the ring of functions over G , the prime ideal I_G corresponds to the unit element in G . Theorem 6.7 then states that Borel cohomology does not take into account “contributions” of the non-trivial elements in G , and hence, in a sense, it is *localised* around the unit element.

There are several approaches to the equivariant Chern character (see refs. [9, 87, 19, 43, 1], for example) which strongly depend on the types of groups involved (discrete, continuous, *etc.*) and on the ring one tensors with (\mathbb{R} , \mathbb{C} , *etc.*). As we are interested in finite groups and real coefficients, we will use the Chern character constructed in [66] and [67].

In the following section we will briefly recall the basic constructions in Bredon cohomology [22, 66, 73], which will turn out to be the best suited equivariant cohomology theory for all of our purposes. We will refer to Appendix A for some pertinent aspects of functor categories.

6.2.1 Bredon cohomology

In the following, G will denote a discrete group. The *orbit category* $\text{Or}(G)$ of G is defined as the category whose objects are homogeneous spaces G/H , with $H < G$, and whose morphisms are G -maps between them. From general considerations [92] it follows that a G -map between two homogeneous spaces G/H and G/K exists if and only if H is conjugate to a subgroup of K , and hence any such map is of the form

$$(gH \longmapsto gaK) \tag{6.2.1}$$

²A *prime ideal* P in a commutative ring R is an ideal such that whenever the product ab of two elements in R lies in P , then a or b lies in P .

for some $a \in G$ such that $a^{-1} H a < K$. If \mathfrak{F} is any family of subgroups of G then there is a subcategory $\text{Or}(G, \mathfrak{F})$ with objects G/H for $H \in \mathfrak{F}$. A simple example is provided by the cyclic groups $G = \mathbb{Z}_p$ with p prime, for which the orbit category has just two objects, $G/e = G$ and $G/G = \text{pt}$.

If Ab denotes the category of abelian groups, then a *coefficient system* is a functor

$$\underline{F} : \text{Or}(G)^{\text{op}} \longrightarrow \text{Ab}$$

where $\text{Or}(G)^{\text{op}}$ denotes the opposite category to $\text{Or}(G)$. With such a functor and any G -complex X ,³ one can define for each $n \in \mathbb{Z}$ the group

$$C_G^n(X, \underline{F}) := \text{Hom}_{\text{Or}(G)}(\underline{C}_n(X), \underline{F}) \quad (6.2.2)$$

where $\underline{C}_n(X) : \text{Or}(G)^{\text{op}} \rightarrow \text{Ab}$ is the projective functor defined by

$$\underline{C}_n(X)(G/H) := C_n(X^H) ,$$

the cellular homology of the fixed point complex

$$X^H := \{x \in X \mid h \cdot x = x \quad \forall h \in H\} . \quad (6.2.3)$$

In equation (6.2.2), $\text{Hom}_{\text{Or}(G)}(-, -)$ denotes the group of natural transformations between two contravariant functors, with the group structure inherited by the images of the functors in Ab . The functoriality property of $\underline{C}_n(X)$ is the natural one induced by the identification $X^H \simeq \text{Map}_G(G/H, X)$. Indeed, the two maps

$$\begin{aligned} X^H &\longrightarrow \text{Map}_G(G/H, X) \quad , & x &\longmapsto f_x([gH]) = g \cdot x \quad , \\ \text{Map}_G(G/H, X) &\longrightarrow X^H \quad , & f &\longmapsto f(H) \end{aligned}$$

are easily seen to be inverse to each other, and the desired homeomorphism is obtained by giving the space $\text{Map}_G(G/H, X)$ the *compact-open* topology. In particular, a G -map (6.2.1) induces a cellular map $X^K \rightarrow X^H$, $x \mapsto a \cdot x$.

These groups can be expressed in terms of the G -complex structure of X . If the n -skeleton X_n is obtained by attaching equivariant cells as in equation (6.1.1) with K_j the stabilizer of an n -cell of X , then the cellular chain complex $C_*(X)$ consists of G -modules $C_n(X) = \bigoplus_{j \in J_n} \mathbb{Z}[G/K_j]$ and hence

$$\underline{C}_n(X)(G/H) \simeq \bigoplus_{j \in J_n} \mathbb{Z}[\text{Mor}_{\text{Or}(G)}(G/H, G/K_j)] .$$

³When G is an infinite discrete group, one should restrict to *proper* G -complexes, i.e., with finite stabilizer for any point of X . Some further minor assumptions are needed when G is a Lie group.

For each $n \geq 0$, the group $C_G^n(X, \underline{F})$ is the direct limit functor over all n -cells of orbit type G/K_j in X of the groups $\underline{F}(G/K_j)$. This follows by restricting equation (6.2.2) to the full subcategory $\text{Or}(G, \mathfrak{F}(X))$, with $\mathfrak{F}(X)$ the family of subgroups of G which occur as stabilizers of the G -action on X [73].

The \mathbb{Z} -graded group $C_G^*(X, \underline{F}) = \bigoplus_{n \in \mathbb{Z}} C_G^n(X, \underline{F})$ inherits a coboundary operator δ , and hence the structure of a cochain complex, from the boundary operator on cellular chains. To a natural transformation $f : \underline{C}_n(X) \rightarrow \underline{F}$, one associates the natural transformation δf defined by

$$\begin{aligned} \delta f(G/H) : C_n(X^H) &\longrightarrow \underline{F}(G/H) \\ \sigma &\longmapsto f(G/H)(\partial\sigma) \end{aligned}$$

for $\sigma \in C_{n-1}(X^H)$, with naturality induced from that of the cellular boundary operator ∂ . Then the *Bredon cohomology* of X with coefficient system \underline{F} is defined as

$$H_G^*(X; \underline{F}) := H(C_G^*(X, \underline{F}), \delta) .$$

This defines a G -cohomology theory. See [65] for the proof that $H_G^*(X; \underline{F})$ is an equivariant cohomology theory, i.e., for the definition of the induction structure. One can also define cohomology groups by restricting the functors in equation (6.2.2) to a subcategory $\text{Or}(G, \mathfrak{F})$. The definition of Bredon cohomology is independent of \mathfrak{F} as long as \mathfrak{F} contains the family $\mathfrak{F}(X)$ of stabilizers [73]. This fact is useful in explicit calculations. In particular, by taking $\mathfrak{F} = H$ to consist of a single subgroup, one shows that the Bredon cohomology of G -homogeneous spaces is given by

$$H_G^*(G/H; \underline{F}) = H_G^0(G/H; \underline{F}) = \underline{F}(G/H) . \quad (6.2.4)$$

Example 6.8 (*Trivial group*). When $G = e$ is the trivial group, i.e., in the non-equivariant case, the functors $\underline{C}_n(X)$ and \underline{F} can be identified with the abelian groups $C_n(X) = \underline{C}_n(X)(e)$ and $F = \underline{F}(e)$. Then

$$C_e^n(X, F) = C^n(X, F)$$

and one has $H_e^n(X; \underline{F}) = H(C^n(X, F), \delta)$, i.e., the ordinary n -th cohomology group of X with coefficients in F .

Example 6.9 (*Free action*). If the G -action on X is *free*, then all stabilizers K_j are trivial and $X^H = \emptyset$ for every $H \leq G$, $H \neq e$. In this case one may take $\mathfrak{F} = e$ to compute the cochain complex

$$C_G^*(X, \underline{F}) \simeq \text{Hom}_G(C_*(X), \underline{F}(G/e))$$

and so the Bredon cohomology $H_G^*(X; \underline{F})$ coincides with the equivariant cohomology

$$H_G^*(X; \underline{F}(G/e))$$

of X with coefficients in the G -module $\underline{F}(G/e) = \underline{F}(G)$. In the case of the constant functor $\underline{F} = \underline{\mathbb{Z}}$, with $\underline{\mathbb{Z}}(G/H) = \mathbb{Z}$ for every $H \leq G$ and the value on morphisms in $\text{Or}(G)^{\text{op}}$ given by the identity homomorphism of \mathbb{Z} , this group reduces to the ordinary cohomology $H^*(X/G; \mathbb{Z})$.

Example 6.10 (*Trivial action*). If the G -action on X is *trivial*, then the collection of isotropy groups K_j for the G -action is the set of all subgroups of G and $X^H = X$ for all $H \leq G$. In this case the functor $\underline{C}_n(X)$ can be decomposed into a sum over n -cells of projective functors \underline{P}_{K_j} with $K_j = G$ [73], and so one has

$$\text{Hom}_{\text{Or}(G)}(\underline{C}_n(X), \underline{F}) \simeq \text{Hom}(C_n(X), \varprojlim_{\text{Or}(G)^{\text{op}}} \underline{F}(G/H))$$

where the inverse limit functor is taken over the opposite category $\text{Or}(G)^{\text{op}}$. It follows that the Bredon cohomology

$$H_G^*(X; \underline{F}) = H^*(X; \underline{F}(G/G))$$

is the ordinary cohomology of X with coefficients in the abelian group $\underline{F}(G/G) = \underline{F}(\text{pt})$.

We will now specialize the coefficient system for Bredon cohomology to the *representation ring functor* $\underline{R}(-)$ defined on the orbit category $\text{Or}(G)$ by sending the left coset G/H to $R(H)$, the representation ring of the group H . A morphism (6.2.1) is sent to the homomorphism $R(K) \rightarrow R(H)$ given by first restricting the representation from K to the subgroup conjugate to H , and then conjugating by a . Since $\underline{R}(-)$ is a functor to rings, the Bredon cohomology $H_G^*(X; \underline{R}(-))$ naturally has a ring structure. By equation (6.2.4), we have

$$H_G^*(G/H; \underline{R}(-)) = \underline{R}(G/H) = R(H) = K_G^*(G/H)$$

which is already an indication that Bredon cohomology is a better relative of equivariant K -theory than Borel cohomology. Indeed, using the induction structure of Example 6.4 one shows that the Borel cohomology

$$H_G^*(G/H) = H^*(BH)$$

coincides with the cohomology of the classifying space $BH = EH/H$, which computes the group cohomology of H and is typically infinite-dimensional (even for finite groups H).

In the construction of the equivariant Chern character in the next section, it will be important to represent the rational Bredon cohomology $H_G^*(X; \mathbb{Q} \otimes \underline{\mathbb{R}}(-))$ as a certain group of homomorphisms of functors, similarly to the cochain groups (6.2.2). For this, we introduce another category $\mathbf{Sub}(G)$. The objects of $\mathbf{Sub}(G)$ are the subgroups of G ,⁴ and the morphisms are given by

$$\text{Mor}_{\mathbf{Sub}(H,K)} := \{f : H \rightarrow K \mid \exists g \in G, gHg^{-1} \leq K, f = \text{Ad}_g\} / \text{Inn}(K) .$$

In particular, there is a functor $\text{Or}(G) \rightarrow \mathbf{Sub}(G)$ which sends the object G/H to H and the morphism (6.2.1) in $\text{Or}(G)$ to the homomorphism $(g \mapsto a^{-1}ga)$ in $\mathbf{Sub}(G)$. If a lies in the centralizer

$$Z_G(H) := \{g \in G \mid g^{-1}hg = h, \forall h \in H\} \quad (6.2.5)$$

of H in G , then the morphism (6.2.1) is sent to the identity map. Any functor $\underline{F} : \mathbf{Sub}(G)^{\text{op}} \rightarrow \mathbf{Ab}$ can be naturally regarded as a functor on $\text{Or}(G)^{\text{op}}$.

Define the quotient functors $\underline{C}_*^{\text{qt}}(X), \underline{H}_*^{\text{qt}}(X) : \mathbf{Sub}(G)^{\text{op}} \rightarrow \mathbf{Ab}$ by

$$\underline{C}_*^{\text{qt}}(X)(H) := C_*(X^H/Z_G(H)) \quad \text{and} \quad \underline{H}_*^{\text{qt}}(X)(H) := H_*(X^H/Z_G(H)) .$$

For any functor $\underline{F} : \mathbf{Sub}(G)^{\text{op}} \rightarrow \mathbf{Ab}$ one has

$$\text{Hom}(C_*(X^H/Z_G(H)), \underline{F}(H)) \simeq \text{Hom}_{Z_G(H)}(C_*(X^H), \underline{F}(H)) .$$

By observing that the centralizer (6.2.5) is precisely the group of automorphisms of G/H in the orbit category $\text{Or}(G)$ sent to the identity map in the subgroup category $\mathbf{Sub}(G)$, we finally have

$$C_G^*(X, \underline{F}) = \text{Hom}_{\text{Or}(G)}(\underline{C}_*(X), \underline{F}) \simeq \text{Hom}_{\mathbf{Sub}(G)}(\underline{C}_*^{\text{qt}}(X), \underline{F}) . \quad (6.2.6)$$

At this point one can apply equation (6.2.6) to the rational representation ring functor $\underline{F} = \mathbb{Q} \otimes \underline{\mathbb{R}}(-)$, which by construction can be regarded as an injective functor $\mathbf{Sub}(G)^{\text{op}} \rightarrow \mathbf{Ab}$, to prove the

Lemma 6.11 ([67]). *For any finite group G and any G -complex X , there exists an isomorphism of rings*

$$\Phi_X : H_G^*(X; \mathbb{Q} \otimes \underline{\mathbb{R}}(-)) \xrightarrow{\simeq} \text{Hom}_{\mathbf{Sub}(G)}(\underline{H}_*^{\text{qt}}(X), \mathbb{Q} \otimes \underline{\mathbb{R}}(-)) .$$

⁴If G is infinite then one should restrict to finite subgroups of G .

6.2.2 Chern character in equivariant K-theory

Before spelling out the definition of the equivariant Chern character given in [67], we recall some basic properties of the equivariant K-theory of a G -complex X . Let H be a subgroup of G , and consider the fixed point subspace of X defined in (6.2.3). The action of G does not preserve X^H , but the action of the normalizer $N_G(H)$ of H in G does. If we denote with $i : X^H \hookrightarrow X$ the inclusion of X^H as a subspace of X , and with $\alpha : N_G(H) \hookrightarrow G$ the inclusion of $N_G(H)$ as a subgroup of G , then we naturally have the equality

$$i(n \cdot x) = \alpha(n) \cdot i(x)$$

for all $n \in N_G(H)$ and $x \in X^H$. It follows that the induced homomorphism on equivariant K-theory is a map [82]

$$i^* : K_G^*(X) \longrightarrow K_{N_G(H)}^*(X^H)$$

which is called a *restriction morphism*.

We also need a somewhat less known property [67]. Let $N < G$ be a finite normal subgroup, and let $\text{Rep}(N)$ be the category of (isomorphism classes of) irreducible complex representations of N . Let X be a (proper) G/N -complex, and let G act on X via the projection map $G \rightarrow G/N$. Then for any complex G -vector bundle $E \rightarrow X$ and any representation $V \in \text{Rep}(N)$, define $\text{Hom}_N(V, E)$ as the vector bundle over X with total space

$$\text{Hom}_N(V, E) := \bigcup_{x \in X} \text{Hom}_N(V, E_x)$$

where N acts on the fibres of E because of the action of G via the projection map. Now if $H \leq G$ is a subgroup which commutes with N , $[H, N] = e$, then one can induce an H -vector bundle from $\text{Hom}_N(V, E)$ by defining $(h \cdot f)(v) = h \cdot f(v)$, $v \in V$ for any $h \in H$ and any $f \in \text{Hom}_N(V, E)$ (remembering that G acts on E). Hence there is a homomorphism of rings

$$\Psi : K_G^*(X) \longrightarrow K_H^*(X) \otimes R(N)$$

defined on G -vector bundles by

$$\Psi([E]) := \sum_{V \in \text{Rep}(N)} [\text{Hom}_N(V, E)] \otimes [V]. \quad (6.2.7)$$

This homomorphism satisfies some naturality properties; see [67]. Note that the sum (6.2.7) is *finite*, since N is a finite subgroup.

We are now ready to construct the equivariant Chern character as a homomorphism

$$\mathrm{ch}_X : K_G^{0,1}(X) \longrightarrow H_G^{\mathrm{even,odd}}(X; \mathbb{Q} \otimes \underline{\mathbb{R}}(-))$$

for any finite proper G -complex X . The strategy used in [67] is to construct \mathbb{Z}_2 -graded homomorphisms

$$\mathrm{ch}_X^H : K_G^*(X) \longrightarrow \mathrm{Hom}(H_*(X^H/Z_G(H)), \mathbb{Q} \otimes \mathbb{R}(H)) \quad (6.2.8)$$

for any finite subgroup H , and then *glue* them together as H varies through the finite subgroups of G . To define the homomorphism (6.2.8), we first compose the ring homomorphisms

$$K_G^*(X) \xrightarrow{i^*} K_{N_G(H)}^*(X^H) \xrightarrow{\Psi} K_{Z_G(H)}^*(X^H) \otimes \mathbb{R}(H) \xrightarrow{\pi_2^* \otimes \mathrm{id}} K_{Z_G(H)}^*(EG \times X^H) \otimes \mathbb{R}(H)$$

where $\pi_2 : EG \times X^H \rightarrow X^H$ is the projection onto the second factor. By using the induction structure of Example 6.5, one then has

$$\begin{aligned} K_{Z_G(H)}^*(EG \times X^H) \otimes \mathbb{R}(H) &\xrightarrow{\cong} K^*(EG \times_{Z_G(H)} X^H) \otimes \mathbb{R}(H) \\ &\xrightarrow{\mathrm{ch} \otimes \mathrm{id}} H^*(EG \times_{Z_G(H)} X^H; \mathbb{Q}) \otimes \mathbb{R}(H) \end{aligned}$$

where ch is the ordinary Chern character. One finally has

$$\begin{aligned} H^*(EG \times_{Z_G(H)} X^H; \mathbb{Q}) \otimes \mathbb{R}(H) &\xrightarrow{\cong} H^*(X^H/Z_G(H); \mathbb{Q}) \otimes \mathbb{R}(H) \\ &\simeq \mathrm{Hom}(H_*(X^H/Z_G(H)), \mathbb{Q} \otimes \mathbb{R}(H)) , \end{aligned}$$

where the first isomorphism follows from the Leray spectral sequence by observing that the fibres of the projection

$$EG \times_{Z_G(H)} X^H \longrightarrow X^H / Z_G(H)$$

are all classifying spaces of finite groups, having trivial reduced cohomology with \mathbb{Q} -coefficients and are therefore \mathbb{Q} -acyclic.

The equivariant Chern character is now defined as⁵

$$\mathrm{ch}_X = \bigoplus_{H \leq G} \mathrm{ch}_X^H . \quad (6.2.9)$$

By using the various naturality properties of the homomorphism (6.2.7) [67], one sees that ch_X takes values in $\mathrm{Hom}_{\mathrm{Sub}(G)}(\underline{H}_*^{\mathrm{qt}}(X), \mathbb{Q} \otimes \underline{\mathbb{R}}(-))$, and by Lemma 6.11 it is thus a \mathbb{Z}_2 -graded map

$$\mathrm{ch}_X : K_G^*(X) \longrightarrow \mathrm{Hom}_{\mathrm{Sub}(G)}(\underline{H}_*^{\mathrm{qt}}(X), \mathbb{Q} \otimes \underline{\mathbb{R}}(-)) \simeq H_G^*(X; \mathbb{Q} \otimes \underline{\mathbb{R}}(-)) .$$

⁵If G is infinite then the direct sum in equation (6.2.9) is understood as the inverse limit functor over the dual subgroup category $\mathrm{Sub}(G)^{\mathrm{op}}$.

This map is well-defined as a ring homomorphism because all maps involved above are homomorphisms of rings. As with the definition of Bredon cohomology, the sum (6.2.9) may be restricted to any family of subgroups of G containing the set of stabilizers $\mathfrak{F}(X)$.

To conclude, we have to prove that this map becomes an isomorphism upon tensoring over \mathbb{Q} . For this, one proves that the morphism ch_X in equation (6.2.9) is an isomorphism on homogeneous spaces G/H , with H a finite subgroup of G , and then uses induction on the number of orbit types of cells in X along with the Mayer-Vietoris sequences for the pushout squares induced by the attaching G -maps (6.1.2). The isomorphism on G/H is a consequence of the isomorphisms (6.2.4) and (6.1.5). The details may be found in [67]. Let $\underline{\pi}_*K_G(-)$ be the functor on $\text{Or}(G)$ defined by $G/H \mapsto K_G^*(G/H)$. Then one has the following

Theorem 6.12. ([67]) *For any finite proper G -complex X , the Chern character ch_X extends to a natural \mathbb{Z} -graded isomorphism of rings*

$$\text{ch}_X \otimes \mathbb{Q} : K_G^*(X) \otimes \mathbb{Q} \xrightarrow{\cong} H_G(X; \mathbb{Q} \otimes \underline{\pi}_*K_G(-))^* .$$

6.3 String theory on orbifolds

The techniques of equivariant cohomology and K -theory illustrated in the previous section play an important role in understanding the behaviour of String theory defined on orbifolds. As mentioned in the introduction to this chapter, we will be interested in good orbifolds, which are obtained as the orbit space of the action of a finite group G on a smooth manifold X . In particular, the action of G will be isometric, proper, and cocompact⁶. It is known that when G acts on X with nontrivial stabilizers, the orbit space X/G cannot be given a differential structure such that the usual projection $\pi : X \rightarrow X/G$ is a smooth map. In the case in which all the stabilizers are trivial, i.e. the group G acts freely, the orbit space naturally carries a manifold structure. We will not attempt to give a definition of nonglobal orbifolds, since we will work in the equivariant “regime”. We direct instead the reader to the seminal paper [81] for a local description of orbifolds, and to [68] for a modern description in terms of groupoids.

The quantum behaviour of String theory on an orbifold $[X/G]$ is different from that of

⁶An action of a group G on a space X is said to be *cocompact* if the orbit space X/G is compact

a quantum particle, as first realized in [35, 36]. Indeed, suppose that G acts freely on X . To describe the quantum mechanics of a point particle propagating on the smooth manifold X/G , one could think of first constructing the Hilbert space of states for a particle on the manifold X , and then restrict to the Hilbert subspace of G invariant states⁷. Following the same logic, the first step to the quantum string propagation on X/G consists in constructing the Hilbert space \mathcal{H}_0 for a string propagating on X , and restrict to the G -invariant states. In contrast to the particle case, this is not yet a complete Hilbert space of states. Indeed, \mathcal{H}_0 does not contain states of strings which are closed on the quotient manifold X/G , but are only modulo a G transformation. More precisely, consider an embedding $f : [0, 1] \times \mathbb{R} \rightarrow X$ of the worldsheet strip, with local coordinates $(\sigma, \tau) \in [0, 1] \times \mathbb{R}$. The open strings obeying

$$f(\sigma + 2\pi, \tau) = h \cdot f(\sigma, \tau) \tag{6.3.1}$$

for some $h \in G$ are closed on the quotient X/G , since the point x and $h \cdot x$ are identified⁸. Hence, the Hilbert space \mathcal{H} for a quantum closed string propagating on X/G is given by

$$\mathcal{H} := \left(\bigoplus_{h \in G} \mathcal{H}_h \right)^G \tag{6.3.2}$$

where the sector \mathcal{H}_h is given by the space of states of an open string satisfying condition (6.3.1).

At this point, notice that the action of G permutes the sectors in the conjugacy classes of the associated element h . Indeed, for any f satisfying condition (6.3.1), for any $g \in G$ we have

$$g \cdot f(\sigma + 2\pi, \tau) = gh \cdot f(\sigma, \tau) = (ghg^{-1})g \cdot f(\sigma, \tau) \tag{6.3.3}$$

We can then define

$$\mathcal{H}_{[h]} := \bigoplus_{l \in [h]} \mathcal{H}_l = \bigoplus_{i=1}^{n_h} \mathcal{H}_{p_i h p_i^{-1}}$$

where n_h is the number of element in the conjugacy class $[h]$, and $\{p_i\}$ is an appropriate set of elements of G . Any element $\xi_{[h]}$ can be expressed as

$$\xi_{[h]} = (\xi_h, \xi_{p_1 h p_1^{-1}}, \dots)$$

⁷A choice of a “lift” of the action of G on the internal degrees of freedom, should be made, if possible. In other words, the vector bundle whose the wave function is a section of must be G -equivariant.

⁸In superstring theory X is a $G - \text{spin}^c$ manifold, and an analogous condition to (6.3.1) should be imposed on the worldsheet fermion fields.

and clearly the action of G preserves the vector space $\mathcal{H}_{[h]}$. Since we are interested in G -invariant states, we have only to consider the action of elements $g \in Z_G(h)$, since these are the only elements of G which do not permute the sectors. More precisely, if we define the action of $g \in Z_G(h)$ on $\mathcal{H}_{[h]}$ as

$$g \cdot \xi_{[h]} := (g\xi_h, p_1 g p_1^{-1} \xi_{p_1 h p_1^{-1}}, \dots)$$

we have

$$\mathcal{H} \simeq \bigoplus_{[h]} \mathcal{H}_{[h]}^{Z_G(h)}$$

The subspaces $\mathcal{H}_{[h]}$ associated to a nontrivial conjugacy class are called *twisted sectors*. The above construction of the Hilbert space of closed strings also ensures that the theory is modular invariant [35, 36]. One expects that these twistor sectors will appear also in the case of a G -action with fixed points, and that the Hilbert space of states can be constructed as above. Moreover, in contrast with ordinary quantum field theory, String theory is usually well defined on the singular orbifold points.

As it is expected, the twisted sectors will play a role in the behaviour of the low-energy limit of type II orbifold String theory and of D-branes. This will be illustrated in the following sections.

We conclude this section with a basic result that will be constantly used later on.

Theorem 6.13. *Let G be a finite group acting via isometries on a smooth Riemannian manifold X . Then the set*

$$X^g := \{x \in X : g \cdot x = x\}$$

is naturally a (possibly disconnected) submanifold of X , for any $g \in G$.

Proof. Let \bar{x} be a fixed point for the action of g . The pushforward g_* acts linearly on the vector space $T_{\bar{x}}X$: denote with $K \subset T_{\bar{x}}X$ the space which is left fixed by g_* . Then the exponential map $\exp : T_{\bar{x}}X \rightarrow X$ maps diffeomorphically the subspace K on the fixed points of g , since g acts by isometries, and hence we can use this coordinate system to define local charts for X^g . □

6.3.1 D-branes and equivariant K-cycles

In this section we will make some remarks concerning the topological classification of D-branes and their charges on global orbifolds of Type II superstring theory with vanishing H -flux. As proposed by Witten in [94] and emphasised in [75, 46], Ramond-Ramond charges in type II String theory on a global orbifold $[X/G]$ are classified by

the equivariant K-theory $K_G^*(X)$ of spacetime. The arguments are essentially the same as those presented in chapter 3, hence we will avoid their restatement. Instead, we will show how equivariant K-homology $K_*^G(X)$, dual to equivariant K-theory, leads to a description of *fractional D-branes* in terms of equivariant K-cycles. In the following we will refer to Appendix D for the definition of equivariant K-homology, both geometric and equivariant.

Similarly to K-homology, the cycles for equivariant K-homology, called G-equivariant K-cycles, live in an additive category $D^G(X)$ whose objects are triples (W, E, f) where W is a G - spin^c manifold without boundary, E is a G -vector bundle over W , and

$$f : W \longrightarrow X \tag{6.3.4}$$

is a G -map. The group $K_*^G(X)$ is the quotient of this category by the equivalence relation generated by bordism, direct sum, and vector bundle modification, as detailed in Appendix B. Note that W need not be a submanifold of spacetime. However, since X is a manifold, we can restrict the bordism equivalence relation to *differential bordism* and assume that the map (6.3.4) is a differentiable G -map in equivariant K-cycles $(W, E, f) \in D^G(X)$. In this way the category $D^G(X)$ extends the standard K-theory classification to include branes supported on non-representable cycles in spacetime. This definition of equivariant K-homology thus gives a concrete geometric model for the topological classification of D-branes (W, E, f) in a global orbifold $[X/G]$ which captures the physical constructions of orbifold D-branes as G -invariant states of branes on the covering space X . In the subsequent sections we will study the pairing of Ramond-Ramond fields with these D-branes.

Consider a D-brane localized on the submanifold X^g of the covering space X . Since the Chan-Paton bundle E is G -equivariant, the fiber of the restriction E to X^g at each point carries a representation of the cyclic group $\langle g \rangle$. In this case the D-brane is said to be *fractional*. Fractional D-branes are stuck at the fixed points and they couple to Ramond-Ramond fields coming from the twisted sector labelled by $[g]$. The term fractional is used since fractional D-branes, in the simple examples known, carry a *fraction* of the corresponding Ramond-Ramond charge.

We can use equivariant K-homology to geometrically describe a particular class of fractional D-branes. Indeed, let G^\vee denote the set of conjugacy classes $[g]$ of elements $g \in G$. There is a natural subcategory $D_{\text{frac}}^G(X)$ of $D^G(X)$ consisting of triples (W, E, f) for which W is a G -fixed space, i.e., for which

$$W^g = W \tag{6.3.5}$$

for all $g \in G$. By G -equivariance this implies $f(W)^g = f(W)$ for all $g \in G$, and so the image of the brane worldvolume lies in the subspace

$$f(W) \subset \bigcap_{g \in G} X^g .$$

This is the set of G -fixed points of X , and so the objects (W, E, f) of the category $D_{\text{frac}}^G(X)$ can naturally be interpreted in terms of fractional branes. More precisely, we call $D_{\text{frac}}^G(X)$ the category of ‘‘maximally fractional D-branes’’.

In this case, an application of Schur’s lemma shows that the Chan-Paton bundle admits an isotopical decomposition and there is a canonical isomorphism of G -bundles

$$E \simeq \bigoplus_{[g] \in G^\vee} E_{[g]} \otimes \mathbb{1}_{[g]} \quad \text{with} \quad E_{[g]} = \text{Hom}_G(\mathbb{1}_{[g]}, E) , \quad (6.3.6)$$

where $E_{[g]}$ is a complex vector bundle with trivial G -action and $\mathbb{1}_{[g]}$ is the G -bundle $W \times V_{[g]}$ with $\gamma : G \rightarrow \text{End}(V_{[g]})$ the irreducible representation corresponding to the conjugacy class $[g] \in G^\vee$.

From the direct sum relation in equivariant K -homology it follows that a fractional D-brane, represented by a K -cycle (W, E, f) in $D_{\text{frac}}^G(X)$, splits into a sum over irreducible fractional branes represented by the K -cycles $(W, E_{[g]} \otimes \mathbb{1}_{[g]}, f)$, $[g] \in G^\vee$, which can then be considered stable.

We then propose that in the framework of equivariant K -homology, the topological charge of a fractional D-brane, in a given closed string twisted sector of the orbifold String theory on a G -spin^{*c*} manifold X , can be computed by using the equivariant Dirac operator theory introduced in Appendix D. The equivariant index of the G -invariant spin^{*c*} Dirac operator \mathcal{D}_E^X coupled to a G -vector bundle $E \rightarrow X$ takes values in $K_G^*(\text{pt}) \simeq R(G)$. We can turn this into a homomorphism on $K_*^G(X)$ with values in \mathbb{Z} by composing with the projection $R(G) \rightarrow \mathbb{Z}$ defined by taking the multiplicity of a given representation

$$\gamma : G \longrightarrow \text{End}(V_\gamma) \quad (6.3.7)$$

of G on a finite-dimensional complex vector space V_γ . There is a corresponding class in the KK -theory group

$$[\gamma] \in KK_*(\mathbb{C}[G], \text{End}(V_\gamma))$$

which is represented by the Kasparov module $(V_\gamma, \gamma, 0)$ associated with the extension of the representation (6.3.7) to a complex representation of group ring $\mathbb{C}[G]$. By Morita invariance, the Kasparov product with $[\gamma]$ is the homomorphism on K -theory

$$K_0(\mathbb{C}[G]) \longrightarrow K_0(\text{End}(V_\gamma)) \simeq K_0(\mathbb{C}) \simeq \mathbb{Z}$$

induced by $\gamma : \mathbb{C}[G] \rightarrow \text{End}(V_\gamma)$ [25]. We may then define a homomorphism

$$\mu_\gamma : K_0^G(X) \longrightarrow \mathbb{Z}$$

of abelian groups by

$$\mu_\gamma([W, E, f]) = \text{Index}_\gamma(f_*[\mathcal{D}_E^W]) := \text{ass}(f_*[\mathcal{D}_E^W]) \otimes_{\mathbb{C}[G]} [\gamma] \quad (6.3.8)$$

on equivariant K-cycles $(W, E, f) \in D^G(X)$ (and extended linearly), where

$$\text{ass} : K_*^G(X) \longrightarrow K_*(\mathbb{C}[G])$$

is the analytic assembly map mentioned in Appendix D. We then naturally interpret 6.3.8 as the topological charge of the D-brane represented by (W, E, f) . Notice that for $G = e$, (6.3.8) reduces to the ordinary expression for the charge of a D-brane.

We may now consider a simple class of examples. Let V be a complex vector space of dimension $\dim_{\mathbb{C}}(V) = d \geq 1$, and let G be a finite subgroup of $SL(V)$. Our spacetime X is the G -space identified with the product

$$X = \mathbb{R}^{p,1} \times V ,$$

where G acts trivially on the Minkowski space $\mathbb{R}^{p,1}$, and p is odd. We will consider fractional D-branes with worldvolume $\mathbb{R}^{p,1} \hookrightarrow \mathbb{R}^{p,1} \times v$, where $v \in V$ is a fixed vector under the linear action of G . In analogy with the nonequivariant case, the group of charge of these fractional D-branes is given by the compact support equivariant K-theory of the normal bundle, which in this case is given by [8]

$$K_{G,\text{cpt}}^*(V) \simeq K_G^*(\text{pt}) \simeq R(G) = \mathbb{Z}^{|G^\vee|} .$$

since V is G -contractible.

The same result can be obtained by considering the equivariant K-theory of the world volume $\mathbb{R}^{p,1}$, which is G -contractible. It follows that the fractional D-branes, as defined by elements of equivariant K-theory, can be identified with representations of the orbifold group

$$\gamma = \bigoplus_{a=1}^{|G^\vee|} N_a \gamma_a$$

consisting of $N_a \geq 0$ copies of the a -th irreducible representation

$$\gamma_a : G \longrightarrow \text{End}(V_a) , \quad a = 1, \dots, |G^\vee| ,$$

which defines the action of G on the fibres of the Chan-Paton bundle. More precisely, each irreducible fractional brane is associated to the G -bundle $\mathbb{R}^{p,1} \times V_a$ over $\mathbb{R}^{p,1}$. We can then consider the K -cycles $(\mathbb{R}^{p,1}, \mathbb{R}^{p,1} \times V_a, i_v)$, where $i_v : \mathbb{R}^{p,1} \rightarrow \mathbb{R}^{p,1} \times v \subset X$. In equivariant K -homology, these cycles can be contracted to $[\text{pt}, V_a, i]$, where i is the inclusion of a point $\text{pt} \subset V$ whose induced homomorphism

$$i_* : K_*^G(\text{pt}) \longrightarrow K_*^G(X)$$

can be taken to be the identity map $R(G) \rightarrow R(G)$. The G -invariant Dirac operator $\mathcal{D}_{V_a}^{\text{pt}}$ is just Clifford multiplication twisted by the G -module V_a , and thus the topological charges (6.3.8) of the corresponding fractional branes in the twisted sector labelled by b are given by

$$\mu_b([\text{pt}, V_a, i]) = \text{Index}_{\gamma_b}([\mathcal{D}_{V_a}^{\text{pt}}]) = [V_a \otimes (\Delta^+ \oplus \Delta^-)] \otimes_{\mathbb{C}[G]} [\gamma_b],$$

where Δ^\pm are the half-spin representations of $\text{SO}(p+1)$ on $\mathbb{C}^{\frac{p+1}{2}}$. Acting on the character ring the projection gives $[W] \otimes_{\mathbb{C}[G]} [\gamma_b] = \gamma_*([W])$, where

$$\gamma_* : K_0(\mathbb{C}[G]) \longrightarrow K_0(\text{End}(V_\gamma))$$

is the map induced by γ .

6.3.2 Delocalization and Ramond-Ramond fields

As discussed in chapter 1, the gauge theory of Ramond-Ramond fields arises as a low-energy limit of type II superstring theory from the Hilbert space of states of closed superstrings. Intuitively, the low-energy limit is the limit in which the string becomes pointlike, i.e. the length of the string goes to zero. As we have seen in the previous section, the Hilbert space for type II superstring theory defined on the good orbifold $[X/G]$ is given by

$$\mathcal{H} := \left(\bigoplus_{h \in G} \mathcal{H}_h \right)^G$$

where the subspaces \mathcal{H}_h are spaces of states of open strings satisfying the boundary condition (6.3.3); we have also noticed that such open strings “look like” closed strings on the submanifolds X^h . We then expect massless fields, in particular Ramond-Ramond fields, arising from each of these sectors, defined on X^h . This is due to the fact that the center of mass of an open string satisfying condition (6.3.3) is constrained to be a point of X^h . We can mathematically “organize” the information about these

Ramond-Ramond fields in the following way. We can associate to each G -manifold the space

$$\hat{X} := \coprod_{h \in G} X^h \quad (6.3.9)$$

Notice that we have an action of G on \hat{X} , with $g \in G$ inducing the diffeomorphism $X^h \rightarrow X^{ghg^{-1}}$. We can then consider Ramond-Ramond fields as elements of the differential complex

$$\Omega_G^*(X; \mathbb{R}) := \Omega^*(\hat{X}; \mathbb{R})^G = \left(\bigoplus_{h \in G} \Omega^*(X^h; \mathbb{R}) \right)^G \quad (6.3.10)$$

equipped with the differential

$$d_G := \bigoplus_{h \in G} d_h$$

where $d_h : \Omega^*(X^h; \mathbb{R}) \rightarrow \Omega^*(X^h; \mathbb{R})$ is the usual deRham exterior derivative. Since X^h is diffeomorphic to $X^{ghg^{-1}}$ for any $g \in G$, by making a choice of submanifolds X^g we have

$$\Omega_G^*(X; \mathbb{R}) \simeq \bigoplus_{[h] \in G^\vee} \Omega^*(X^h; \mathbb{R})^{Z_G(h)} \quad (6.3.11)$$

The cohomology of the complex (6.3.10) with respect to the differential d_G is given by

$$H^*(\Omega_G^*(X; \mathbb{R}); d_G) = \left(\bigoplus_{h \in G} H^*(X^h; \mathbb{R}) \right)^G \simeq \bigoplus_{[h] \in G^\vee} H^*(X^h; \mathbb{R})^{Z_G(h)}$$

where we have used

$$H^*(X^h/Z_G(h); \mathbb{R}) \simeq H^*(X^h; \mathbb{R})^{Z_G(h)}$$

The cohomology groups above correspond to the *delocalized equivariant cohomology* theory defined by Baum and Connes [12]. Notice that the group $H^*(\Omega_G^*(X; \mathbb{R}); d_G)$ is non-canonically isomorphic to $H^*(X; \mathbb{R}) \otimes R(G)$ when the G -action on X is trivial.

We will now show how Bredon cohomology can be used to compute the cohomology of the complex (6.3.10) of orbifold Ramond-Ramond fields by giving a delocalized description of Bredon cohomology with *real* coefficients, following [73] and [67] where further details can be found.

Denote with $\underline{\mathbb{R}}(-)$ the real representation ring functor $\mathbb{R} \otimes \underline{R}(-)$ on the orbit category $\text{Or}(G)$. Let $\langle G \rangle$ denote the set of conjugacy classes $[C]$ of cyclic subgroups C of G . Let $\underline{\mathbb{R}}_C(-)$ be the contravariant functor on $\text{Or}(G)$ defined by $\underline{\mathbb{R}}_C(G/H) = 0$ if $[C]$ contains no representative $g C g^{-1} < H$, and otherwise $\underline{\mathbb{R}}_C(G/H)$ is isomorphic to the

cyclotomic field $\mathbb{R}(\zeta_{|C|})$ over \mathbb{R} generated by the primitive root of unity $\zeta_{|C|}$ of order $|C|$. A standard result from the representation theory of finite groups then gives a natural splitting

$$\underline{\mathbb{R}}(-) = \bigoplus_{[C] \in \langle G \rangle} \underline{\mathbb{R}}_C(-).$$

By definition, for any module $\underline{M}(-)$ over the orbit category one has

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Or}(G)}(\underline{M}(-), \underline{\mathbb{R}}_C(-)) &\simeq \mathrm{Hom}_{N_G(C)}(\underline{M}(G/C), \underline{\mathbb{R}}_C(G/C)) \\ &\simeq \mathrm{Hom}(\underline{M}(G/C); \mathbb{Z}) \otimes_{N_G(C)} \underline{\mathbb{R}}_C(G/C) \end{aligned}$$

where the normalizer subgroup $N_G(C)$ acts on $\underline{\mathbb{R}}_C(G/C) \simeq \mathbb{R}(\zeta_{|C|})$ via identification of a generator of C with $\zeta_{|C|}$.

These facts together imply that the cochain groups (6.2.2) with $\underline{F} = \underline{\mathbb{R}}(-)$ admit a splitting given by

$$C_G^*(X, \underline{\mathbb{R}}(-)) \simeq \bigoplus_{[C] \in \langle G \rangle} C^*(X^C) \otimes_{N_G(C)} \underline{\mathbb{R}}_C(G/C).$$

As the centralizer $Z_G(C)$ acts properly on X^C , the natural map

$$\bigoplus_{[C] \in \langle G \rangle} C^*(X^C) \otimes_{N_G(C)} \underline{\mathbb{R}}_C(G/C) \longrightarrow \bigoplus_{[C] \in \langle G \rangle} C^*(X^C/Z_G(C)) \otimes_{W_G(C)} \underline{\mathbb{R}}_C(G/C)$$

is a cohomology isomorphism, where $W_G(C) := N_G(C)/Z_G(C)$ is the Weyl group of $C < G$ which acts by translation on $X^C/Z_G(C)$. Since $\underline{\mathbb{R}}_C(G/C)$ is a projective $\mathbb{R}[W_G(C)]$ -module, it follows that for any proper G -complex X the Bredon cohomology of X with coefficient system $\mathbb{R} \otimes \underline{\mathbb{R}}(-)$ has a splitting

$$H_G^*(X; \mathbb{R} \otimes \underline{\mathbb{R}}(-)) \simeq \bigoplus_{[C] \in \langle G \rangle} H^*(X^C/Z_G(C); \mathbb{R}) \otimes_{W_G(C)} \underline{\mathbb{R}}_C(G/C). \quad (6.3.12)$$

At this point, we note that the dimension of the \mathbb{R} -vector space

$$\underline{\mathbb{R}}_C(G/C)^{W_G(C)} \simeq \mathbb{R} \otimes_{W_G(C)} \underline{\mathbb{R}}_C(G/C)$$

is equal to the number of G -conjugacy classes of generators for C . We also use the fact that for a finite group G a sum over conjugacy classes of cyclic subgroups is equivalent to a sum over conjugacy classes of elements in G , and that $X^{\langle g \rangle} = X^g$ and $Z_G(\langle g \rangle) = Z_G(g)$. One finally obtains a splitting of real Bredon cohomology groups⁹

$$H_G^*(X; \mathbb{R} \otimes \underline{\mathbb{R}}(-)) \simeq \bigoplus_{[g] \in G^\vee} H^*(X^g; \mathbb{R})^{Z_G(g)} \quad (6.3.13)$$

⁹This splitting in fact holds over \mathbb{Q} [73].

which is the cohomology of the differential complex (6.3.10).

By using Theorem 6.12, one also has a decomposition for equivariant K-theory with real coefficients given by

$$K_G^*(X) \otimes \mathbb{R} \simeq \bigoplus_{[g] \in G^\vee} (K^*(X^g) \otimes \mathbb{R})^{Z_G(g)} .$$

6.3.3 Delocalization of the equivariant Chern character

It is well known that the ordinary Chern character, when tensored over \mathbb{R} , admits a Chern-Weyl refinement, expressed in terms of the curvature of an arbitrary connection on the given vector bundle. This is not the case for the equivariant Chern character defined in section 6.2.2. This is expected, since the equivariant Chern character was constructed in terms of homomorphisms between K-theory and cohomology, without any reference to any geometric description. However, when tensored over \mathbb{C} , the equivariant Chern character admits a more geometric description. We will now explain this construction, referring the reader to [25] for the technical details. Consider a complex G -bundle E over X equipped with a G -invariant hermitean metric and a G -invariant metric connection ∇^E . One can then define a closed G -invariant differential form

$$\text{ch}(E) \in \Omega^*(X; \mathbb{C})^G$$

in the usual way by the Chern-Weil construction

$$\text{ch}(E) := \text{Tr} \left(\exp(-F^E/2\pi i) \right)$$

where F^E is the curvature of the connection ∇^E . It represents a cohomology class

$$[\text{ch}(E)] \in H^*(X; \mathbb{C})^G$$

in the fixed point subring of the action of G as automorphisms of $H^*(X; \mathbb{C})$. By using the definition of the homomorphisms (6.2.8), with \mathbb{Q} substituted by \mathbb{C} and $H = e$, one can establish the equality

$$[\text{ch}(E)] = \text{ch}_X^e([E]) .$$

Let $C < G$ be a cyclic subgroup, and define the cohomology class

$$[\text{ch}(g, E)] \in H^*(X^C; \mathbb{C})^{Z_G(C)} \simeq H^*(X^C/Z_G(C); \mathbb{C}) \simeq H(\Omega^*(X^C; \mathbb{C})^{Z_G(C)}, d)$$

represented by

$$\text{ch}(g, E) := \text{Tr} \left(\gamma(g) \exp(-F_C^E/2\pi i) \right)$$

where g is a generator of C , F_C^E is the restriction of the invariant curvature two-form F^E to the fixed point subspace X^C , and γ is a representation of C on the fibres of the restriction bundle $E|_{X^C}$ which is an $N_G(C)$ -bundle over X^C . The character χ_C naturally identifies $R(C) \otimes \mathbb{C}$ with the \mathbb{C} -vector space of class functions $C \rightarrow \mathbb{C}$. By using the splitting (6.3.12) for complex Bredon cohomology, one can then show that

$$\mathrm{ch}_X^{\mathbb{C}}([E])(g) = [\mathrm{ch}(g, E)]$$

up to the restriction homomorphism $R(C) \otimes \mathbb{C} \rightarrow \underline{\mathbb{C}}_C(G/C)$ of rings with kernel the ideal of elements whose characters vanish on all generators of C .

Using (6.2.9) we can then define the map

$$\mathrm{ch}^{\mathbb{C}} : \mathrm{Vect}_G^{\mathbb{C}}(X) \longrightarrow \Omega_G^{\mathrm{even}}(X; \mathbb{C})$$

from complex G -bundles $E \rightarrow X$ given by

$$\mathrm{ch}^{\mathbb{C}}(E) = \bigoplus_{[g] \in G^{\vee}} \mathrm{Tr}(\gamma(g) \exp(-F_g^E/2\pi i)) . \quad (6.3.14)$$

At the level of equivariant K -theory, from Theorem 6.12 it follows that this map induces an isomorphism

$$\mathrm{ch}^{\mathbb{C}} : K_G^*(X) \otimes \mathbb{C} \xrightarrow{\cong} H_G(X; \mathbb{C} \otimes \underline{\pi_{-*}} K_G(-))^* \quad (6.3.15)$$

where we have used the splitting (6.3.13). The map (6.3.14) coincides with the equivariant Chern character defined in [9].

6.3.4 Ramond-Ramond couplings with D-branes

We now have all the necessary ingredients to define a coupling of the Ramond-Ramond fields to a D-brane in the orbifold $[X/G]$. In this section we will only consider Ramond-Ramond fields which are topologically trivial, i.e., elements of the differential complex (6.3.10), and use the delocalized cohomology theory above by working throughout with complex coefficients. Moreover, we will only consider electric couplings to D-branes, i.e. we will not impose selfduality. Under these conditions we can straightforwardly make contact with existing examples in the physics literature and write down their appropriate generalizations.

To this aim, we introduce the bilinear product

$$\wedge_G : \Omega_G^*(X; \mathbb{R}) \otimes \Omega_G^*(X; \mathbb{R}) \longrightarrow \Omega_G^*(X; \mathbb{R})$$

defined on $\omega = \bigoplus_{g \in G} \omega_g$ and $\eta = \bigoplus_{g \in G} \eta_g$ by

$$\omega \wedge_G \eta := \bigoplus_{g \in G} \omega_g \wedge_g \eta_g \quad (6.3.16)$$

where $\wedge_g = \wedge$ is the usual exterior product on $\Omega^*(X^g; \mathbb{R})$. There is also an integration

$$\int_X^G : \Omega_G^*(X; \mathbb{R}) \longrightarrow \mathbb{R} .$$

If $\omega \in \Omega_G^*(X; \mathbb{R})$ then we set

$$\int_X^G \omega := \frac{1}{|G^\vee|} \sum_{[g] \in G^\vee} \int_{X^g} \omega_{[g]} .$$

where we have used in the above construction that for a G -manifold X admitting a G -equivariant spin structure, the fixed point manifold X^g is naturally oriented, for any $g \in G$ [17].

The normalization ensures that $\int_X^G \omega = \int_X \omega$ when G acts trivially on X , and ω is “diagonal” in $\Omega^*(X) \otimes R(G)$.

Suppose now that $f : W \rightarrow X$ is the smooth immersed worldvolume cycle of a wrapped D-brane state $(W, E, f) \in D^G(X)$ in the orbifold $[X/G]$, i.e., W is a G -spin^c manifold equipped with a G -bundle $E \rightarrow W$ and an invariant connection ∇^E on E . We define the *Wess-Zumino pairing*

$$\text{WZ} : D^G(X) \times \Omega_G^*(X; \mathbb{C}) \longrightarrow \mathbb{C}$$

between such D-branes and Ramond-Ramond fields as

$$\text{WZ}((W, E, f), C) = \int_W^G \tilde{C} \wedge_G \text{ch}^C(E) \wedge_G \mathcal{R}(W, f) , \quad (6.3.17)$$

where $\tilde{C} = f^*C$ is the pullback along $f : W \rightarrow X$ of the total Ramond-Ramond field

$$C = \bigoplus_{[g] \in G^\vee} C_{[g]}$$

and the equivariant Chern character is given by (6.3.14) with γ giving the action of G on the Chan-Paton factors of the D-brane. The closed worldvolume form $\mathcal{R}(W, f) \in \Omega_{G, \text{cl}}^{\text{even}}(W; \mathbb{C})$ represents a complex Bredon cohomology class which accounts for gravitational corrections due to curvature in the spacetime X and depends only on the bordism class of (W, f) . We refer the reader to [91], where a construction of $\mathcal{R}(W, f)$ can be found.

Modulo the curvature contribution $\mathcal{R}(W, f)$, the very natural expression (6.3.17) reduces to the usual Wess-Zumino coupling of topologically trivial Ramond-Ramond fields to D-branes in the case $G = e$. But even if a group $G \neq e$ acts trivially on the brane worldvolume W (or on the spacetime X), there can still be additional contributions to the usual Ramond-Ramond coupling if E is a *non-trivial* G -bundle. This is the situation, for instance, for fractional D-branes

$$(W, E, f) \in \mathbf{D}_{\text{frac}}^G(X)$$

placed at orbifold singularities. In this case, we may use the isotopical decomposition (6.3.6) of the Chan-Paton bundle along with (6.3.5). Then the Wess-Zumino pairing (6.3.17) descends to a pairing

$$\text{WZ}_{\text{frac}} : \mathbf{D}_{\text{frac}}^G(X) \times \Omega_G^*(X; \mathbb{C}) \longrightarrow \mathbb{C}$$

with the additive subcategory of fractional branes at orbifold singularities.

Example 6.14. We will now “test” our definition (6.3.17) on the class of examples considered in section 6.3.1. These are flat orbifolds for which there are no non-trivial curvature contributions, i.e., $\mathcal{R}(W, f) = 1$. Let us specialize to the case of cyclic orbifolds having twist group $G = \mathbb{Z}_n$ with $n \geq d$. In this case, as \mathbb{Z}_n is an abelian group, one has $\mathbb{Z}_n^\vee = \mathbb{Z}_n$ (setwise) and we can label the non-trivial twisted sectors of the orbifold String theory on X by $k = 1, \dots, n - 1$. The untwisted sector is labelled by $k = 0$. We take a generator g of \mathbb{Z}_n whose action on $V \simeq \mathbb{C}^d$ is given by

$$g \cdot (z^1, \dots, z^d) := (\zeta^{a_1} z^1, \dots, \zeta^{a_d} z^d),$$

where $\zeta = \exp(2\pi i/n)$ and a_1, \dots, a_d are integers satisfying $a_1 + \dots + a_d \equiv 0 \pmod{n}$ ¹⁰. In this case the action of any element in \mathbb{Z}_n has only one fixed point, an orbifold singularity at the origin $(0, \dots, 0)$. Hence for any $g \neq e$ one has

$$X^g \simeq \mathbb{R}^{p,1}$$

and the differential complex (6.3.10) of orbifold Ramond-Ramond fields is given by

$$\Omega_{\mathbb{Z}_n}^*(X; \mathbb{R}) = \Omega^*(X; \mathbb{R}) \oplus \left(\bigoplus_{k=1}^{n-1} \Omega^*(\mathbb{R}^{p,1}; \mathbb{R}) \right).$$

¹⁰Both the requirement that the representation V be complex and the form of the G -action are physical inputs ensuring that the closed string background X preserves a sufficient amount of supersymmetry after orbifolding.

Consider now a fractional D-brane $(W, E, f) \in \mathbf{D}_{\text{frac}}^{\mathbb{Z}_n}(X)$ with worldvolume cycle $f(W) \subset \mathbb{R}^{p,1}$ placed at the orbifold singularity, i.e., $f : W \rightarrow \mathbb{R}^{p,1} \times (0, \dots, 0) \subset X$. Let the generator g act on the fibres of the Chan-Paton bundle $E \rightarrow W$ in the n -dimensional regular representation $\gamma(g)_{ij} = \zeta^i \delta_{ij}$. The action on worldvolume fermion fields is determined by a lift $\hat{\mathbb{Z}}_n$ acting on the spinor bundle $S \rightarrow W$. Then the pairing (6.3.17) contains the following terms. First of all, we have the coupling of the untwisted Ramond-Ramond fields to W given by

$$\frac{1}{n} \int_W \tilde{C} \wedge \text{Tr} \left(\exp(-F^E/2\pi i) \right),$$

which is just the usual Wess-Zumino coupling and hence the untwisted Ramond-Ramond charge of the brane is 1. Then there are the contributions from the twisted sectors, which by recalling (6.3.5) are given by the expression

$$\frac{1}{n} \int_W \sum_{k=1}^{n-1} \tilde{C}_k \wedge \text{Tr} \left(\gamma(g^k) \exp(-F^E/2\pi i) \right)$$

where g^k is an element of \mathbb{Z}_n of order k . Since $\gamma(g^k)_{ii} = \zeta^{ik}$, the brane associated with the i -th irreducible representation of \mathbb{Z}_n has charge ζ^{ik}/n with respect to the top Ramond-Ramond field in the k -th twisted sector. For $d = 2$ and $d = 3$, the form of these couplings agrees with those computed in [37].

6.4 An equivariant Riemann-Roch formula

Let X, W be smooth compact G -manifolds, and $f : W \rightarrow X$ a smooth proper G -map. If we want to make sense of the equations of motion for the Ramond-Ramond field C , which is a quantity defined on the spacetime X , then we need to pushforward classes defined on the brane worldvolume W to classes defined on the spacetime. This will enable the construction of Ramond-Ramond currents in a later section induced by the background and D-branes which appear as source terms in the Ramond-Ramond field equations.

As we have seen in chapter 3, given a smooth embedding $f : W \rightarrow X$ with normal bundle $\nu \rightarrow W$ equipped with a spin^c structure, we have

$$\text{ch}(f_!^K(\xi)) = f_!^H(\text{ch}(\xi) \cup \text{Todd}(\nu)^{-1}) \tag{6.4.1}$$

for any class $\xi \in K^*(W)$, where $f_!$ denote the Gysin homomorphism defined in chapter 3, and $\text{Todd}(E) \in \Omega_{\text{cl}}^{\text{even}}(W; \mathbb{C})$ denotes the Todd genus characteristic class of a

hermitean vector bundle E over W , whose Chern-Weil representative is

$$\text{Todd}(E) = \sqrt{\det \left(\frac{F^E/2\pi i}{\tanh(F^E/2\pi i)} \right)}$$

where F^E is the curvature of a hermitean connection ∇^E on E . The important aspect is that the Chern character does not commute with the Gysin pushforward maps, and the defect in the commutation relation is precisely the Todd genus of the bundle ν . This “twisting” by the bundle ν over the D-brane contributes in a crucial way to the Ramond-Ramond current in the non-equivariant case [29, 71, 75].

We will now attempt to find an equivariant version of the Riemann-Roch theorem. We will consider G -equivariant embeddings $f : W \rightarrow X$: in this case normal bundle ν is itself a G -bundle. We assume that ν is K_G -oriented. This requirement is just the Freed-Witten anomaly cancellation condition [45] in this case, generalized to global worldsheet anomalies for D-branes represented by generic G -equivariant K -cycles. It enables, analogously to the non-equivariant case, the construction of an equivariant Gysin homomorphism

$$f_!^{K_G} : K_G^*(W) \longrightarrow K_G^*(X) . \tag{6.4.2}$$

We will show that, under some very special conditions, one can construct a complex Bredon cohomology class which is analogous to the Todd genus and which plays the role of the equivariant commutativity defect as above.

For this we will need the following¹¹

Lemma 6.15. *Let $\pi : E \rightarrow X$ be an equivariant G -vector bundle, where G is a finite group acting properly. Then for any $g \in G$ the map $\pi|_{E^g} : E^g \rightarrow X^g$ is vector bundle projection.*

Proof. Consider a point $w \in E$. Then $d\pi : T_w E \rightarrow T_{\pi(w)} X$ is surjective, since it is the projection map of a fiber bundle. If $w \in E^g$, then $\pi(w)$ belongs to X^g , and the element g acts linearly on $T_{\pi(w)} X$. If $v \in T_{\pi(w)} X$ is a g -fixed vector, one can then find a g -fixed preimage in $T_w E$ by taking any preimage and averaging with respect to the action of g . By using that $(TX)^g = TX^g$, we have that the map $T_w E^g \rightarrow T_{\pi(w)} X^g$ is surjective, hence the restricted map $\pi|_{E^g}$ is a proper surjection. Any proper submersion is automatically a fibre bundle, and the linear structure on the fibers of $E^g \rightarrow X^g$ is inherited by that on the fibers of $E \rightarrow X$. \square

¹¹We are grateful to U.Bunke for suggesting this result to us.

Notice that the above lemma strongly use the properties that the group G is finite and acting properly. To see what can happen when these conditions are not satisfied¹², let $X = \mathbb{R}$ and $G = \mathbb{R}^+$ the group of positive reals under multiplication. Consider the G -bundle $X \times V \rightarrow X$ given by projection onto the first factor, where V is a finite-dimensional real vector space and the G -action is

$$g \cdot (x, v) = (x, g^x v)$$

for all $g \in G$. For any $g \neq 1$, $(X \times V)^g$ is not a fibre bundle over $X^g = X$, as the G -invariant fibre space over $x = 0$ is V while it is the null vector over any other point. In particular, $(X \times V)^g$ is not even a manifold.

Consider now two G -manifolds W and X equipped with G -invariant spin^c structure, and a proper G -equivariant embedding $f : W \rightarrow X$. For any $g \in G$, the vector bundle $\nu^g = \nu(W; f)^g \rightarrow W^g$ is the normal bundle $\nu^g = f^*|_{W^g}(\mathbb{T}_{X^g}) \oplus \mathbb{T}_{W^g}$ over the immersion $f|_{W^g} : W^g \rightarrow X^g$. Recall that this is a $Z_G(g)$ -bundle. We will suppose that ν^g is equipped with a $Z_G(g)$ -invariant spin^c structure.

As we have seen in section 6.3.2, equivariant K-theory “delocalizes” when tensored over \mathbb{R} , thanks to the equivariant Chern character. We will show in the following that when working over \mathbb{C} the delocalization of equivariant K-theory is compatible, in a certain sense, with the complex equivariant Chern character defined in (6.3.14).

Let E be a G -vector bundle over a X . For any $g \in G$, the restriction E_g of E to the fixed point subspace X^g gives a $Z_G(g)$ -invariant vector bundle carrying a representation γ of $\langle g \rangle$ on the fibers, where $\langle g \rangle$ is the cyclic group generated by g . The decomposition of the representation γ in terms of irreducible representations γ_α of $\langle g \rangle$ induces the homomorphism

$$K_G^*(X) \rightarrow K^*(X^g) \otimes R(\langle g \rangle)$$

given on vector bundles by

$$E \rightarrow E_g \simeq \bigoplus_{\gamma_\alpha} E_\alpha \otimes \gamma_\alpha$$

where $E_\alpha := \text{Hom}_{\langle g \rangle}(E, \mathbb{1}_\alpha)$, where $\mathbb{1}_\alpha = X^g \times V_\alpha$, with V_α carrying the irreducible representation γ_α . We can then define the morphism

$$K^*(X^g) \otimes R(\langle g \rangle) \rightarrow K^*(X^g) \otimes \mathbb{C}$$

by tracing over the second factor, i.e.

$$[E_g] \rightarrow \sum_{\alpha} [E_\alpha] \text{Tr}(\gamma_\alpha(g))$$

¹²We are grateful to J.Figueroa-O’Farrill for suggesting this example to us

Denoting ϕ_g the composition of the above morphism, we can define

$$\phi := \bigoplus_{[g] \in G^\vee} \phi_g$$

Recalling that E_g is a $Z_G(g)$ -invariant bundle, we have the isomorphism [9]

$$\phi : K_G^*(X) \otimes \mathbb{C} \simeq \bigoplus_{[g] \in G^\vee} (K^*(X^g) \otimes \mathbb{C})^{Z_G(g)} \quad (6.4.3)$$

By using that [25]

$$\mathrm{Tr}(\gamma(g) \exp(-F_g^E/2\pi i)) = \sum_{\alpha} \mathrm{Tr}(\exp(-F_{\alpha}^E/2\pi i)) \mathrm{Tr}(\gamma_{\alpha}(g))$$

where F_{α}^E is the restriction of the curvature form F_g^E to the subbundle E_{α} , we have that the complex equivariant Chern character (6.3.14) coincides on the components of the decomposition (6.4.3) with the ordinary Chern character.

Suppose now that the equivariant Thom class $\mathrm{Thom}_G(\nu) \in K_{G,\mathrm{cpt}}^*(\nu)$ can be decomposed according to the splitting (6.4.3) in such a way that the component in any subgroup

$$\mathrm{Thom}(\nu^g) \in (K_{\mathrm{cpt}}^*(\nu^g) \otimes \mathbb{C})^{Z_G(g)}$$

coincides with the (ordinary) Thom class of the vector bundle $\nu^g \rightarrow W^g$. Under these conditions, the equivariant Gysin homomorphism (6.4.2) decomposes according to the splitting

$$f_!^{\mathrm{KG}} = \bigoplus_{[g] \in G^\vee} f_g^{\mathrm{K}}$$

where f_g^{K} is the K-theoretic Gysin homomorphism associated to the smooth map

$$f|_{W^g} : W^g \longrightarrow X^g .$$

Define the characteristic class Todd_G by

$$\mathrm{Todd}_G(\nu) := \bigoplus_{[g] \in G^\vee} \mathrm{Todd}(\nu^g) \quad (6.4.4)$$

This class defines an element of the even degree complex Bredon cohomology of the brane worldvolume W . Under the conditions spelled out above, we can now use the equivariant Chern character (6.3.15) and the usual Riemann-Roch theorem for each pair (W^g, X^g) to prove the identity

$$f_!^{\mathrm{HG}}(\mathrm{ch}^{\mathbb{C}}(\xi) \cup_G \mathrm{Todd}_G(\nu)^{-1}) = \mathrm{ch}^{\mathbb{C}}(f_!^{\mathrm{KG}}(\xi)) \quad (6.4.5)$$

for any class $\xi \in K_G^*(W) \otimes \mathbb{C}$, as all quantities involved in the expression (6.4.5) are compatible with the G-equivariant decompositions given above.

This equivariant Riemann-Roch formula will be important when we will argue the flux quantization for Ramond-Ramond fields on orbifolds.

6.5 Orbifold differential K-theory and flux quantization

In this section we will propose an extension of differential K-theory as defined in section 5.4 to incorporate the case of a G-manifold. These are the groups needed to extend the analysis of the previous section to topologically non-trivial, real-valued Ramond-Ramond fields. While we do not have formal arguments that this is a proper definition of an equivariant differential cohomology theory, we will see that it matches exactly with expectations from String theory on orbifolds and also has the correct limiting properties. For this reason we dub the theory that we define ‘orbifold’ differential K-theory, deferring the terminology ‘equivariant’ to a more thorough treatment of our model. We will use the Riemann-Roch formula developed in the last section to argue that the Dirac quantization condition for Ramond-Ramond field strengths on a good orbifold is dictated by equivariant K-theory via the equivariant Chern character, suggesting that our orbifold differential K-theory is the correct guess for the space gauge equivalent classes of Ramond-Ramond fields. Finally, we will study the classification of Ramond-Ramond fields on the linear orbifolds considered in section 6.3.1, and give a definition for the group of flat Ramond-Ramond fields on more general good orbifolds.

6.5.1 Orbifold differential K-groups

As mentioned above, we will generalize the definition of differential K-theory given in section 5.4 to accommodate the action of a finite group.

First, let us recall some further basic facts about equivariant K-theory. Similarly to ordinary K-theory, a model for the classifying space of the functor K_G^0 is given by the G-algebra of Fredholm operators Fred_G acting on a separable Hilbert space which is a representation space for G in which each irreducible representation occurs with infinite multiplicity [4]. Then there is an isomorphism

$$K_G^0(X) \simeq [X, \text{Fred}_G]_G$$

where $[-, -]_G$ denotes the set of equivalence classes of G-homotopic maps, and the isomorphism is given by taking the index bundle.

There is also a universal space Vect_G^n , equipped with a universal G-bundle \tilde{E}_G^n , such that $[X, \text{Vect}_G^n]_G$ corresponds to the set of isomorphism classes of n -dimensional G-vector bundles over X [67]. These spaces are constructed as follows. Let EG be the category whose objects are the elements of G and with exactly one morphism between each pair of objects. The geometric realization (or nerve) of the set of isomorphism classes in EG is, as a simplicial space, the total space of the classifying principal G-bundle EG . With $\text{Vect}^n(\text{pt})$ the category of n -dimensional complex vector spaces $V \simeq \mathbb{C}^n$, the universal space Vect_G^n is defined to be the geometric realization of the functor category $[EG, \text{Vect}^n(\text{pt})]$. The universal n -dimensional G-vector bundle \tilde{E}_G^n is then defined as

$$\tilde{E}_G^n = \widetilde{\text{Vect}}_G^n \times_{\text{GL}(n, \mathbb{C})} \mathbb{C}^n \longrightarrow \text{Vect}_G^n, \quad (6.5.1)$$

where $\widetilde{\text{Vect}}_G^n$ is the geometric realization of the functor category defined as above but with $\text{Vect}^n(\text{pt})$ replaced with the category consisting of objects V in $\text{Vect}^n(\text{pt})$ together with an oriented basis of V .

We assume sufficient regularity conditions on the infinite-dimensional spaces Fred_G and \tilde{E}_G^n . Since Fred_G and the group completion Vect_G are both classifying spaces for equivariant K-theory, they are G-homotopic and we can thereby choose a cocycle

$$u_G \in Z_G^{\text{even}}(\text{Fred}_G; \mathbb{R})$$

representing the equivariant Chern character of the universal G-bundle (6.5.1). Generally, the group $Z_G^{\text{even}}(X; \mathbb{R})$ is the subgroup of closed cocycles in the complex

$$C_G^{\text{even}}(X; \mathbb{R}) := \bigoplus_{[g] \in G^\vee} C^{\text{even}}(X^g; \mathbb{R})^{Z_G(g)} \quad (6.5.2)$$

which, by the results of section 6.3.2, is a cochain model for the Bredon cohomology group $H_G^{\text{even}-1}(X; \mathbb{R} \otimes \underline{\mathbb{R}}(-))$. The equivariant Chern character is understood to be composed with the delocalizing isomorphism of section 6.3.2. Since it is a natural homomorphism, for any G-bundle $E \rightarrow X$ classified by a G-map $f : X \rightarrow \text{Fred}_G$ one has

$$\text{ch}_X([E]) = [f^* u_G].$$

Definition 6.16. The *orbifold differential K-theory* $\check{K}_G^0(X)$ of the (global) orbifold $[X/G]$ is the group of triples (c, h, ω) , where $c : X \rightarrow \text{Fred}_G$ is a G-map, ω is an element in $\Omega_{G, \text{cl}}^{\text{even}}(X; \mathbb{R})$, and h is an element in $C_G^{\text{even}-1}(X; \mathbb{R})$ satisfying

$$\delta h = \omega - c^* u_G. \quad (6.5.3)$$

Two triples (c_0, h_0, ω_0) and (c_1, h_1, ω_1) are said to be *equivalent* if there exists a triple (c, h, ω) on $X \times [0, 1]$, with the group G acting trivially on the interval $[0, 1]$ and with ω constant along $[0, 1]$, such that

$$(c, h, \omega)|_{t=0} = (c_0, h_0, \omega_0) \quad \text{and} \quad (c, h, \omega)|_{t=1} = (c_1, h_1, \omega_1) .$$

In (6.5.3) the closed orbifold differential form ω is regarded as an orbifold cochain in the complex (6.5.2) by applying the de Rham map componentwise on the fixed point submanifolds X^g , $g \in G$. The higher orbifold differential K-theory groups $\check{K}_G^{-n}(X)$ are defined analogously to those of section 5.4. To confirm that this is a suitable extension of the ordinary differential K-theory of X , we should show that the orbifold differential K-theory groups fit into exact sequences which reduce to those given by (5.4.7) when G is taken to be the trivial group. For this, we define the group

$$A_{K_G}^0(X) := \{(\xi, \omega) \in K_G^0(X) \times \Omega_{G, \text{cl}}^{\text{even}}(X; \mathbb{R}) \mid \text{ch}_X(\xi) = [\omega]_{G\text{-dR}}\} .$$

Theorem 6.17. *The orbifold differential K-theory group $\check{K}_G^0(X)$ satisfies the exact sequence*

$$0 \longrightarrow \frac{H_G^{\text{even}-1}(X; \mathbb{R} \otimes \underline{\mathbb{R}}(-))}{\text{ch}_X(K_G^{-1}(X))} \longrightarrow \check{K}_G^0(X) \longrightarrow A_{K_G}^0(X) \longrightarrow 0 \quad (6.5.4)$$

Proof. Consider the subgroup of $H_G^{\text{even}-1}(X; \mathbb{R} \otimes \underline{\mathbb{R}}(-))$ defined as the image of the equivariant K-theory group $K_G^{-1}(X)$ under the Chern character ch_X . It consists of Bredon cohomology classes of the form $[\tilde{c}^* u_G^{-1}]$, where $\tilde{c} : X \rightarrow \Omega \text{Fred}_G$. There is a surjective map

$$\begin{aligned} f : \check{K}_G^0(X) &\longrightarrow A_{K_G}^0(X) \\ [(c, h, \omega)] &\longmapsto ([c], \omega) \end{aligned}$$

which is a well-defined homomorphism, i.e., it does not depend on the chosen representative of the orbifold differential K-theory class. By definition, the kernel of f consists of triples of the form $(\underline{c}, h, 0)$, where \underline{c} is G -homotopic to the constant (identity) map. We also define the map

$$\begin{aligned} g : H_G^{\text{even}-1}(X; \mathbb{R} \otimes \underline{\mathbb{R}}(-)) &\longrightarrow \check{K}_G^0(X) \\ [h] &\longmapsto [(\underline{c}, h, 0)] , \end{aligned}$$

which is a well-defined homomorphism because the class $[(\underline{c}, h, 0)]$ depends only on the Bredon cohomology class $[h] \in H_G^{\text{even}-1}(X; \mathbb{R} \otimes \underline{\mathbb{R}}(-))$. Then by construction one has $\text{im}(g) = \ker(f)$.

The homomorphism g is not injective. To determine the kernel of g , we use the fact that the zero element in $\check{K}_G^0(X)$ can be represented as

$$[(\underline{c}, 0, 0)] = [(\underline{c}, \pi_* F^* u_G + d_G \sigma, 0)]$$

with $F : X \times S^1 \rightarrow \text{Fred}_G$ and $\sigma \in \Omega_G^{\text{even}-2}(X; \mathbb{R})$. To the map F we can associate a map $\tilde{c} : X \rightarrow \Omega \text{Fred}_G$ such that $F = \text{ev} \circ (\tilde{c} \times \text{id}_{S^1})$. This follows from the isomorphism

$$K_G^{-1}(X) \simeq \ker(i^* : K_G^0(X \times S^1) \rightarrow K_G^0(X))$$

where i is the inclusion $i : X \hookrightarrow X \times \text{pt} \subset X \times S^1$. Now use the fact that at the level of (real) Bredon cohomology one has an equality

$$\pi_* (\tilde{c} \times \text{id}_{S^1})^* = \tilde{c}^* \Pi_*$$

since the projection homomorphisms π_* and Π_* both correspond to integration (slant product) along the S^1 fibre. Then one has the identity

$$[\pi_* F^* u_G] = [\pi_* (\tilde{c} \times \text{id}_{S^1})^* \text{ev}^* u_G] = [\tilde{c}^* \Pi_* \text{ev}^* u_G] = [\tilde{c}^* u_G^{-1}].$$

It follows that $\ker(g)$ is exactly the group $\text{ch}_X(K_G^{-1}(X))$, and putting everything together we arrive at (6.5.4). \square

The torus $H_G^{\text{even}-1}(X; \mathbb{R} \otimes \underline{\mathbb{R}}(-)) / \text{ch}_X(K_G^{-1}(X)) \simeq K_G^{-1}(X) \otimes \mathbb{R} / \mathbb{Z}$ is called the group of *topologically trivial flat fields*.

Consider now the characteristic class map

$$\begin{aligned} f_{cc} : \check{K}_G^0(X) &\longrightarrow K_G^0(X) \\ [(c, h, \omega)] &\longmapsto [c] \end{aligned}$$

and the map

$$\begin{aligned} g_{cc} : \Omega_G^{\text{even}-1}(X; \mathbb{R}) &\longrightarrow \check{K}_G^0(X) \\ h &\longmapsto [(\underline{c}, h, d_G h)]. \end{aligned}$$

Let $\Omega_{K_G}^{\text{even}-1}(X; \mathbb{R})$ be the subgroup of elements in $\Omega_{G, \text{cl}}^{\text{even}-1}(X; \mathbb{R})$ whose Bredon cohomology class lies in $\text{ch}_X(K_G^{-1}(X))$. Then by using arguments similar to those used in arriving at the sequence (6.5.4), one finds the

Corollary 6.18 (Characteristic class exact sequence). *In analogy with the ordinary case, the orbifold differential K-theory group $\check{K}_G^0(X)$ satisfies the exact sequence*

$$0 \longrightarrow \frac{\Omega_G^{\text{even}-1}(X; \mathbb{R})}{\Omega_{K_G}^{\text{even}-1}(X; \mathbb{R})} \longrightarrow \check{K}_G^0(X) \longrightarrow K_G^0(X) \longrightarrow 0. \quad (6.5.5)$$

The quotient space of orbifold differential forms in the exact sequence (6.5.5) is called the group of topologically trivial fields.

Finally, consider the field strength map

$$\begin{aligned} f_{\text{fs}} : \check{K}_G^0(X) &\longrightarrow \Omega_{G, \text{cl}}^{\text{even}}(X; \mathbb{R}) \\ [(c, h, \omega)] &\longmapsto \omega. \end{aligned} \quad (6.5.6)$$

The kernel of the homomorphism f_{fs} is the group which classifies the flat fields (which are not necessarily topologically trivial) and is denoted $K_G^{-1}(X; \mathbb{R}/\mathbb{Z})$. This group will be described in more detail in the next section, where we shall also conjecture an essentially purely algebraic definition of $K_G^{-1}(X; \mathbb{R}/\mathbb{Z})$ which explains the notation. In any case, we have the

Corollary 6.19 (Field strength exact sequence). *The orbifold differential K-theory group $\check{K}_G^0(X)$ satisfies the exact sequence*

$$0 \longrightarrow K_G^{-1}(X; \mathbb{R}/\mathbb{Z}) \longrightarrow \check{K}_G^0(X) \longrightarrow \Omega_{K_G}^{\text{even}}(X; \mathbb{R}) \longrightarrow 0 \quad (6.5.7)$$

Higher orbifold differential K-theory groups satisfy analogous exact sequences, with the appropriate degree shifts throughout. It is clear from our definition that one recovers the ordinary differential K-theory groups in the case of the trivial group $G = e$, and in this sense our orbifold differential K-theory may be regarded as its equivariant generalization. At this point we hasten to add that, although our groups are well-defined and satisfy desired properties which are useful for physical applications such as the various exact sequences above, we have not showed that our orbifold theory generalizes *all* the properties of an ordinary differential cohomology theory. For instance, it would be interesting to define a ring structure and an integration on $\check{K}_G^*(X)$. We have not developed these constructions in this thesis.

6.5.2 Flux quantization of orbifold Ramond-Ramond fields

In this section we will argue the flux quantization condition for Ramond-Ramond fieldstrengths on orbifolds of Type II superstring theory with vanishing H -flux, showing that it is dictated by equivariant K-theory via the complex equivariant Chern character. Essentially, we will closely follow the Moore-Witten argument discussed in section 5.3.

Suppose that our spacetime X is a non-compact G -manifold. Suppose further that there are D-branes present in Type II superstring theory on X/G . Their Ramond-Ramond charges are classified by the equivariant K-theory $K_{G,\text{cpt}}^i(X)$ with compact support, where $i = 0$ in Type IIB theory and $i = -1$ in Type IIA theory.

We require that the brane be a source for the equation of motion for the total Ramond-Ramond field strength ω . This means that it creates a Ramond-Ramond current J . If we require that the worldvolume W be compact in equivariant K-cycles $(W, E, f) \in D^G(X)$, then J is supported in the interior $\overset{\circ}{X}$ of X . Let X_∞ be the “boundary of X at infinity”, which we assume is preserved by the action of G . Then $K_{G,\text{cpt}}^*(X) \simeq K_G^*(X, X_\infty)$. Since J is trivialized by ω in $\overset{\circ}{X}$, the D-brane charge lives in the kernel of the natural forgetful homomorphism

$$f^* : K_{G,\text{cpt}}^*(X) \longrightarrow K_G^*(X) \quad (6.5.8)$$

induced by the inclusion $(X, \emptyset) \hookrightarrow (X, X_\infty)$. We denote by $i : X_\infty \hookrightarrow X$ the canonical inclusion.

The long exact sequence for the pair (X, X_∞) in equivariant K-theory truncates, by Bott periodicity, to the six-term exact sequence

$$\begin{array}{ccccc} K_G^{-1}(X_\infty) & \longrightarrow & K_G^0(X, X_\infty) & \xrightarrow{f^0} & K_G^0(X) \\ & & \uparrow i^* & & \downarrow i^* \\ K_G^{-1}(X) & \xleftarrow{f^{-1}} & K_G^{-1}(X, X_\infty) & \longleftarrow & K_G^0(X_\infty) . \end{array}$$

It follows that the charge groups are given by

$$\ker(f^0) \simeq \frac{K_G^{-1}(X_\infty)}{i^*(K_G^{-1}(X))} \quad \text{and} \quad \ker(f^{-1}) \simeq \frac{K_G^0(X_\infty)}{i^*(K_G^0(X))} .$$

This formula means that the group of Type IIB (resp. Type IIA) brane charges is measured by the group $K_G^{-1}(X_\infty)$ (resp. $K_G^0(X_\infty)$) of “orbifold Ramond-Ramond fluxes at infinity” which cannot be extended to all of spacetime X . We may then interpret, for arbitrary spacetimes X , the group $K_G^{-1}(X)$ (resp. $K_G^0(X)$) as the group classifying Ramond-Ramond fields in the orbifold X/G in absence of branes in Type IIB (resp. Type IIA) String theory.

The Ramond-Ramond current can be described explicitly by using the delocalization of Bredon cohomology over \mathbb{C} . The Wess-Zumino pairing (6.3.17) between a topologically trivial, complex Ramond-Ramond potential and a D-brane represented by an equivariant K-cycle $(W, E, f) \in D^G(X)$ contributes a source term to the Ramond-Ramond equations of motion, which is the class

$$[Q(W, E, f)] \in H_G^{\text{even}}(X; \mathbb{C} \otimes \underline{\mathbb{R}}(-))$$

represented by the pushforward

$$Q(W, E, f) = f_!^{\text{Hg}}(\text{ch}^{\mathbb{C}}(E) \wedge_G \mathcal{R}(W, f)) .$$

We now use the Riemann-Roch formula (6.4.5) and the fact that $f_!^{\text{Hg}} \circ f^* = \text{id}_{H_G^*(X; \mathbb{C}(-))}$. Using the explicit expression for the curvature form and the definition for $\Lambda_G(X)$ given in [91], we can then rewrite this class as

$$Q(W, E, f) = \text{ch}^{\mathbb{C}}(f_!^{\text{Kg}}(E)) \wedge_G \sqrt{\text{Todd}_G(T_X) \wedge_G \Lambda_G(X)} . \quad (6.5.9)$$

This is the complex Bredon cohomology class of the Ramond-Ramond current J created by the D-brane (W, E, f) . In the case $G = e$, the expression (6.5.9) reduces to the standard class of the current for Ramond-Ramond fields in Type II superstring theory on X [29, 71, 74, 75]. We can then formally conclude, in analogy with the non-equivariant case, that the complex Bredon cohomology class associated to a class $\xi \in K_G^*(X) \otimes \mathbb{C}$ representing a Ramond-Ramond field is assigned by the equivariant Chern character, and that the total Ramond-Ramond fieldstrength ω associated to ξ satisfies

$$\frac{[\omega(\xi)]}{2\pi [\sqrt{\text{Todd}_G(T_X) \wedge_G \Lambda_G(X)}]} = [\text{ch}^{\mathbb{C}}(\xi)] . \quad (6.5.10)$$

The above expression suggests that the orbifold differential K-theory developed in the previous section is a suitable candidate to describe Ramond-Ramond fields with Dirac quantization condition, at least in the complex case. Moreover, we should stress that this analysis of the delocalized theory assumes the strong conditions spelled out in section 6.4, which require a deep geometrical compatibility of the equivariant K-cycle (W, E, f) with the orbifold structure of $[X/G]$. The example of the linear orbifolds considered in section 6.3.1 is simple enough to satisfy these conditions. It would be very interesting to find a geometrically non-trivial explicit example to test these requirements on. However, the linear orbifolds case is very useful to understand

certain aspects of the orbifold differential K-theory groups.

Since the \mathbb{C} -linear G -module X is equivariantly contractible, one has

$$H_G^{\text{odd}}(X; \mathbb{R} \otimes \underline{\mathbb{R}}(-)) = 0$$

and $K_G^0(X) = R(G)$. From Theorem 6.17 it then follows that

$$\check{K}_G^0(X) \simeq A_{K_G}^0(X) \simeq \{(\gamma, \omega) \in R(G) \times \Omega_{G,\text{cl}}^{\text{even}}(X; \mathbb{R}) \mid \text{ch}_{\text{pt}}(\gamma) = [\omega]_{G-\text{dR}}\}.$$

Since the equivariant Chern character $\text{ch}_{G/H} : R(H) \rightarrow R(H)$ for $H \leq G$ is the identity map, the setwise fibre product truncates to the lattice of quantized orbifold differential forms and one has

$$\check{K}_G^0(V) = \Omega_{K_G}^{\text{even}}(V; \mathbb{R}). \quad (6.5.11)$$

This is the group of Type IIA Ramond-Ramond fieldstrengths on X . It naturally contains those fields which trivialize the Ramond-Ramond currents sourced by the stable fractional D0-branes of the Type IIA theory, corresponding to characteristic classes $[c]$ in the representation ring $R(G)$.

This can be explicitly described as an extension of the equivariant K-theory of X by the group of topologically trivial Ramond-Ramond fields C of odd degree, as implied by Corollary 6.18. Since X is connected and G -contractible, one has $\Omega_{G,\text{cl}}^0(X; \mathbb{R}) = \mathbb{R} \otimes R(G)$ and the group (6.5.11) has a natural splitting

$$\check{K}_G^0(X) = R(G) \oplus \left(\bigoplus_{k=1}^d \Omega_{G,\text{cl}}^{2k}(X; \mathbb{R}) \right). \quad (6.5.12)$$

Any closed orbifold form ω on X of positive degree is exact, $\omega = d_G C$, with the gauge invariance $C \mapsto C + d_G \xi$. It follows that there is a natural map

$$\bigoplus_{k=1}^d \Omega_{G,\text{cl}}^{2k}(X; \mathbb{R}) \longrightarrow \frac{\Omega_G^{\text{even}-1}(X; \mathbb{R})}{\Omega_{K_G}^{\text{even}-1}(X; \mathbb{R})}$$

which associates to the field strength ω the corresponding globally well-defined Ramond-Ramond potential C .

On the other hand, the orbifold differential K-theory group $\check{K}_G^{-1}(X)$ of Type IIB Ramond-Ramond fields on X can be computed by using the characteristic class exact sequence (6.5.5) with degree shifted by -1 . Using $K_G^{-1}(X) = 0$, one finds

$$\check{K}_G^{-1}(X) = \frac{\Omega_G^{\text{even}}(X; \mathbb{R})}{\Omega_{K_G}^{\text{even}}(X; \mathbb{R})}. \quad (6.5.13)$$

This result reflects the fact that the Type IIB theory has no stable fractional D0-branes. Hence there is no extension and the Ramond-Ramond fields are induced solely by the closed string background. Their field strengths $\omega = d_G C$ are determined entirely by the potentials C , which are globally defined differential forms of even degree.

Setting $X = \text{pt}$, we obtain the orbifold differential K-theory of the point, which are given by

$$\check{K}_G^0(\text{pt}) \simeq \mathbb{R}(G)$$

and

$$\check{K}_G^{-1}(\text{pt}) \simeq \mathbb{R}(G) \otimes \mathbb{R}/\mathbb{Z}$$

For $G = e$, these groups reduce to the usual differential K-theory groups of the point.

In the previous section we defined the group of flat Ramond-Ramond fields on a general orbifold $[X/G]$ as the subgroup of orbifold differential K-theory with vanishing curvature. In the following we will conjecture a very natural algebraic definition of these groups which ties them somewhat more directly to equivariant K-theory groups. To motivate this conjecture, we first compute the groups $K_G^*(X; \mathbb{R}/\mathbb{Z})$ for the linear orbifolds above, wherein the associated differential K-theory groups were determined explicitly. Using the field strength exact sequence (6.5.7), by definition one has

$$K_G^{-1}(X; \mathbb{R}/\mathbb{Z}) \simeq \ker(f_{\text{fs}} : \check{K}_G^0(X) \rightarrow \Omega_{K_G}^{\text{even}}(X; \mathbb{R}))$$

which from the natural isomorphism (6.5.11) trivially gives

$$K_G^{-1}(X; \mathbb{R}/\mathbb{Z}) = 0 . \quad (6.5.14)$$

Similarly, using $K_G^{-1}(X) = 0$ one has

$$K_G^0(X; \mathbb{R}/\mathbb{Z}) \simeq \ker(f_{\text{fs}} : \check{K}_G^{-1}(X) \rightarrow \Omega_{G, \text{cl}}^{\text{odd}}(X; \mathbb{R})) .$$

Using the natural isomorphism (6.5.13), the field strength map is

$$f_{\text{fs}}([C]) = d_G C \quad \text{for } C \in \Omega_G^{\text{even}}(X; \mathbb{R}) ,$$

giving

$$K_G^0(X; \mathbb{R}/\mathbb{Z}) \simeq \frac{\Omega_{G, \text{cl}}^{\text{even}}(X; \mathbb{R})}{\Omega_{K_G}^{\text{even}}(X; \mathbb{R})} .$$

Similarly to (6.5.12), there is a natural splitting of the vector space of closed orbifold differential forms given by

$$\Omega_{G, \text{cl}}^{\text{even}}(X; \mathbb{R}) = (\mathbb{R}(G) \otimes \mathbb{R}) \oplus \left(\bigoplus_{k=1}^d \Omega_{G, \text{cl}}^{2k}(X; \mathbb{R}) \right)$$

and we arrive finally at

$$K_G^0(X; \mathbb{R}/\mathbb{Z}) = R(G) \otimes \mathbb{R}/\mathbb{Z} . \quad (6.5.15)$$

These results of course simply follow from the fact that X is G -contractible, so that every d_G -closed Ramond-Ramond field is trivial, except in degree zero where the gauge equivalence classes are naturally parametrized by the twisted sectors of the String theory in (6.5.15). Note that both groups of flat fields (6.5.14) and (6.5.15) are unchanged by (equivariant) contraction of the G -module X to a point, as an analogous (but simpler) calculation shows. This suggests that the groups $K_G^*(X; \mathbb{R}/\mathbb{Z})$ have at least some G -homotopy invariance properties, unlike the differential K_G -theory groups. This motivates the following conjectural algebraic framework for describing these groups.

We will propose that the group $K_G^*(X; \mathbb{R}/\mathbb{Z})$ is an *extension* of the torus of topologically trivial flat orbifold Ramond-Ramond fields by the torsion elements in $K_G^{*+1}(X)$, as they have vanishing image under the equivariant Chern character ch_X . The resulting group may be called the “equivariant K-theory with coefficients in \mathbb{R}/\mathbb{Z} ”. The short exact sequence of coefficient groups

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0$$

induces a long exact sequence of equivariant K-theory groups which, by Bott periodicity, truncates to the six-term exact sequence

$$\begin{array}{ccccc} K_G^0(X) & \longrightarrow & K_G^0(X; \mathbb{R}) & \longrightarrow & K_G^0(X; \mathbb{R}/\mathbb{Z}) \\ \beta \uparrow & & & & \downarrow \beta \\ K_G^{-1}(X; \mathbb{R}/\mathbb{Z}) & \longleftarrow & K_G^{-1}(X; \mathbb{R}) & \longleftarrow & K_G^{-1}(X) . \end{array} \quad (6.5.16)$$

The connecting homomorphism β is a suitable variant of the usual Bockstein homomorphism. We assume that the equivariant K-theory with real coefficients is defined simply by the \mathbb{Z}_2 -graded ring

$$K_G^*(X; \mathbb{R}) = K_G^*(X) \otimes \mathbb{R} \simeq H_G^*(X; \mathbb{R} \otimes \underline{\mathbb{R}}(-)) ,$$

where we have used Theorem 6.12. The maps to real K-theory in (6.5.16) may then be identified with the equivariant Chern character ch_X , whose image is a full lattice in the Bredon cohomology group $H_G^*(X; \mathbb{R} \otimes \underline{\mathbb{R}}(-))$. Then the abelian group $K_G^*(X; \mathbb{R}/\mathbb{Z})$ sits in the exact sequence

$$0 \longrightarrow K_G^*(X) \otimes \mathbb{R}/\mathbb{Z} \longrightarrow K_G^*(X; \mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} \text{Tor}(K_G^{*+1}(X)) \longrightarrow 0 . \quad (6.5.17)$$

When $G = e$, (6.5.16) is the usual Bockstein exact sequence for K-theory. In this case, an explicit geometric realization of the groups $K^*(X; \mathbb{R}/\mathbb{Z})$ in terms of bundles with connection has been given by Lott [64]. Moreover, in [54] a geometric construction of the map $K^{-1}(X; \mathbb{R}/\mathbb{Z}) \rightarrow \check{K}^0(X)$ in the field strength exact sequence is given. Unfortunately, no such geometrical description is immediately available for our equivariant differential K-theory, due to the lack of a Chern-Weil theory for the homotopy theoretic equivariant Chern character. Our conjectural definition (6.5.17) is satisfied by the linear orbifold groups (6.5.14) and (6.5.15).

In [32] a very different definition of the groups $K_G^*(X; \mathbb{R}/\mathbb{Z})$ is given, by defining both equivariant K-theory and cohomology using the Borel construction of Example 6.4. Then the Bockstein exact sequence (6.5.16) is written for the ordinary K-theory groups of the homotopy quotient $X_G = EG \times_G X$. While these groups reduce, like ours, to the usual K-theory groups of flat fields when $G = e$, they do not obey the exact sequence (6.5.17). The reason is that the equivariant Chern character used is *not* an isomorphism over the reals, as explained in section 6.2. Moreover, an associated differential K-theory construction would directly involve differential forms on the infinite-dimensional space X_G which is only homotopic to the finite-dimensional CW-complex X/G . The physical interpretation of such fields is not clear. Even in the simple case of the linear orbifolds X studied above, this description predicts an infinite set of equivariant fluxes of arbitrarily high dimension on the infinite-dimensional classifying space BG , and one must perform some non-canonical quotients in order to try to isolate the physical fluxes. In contrast, with our constructions the relation between orbifold flux groups and Bredon cohomology is much more natural, and it involves Ramond-Ramond fields defined on submanifolds of the covering space X .

6.6 K-homology and flat fields in Type II String theory

In this final section, we will briefly comment on how the flat Ramond-Ramond fields can in principle couple to the wrapped D-branes defined in section 4.8. The analysis will be restricted to type IIA String theory.

As we have seen in the previous sections, the Ramond-Ramond fields in ordinary type IIA String theory are described by differential K theory. Hence, the group of flat Ramond-Ramond fields is given by $K^{-1}(X; \mathbb{R}/\mathbb{Z})$. Consider the short exact sequence

of coefficient groups given by

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 1 ,$$

It induces the corresponding long exact sequence

$$\dots \longrightarrow K^i(X) \longrightarrow K^i(X) \otimes \mathbb{R} \longrightarrow K^i(X; \mathbb{R}/\mathbb{Z}) \longrightarrow K^{i+1}(X) \longrightarrow \dots .$$

By truncating the above long exact sequence, we have

$$0 \longrightarrow K^{-1}(X) \otimes \mathbb{R}/\mathbb{Z} \longrightarrow K^{-1}(X; \mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} \text{Tor}(K^0(X)) \longrightarrow 0 \quad (6.6.1)$$

where β is the Bockstein homomorphism. Thus the identity component of the circle coefficient K-theory group is the torus $K^{-1}(X; \mathbb{R}/\mathbb{Z})$. Suppose now that $K^{-1}(X)$ is pure torsion. In this case, $K^{-1}(X; \mathbb{R}/\mathbb{Z}) \simeq \text{Tor}(K^0(X))$, and the corresponding flat Ramond-Ramond fields can be represented by virtual vector bundles over X . A torsion Ramond-Ramond flat field $\xi \in K^0(X)$ gives an additional phase factor to a D-brane in the String theory path integral [32]. Generally, the origin of these phases can be understood from the topological classification of the physical coupling between Type I D-branes and Ramond-Ramond fields.

Let W be a compact spin^c submanifold of X of dimension $p+1$, and let the spacetime manifold X be a spherical spin^c fibration $\pi : X \rightarrow W$ such that $X/W \simeq S^{9-p}$.

The group of flat Ramond-Ramond fields is given by

$$K^{-1}(X; \mathbb{R}/\mathbb{Z}) = \text{Hom}(K_{-1}^t(X), \mathbb{R}/\mathbb{Z}) \simeq \text{Hom}(K_{p-10}^t(W), \mathbb{R}/\mathbb{Z})$$

where we have use the following exact sequence [95, 80]

$$0 \longrightarrow \text{Ext}(K_{i-1}^t(X), G) \longrightarrow K^i(X; G) \longrightarrow \text{Hom}(K_i^t(X), G) \longrightarrow 0$$

for $G = \mathbb{R}/\mathbb{Z}$, and the Thom isomorphism.

Using Bott periodicity, we have finally

$$K^{-1}(X; \mathbb{R}/\mathbb{Z}) \simeq \text{Hom}(K_{p+2}^t(W), \mathbb{R}/\mathbb{Z}) . \quad (6.6.2)$$

The K-homology group $K_{p+2}^t(W)$ consists of wrapped D-branes $[M, E, \phi]$ with the properties $\dim M = p+2$ and $\phi(M) \subset W$. The dimension shift is related to the topological anomaly in the worldvolume fermion path integral [74], as the following argument seems to suggest.

Consider a one-parameter family of $p+1$ -dimensional brane worldvolumes specified by a circle bundle $U \rightarrow W$ whose total space U is a $p+2$ -dimensional submanifold

of spacetime X with the topology of $W \times S^1$. Complex vector bundles E_g of rank n over generic fibres $U/W \simeq S^1$ are determined by elements $g \in U(n)$ by the clutching construction. Thus the family of self-adjoint Atiyah-Singer operators $\mathfrak{D}_{E_g}^{S^1}$ determined by (4.9.1) is parametrized by the group $U(n)$. The anomaly [74] arises as the determinant line bundle of this family, which is essentially defined as the highest exterior power of the kernel of the family. This defines a non-trivial line bundle on the group $U(n)$ called the *Pfaffian line bundle*, which has the property that its lift to $\text{Spin}(n)$ is the trivial line bundle. One can also construct a connection and holonomy of the Pfaffian line bundle [45]. The manifold U is wrapped by D-branes in $K_{p+2}^t(U)$. One can now restrict to the subgroup $K_{p+2}^t(W) \subset K_{p+2}^t(U)$ by keeping only those D-branes which are wrapped on the embedding $W \hookrightarrow U$ by the zero section of $U \rightarrow W$. The isomorphism (6.6.2) reflects the fact that the topological anomaly could in principle be cancelled by coupling D-branes to the Ramond-Ramond fields through a phase factor.

Appendix A

Linear algebra in Functor categories

In this appendix we will summarize some notions about algebra in functor categories that were used in the main text of the paper. They generalize the more commonly used concepts for modules over a ring. For further details see [92].

Let R be a commutative ring, and denote the category of (left) R -modules by $R\text{-Mod}$. Let Γ be a *small* category, *i.e.*, its class of objects $\text{Obj}(\Gamma)$ is a set. If \mathcal{C} is another category, then one denotes by

$$[\Gamma, \mathcal{C}]$$

the *functor category* of (covariant) functors $\Gamma \rightarrow \mathcal{C}$. The objects of $[\Gamma, \mathcal{C}]$ are (covariant) functors $\phi : \Gamma \rightarrow \mathcal{C}$ and a morphism from ϕ_1 to ϕ_2 is a natural transformation $\alpha : \phi_1 \rightarrow \phi_2$ between functors.

In particular, in the main text we used the functor category

$$R\Gamma\text{-Mod} := [\Gamma, R\text{-Mod}]$$

whose objects are called *left $R\Gamma$ -modules*. If one denotes with Γ^{op} the dual category to Γ , then there is also the functor category

$$\text{Mod-}R\Gamma := [\Gamma^{\text{op}}, R\text{-Mod}]$$

of contravariant functors $\Gamma \rightarrow R\text{-Mod}$, whose objects are called *right $R\Gamma$ -modules*. As an example, let G be a discrete group regarded as a category with a single object and a morphism for each element of G . A covariant functor $G \rightarrow R\text{-Mod}$ is then the same thing as a left module over the group ring $R[G]$ of G over R .

As the name itself suggests, all standard definitions from the linear algebra of modules have extensions to this more general setting. For instance, the notions of *submodule*, *kernel*, *cokernel*, *direct sum*, *coproduct*, *etc.* can be naturally defined objectwise. If M and N are $R\Gamma$ -modules, then $\text{Hom}_{R\Gamma}(M, N)$ is the R -module of all natural transformations $M \rightarrow N$. This notation should not be confused with the one used for the set of all morphisms between two objects in Γ , and usually it is clear from the context.

If M is a right $R\Gamma$ -module and N is a left $R\Gamma$ -module, then one can define their categorical *tensor product*

$$M \otimes_{R\Gamma} N$$

in the following way. It is the R -module given by first forming the direct sum

$$F = \bigoplus_{\lambda \in \text{Obj}(\Gamma)} M(\lambda) \otimes_R N(\lambda)$$

and then quotienting F by the R -submodule generated by all relations of the form

$$f^*(m) \otimes n - m \otimes f_*(n) = 0 ,$$

where $(f : \lambda \rightarrow \rho) \in \text{Mor}(\Gamma)$, $m \in M(\rho)$, $n \in N(\lambda)$ and $f^*(m) = M(f)(m)$, $f_*(n) = N(f)(n)$. This tensor product commutes with coproducts. If M and N are functors from Γ to the category of vector spaces over a field \mathbb{K} , then their tensor product is naturally equipped with the structure of a vector space over \mathbb{K} . When Γ is the orbit category $\text{Or}(G)$ and $R = \mathbb{Z}$, the tensor product has precise limiting cases. For an arbitrary contravariant module M and the constant covariant module N , the categorical product $M \otimes_{\mathbb{Z}\text{Or}(G)} N$ is the tensor product of the right $\mathbb{Z}[G]$ -module $M(G/e)$ with the constant left $\mathbb{Z}[G]$ -module $N(G/e)$, $M(G/e) \otimes_{\mathbb{Z}[G]} N(G/e)$. On the other hand, if the contravariant module M is constant and the covariant module N is arbitrary, then $M \otimes_{\mathbb{Z}\text{Or}(G)} N$ is just $N(G/G)$.

Appendix B

Clifford algebras and Spin manifolds

In this appendix we will briefly recall some basic properties of Clifford algebras and Spin manifolds used throughout this thesis. We direct the reader to [5, 63] for an extensive treatment of these topics.

Let V be a real vector space, and let $q : V \rightarrow \mathbb{R}$ be a quadratic form. Consider the tensor algebra of V

$$\mathcal{F}(V) := \sum_{r=0}^{\infty} \bigotimes^r V$$

and denote with $I_q(V)$ the two-sided ideal in $\mathcal{F}(V)$ generated by all elements of the form

$v \otimes v + q(v)1$ for $v \in V$. Then the *Clifford algebra* $Cl(V; q)$ associated to V and q is the associative algebra with unit defined as

$$Cl(V; q) := \mathcal{F}(V)/I_q(V)$$

The algebra $Cl(V; q)$ is generated by the vector space $V \subset Cl(V; q)$ subject to the relations

$$v \cdot v = -q(v)1$$

which give a “universal” characterization of the algebra.

Given two vector spaces V, V' , equipped with quadratic forms q, q' , respectively, any linear map $f : V \rightarrow V'$ preserving the quadratic forms induces a homomorphism $\tilde{f} : Cl(V; q) \rightarrow Cl(V'; q')$. An important example is given by the homomorphism $\tilde{\alpha}$ induced by the map $\alpha(v) = -v$ on V . Since $\alpha^2 = \text{id}$, there is a decomposition

$$Cl(V; q) = Cl^0(V; q) \oplus Cl^1(V; q)$$

where $Cl^i(V; q) := \{\varphi \in Cl(V; q) : \alpha(\varphi) = (-1)^i \varphi\}$. The vector space $Cl^0(V; q)$ is a subalgebra of $Cl(V; q)$, and it is called the *even part*, while the subspace $Cl^1(V; q)$ is called the *odd part* of $Cl(V; q)$.

Given a Clifford algebra $Cl(V; q)$, we can define the *multiplicative group of units* as

$$Cl^*(V; q) := \{\varphi \in Cl(V; q) : \exists \varphi^{-1} \in Cl(V; q) \text{ such that } \varphi^{-1} \cdot \varphi = \varphi \cdot \varphi^{-1} = 1\}$$

When $\dim V = n < \infty$, $Cl^*(V; q)$ is a Lie group of dimension 2^n . Consider the subgroup $P(V; q) \subset Cl^*(V; q)$ generated by the elements $v \in V$ with $q(v) \neq 0$.

Then the *Pin group* associated to the pair $(V; q)$ is the subgroup $\text{Pin}(V; q)$ of $P(V; q)$ generated by elements $v \in V$ with $q(v) = \pm 1$. The associated *Spin group* of $(V; q)$ is defined as

$$\text{Spin}(V; q) := \text{Pin}(V; q) \cap \text{Cl}^0(V; q)$$

We will now restrict to the case in which $V = \mathbb{R}^n$, and

$$q(x) := x_1^2 + x_2^2 + \cdots + x_n^2$$

is the quadratic form induced by the usual scalar product. Denote with Cl_n the Clifford algebra $\text{Cl}(\mathbb{R}^n; q)$. If e_1, e_2, \dots, e_n is any orthonormal basis of \mathbb{R}^n , then Cl_n is generated as an algebra by e_1, e_2, \dots, e_n and 1 subject to the relations

$$e_i \cdot e_j + e_j \cdot e_i = -2\delta_{ij} 1$$

It is straightforward to see that $\text{Cl}_1 \simeq \mathbb{C}$, and $\text{Cl}_2 \simeq \mathbb{H}$. Moreover, one can prove that $\dim_{\mathbb{R}}(\text{Cl}_n) = 2^n$. The decomposition of Cl_n in even and odd part induces the following isomorphism

$$\text{Cl}_n \simeq \text{Cl}_{n+1}^0$$

An important result concerning the Clifford algebra Cl_n is the following. There is a canonical isomorphism $\text{Cl}_n \simeq \Lambda^* \mathbb{R}^n$, according to which the Clifford multiplication between $v \in \mathbb{R}^n$ and any $\varphi \in \text{Cl}_n$ can be written as

$$v \cdot \varphi \simeq v \wedge \varphi - v \lrcorner \varphi$$

where we have identified \mathbb{R}^n with its dual via the scalar product, and \lrcorner denotes the contraction of the vector v with an element of $\Lambda^* \mathbb{R}^n$. The Clifford algebras can be described as matrix algebras over \mathbb{R}, \mathbb{C} , or \mathbb{H} . Indeed, denote with $\mathbb{C}\text{Cl}_n$ the complexification $\text{Cl}_n \otimes_{\mathbb{R}} \mathbb{C}$ of Cl_n . Then for all $n \leq 0$ we have the *periodicity isomorphisms*

$$\begin{aligned} \text{Cl}_{n+8} &\simeq \text{Cl}_n \otimes \text{Cl}_8 \\ \mathbb{C}\text{Cl}_{n+2} &\simeq \mathbb{C}\text{Cl}_n \otimes_{\mathbb{C}} \mathbb{C}\text{Cl}_2 \end{aligned}$$

The above isomorphisms allow to deduce all the algebras Cl_n and $\mathbb{C}\text{Cl}_n$ from the following table

	1	2	3	4	5	6	7	8
Cl_n	\mathbb{C}	\mathbb{H}	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
$\mathbb{C}\text{Cl}_n$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$	$\mathbb{C}(16)$

where $K(n)$ denotes the algebra of $n \times n$ matrices over the field K , with $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

The above table also dictates the theory of representations of Clifford algebras. Indeed, if we see $K(n)$ as an algebra over \mathbb{R} , the natural representation of $K(n)$ on the vector space K^n is the only irreducible real representation of $K(n)$ up to isomorphism, while the algebras $K(n) \oplus K(n)$ have exactly two equivalence classes of irreducible real representations [63].

Given a module An important role is played by the Spin groups associated to \mathbb{R}^n with the Euclidean quadratic form. Indeed, if we denote with Spin_n the group $\text{Spin}(\mathbb{R}^n; q)$, we have the following exact sequence

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}_n \xrightarrow{\xi_0} \text{SO}_n \rightarrow 0$$

for all $n \geq 3$. In particular, the map ξ_0 denotes the universal covering homomorphism of SO_n . The representation theory of Clifford algebras can be used to construct representations of $\text{Spin}_n \subset \mathbb{C}\ell_n^0$ which are not trivial on the element -1, hence they do not arise from representation of the orthogonal group SO_n . Indeed, we can define the *real spinor representation* of Spin_n as the homomorphism

$$\Delta_n : \text{Spin}_n \rightarrow \text{GL}(\mathbb{S})$$

induced by restricting an irreducible real representation $\mathbb{C}\ell_n \rightarrow \text{Hom}_{\mathbb{R}}(\mathbb{S}, \mathbb{S})$ to Spin_n .

We can also define the *complex spinor representation* of Spin_n as the homomorphism

$$\Delta_n^{\mathbb{C}} : \text{Spin}_n \rightarrow \text{GL}_{\mathbb{C}}(\mathbb{S})$$

induced by restricting an irreducible complex representation $\mathbb{C}\ell_n \rightarrow \text{Hom}_{\mathbb{C}}(\mathbb{S}, \mathbb{S})$ to $\text{Spin}_n \subset \mathbb{C}\ell_n^0 \subset \mathbb{C}\ell_n$. In particular, for n odd, the complex spinor representation $\Delta_n^{\mathbb{C}}$ is irreducible, and independent of which irreducible representation of $\mathbb{C}\ell_n$ is used. When $n = 2m$, there is a decomposition

$$\Delta_{2m}^{\mathbb{C}} = \Delta_{2m}^{\mathbb{C}^+} \oplus \Delta_{2m}^{\mathbb{C}^-}$$

into a direct sum of irreducible complex representations of Spin_n . The representation $\Delta_{2m}^{\mathbb{C}^{\pm}}$ is given by the composition of $\Delta_{2m}^{\mathbb{C}}$ with the projection $1 \pm \omega_{\mathbb{C}}$, where

$$\omega_{\mathbb{C}} = i^m e_1 \cdot e_2 \cdots e_{2m}$$

is the complex volume element. Notice that $\Delta_{2m}^{\mathbb{C}^{\pm}}$ is *not* a representation of $\mathbb{C}\ell_n$. Similar results hold for the real spinor representation.

Finally, a \mathbb{Z}_2 -graded module for $\mathbb{C}\ell_n$ is a module W with a decomposition $W = W^0 \oplus W^1$ such that

$$\mathbb{C}\ell_n^i W^j \subseteq W^{(i+j) \pmod{2}}$$

Given a \mathbb{Z}_2 -graded module W over Cl_n , the even part W^0 is a module over Cl_n^0 . Conversely, given a module W^0 over Cl_n^0 , we can form the \mathbb{Z}_2 -graded module

$$W := Cl_n \otimes_{Cl_n^0} W^0$$

Hence there is an equivalence between the category of \mathbb{Z}_2 -graded modules over Cl_n and the category of ungraded modules over Cl_{n-1} .

Let E be an oriented n -dimensional Riemannian vector bundle over a manifold X , and suppose $n \geq 3$. Let $P_{SO}(E)$ denote the orthonormal frame bundle of E . Then a *spin structure* on E is a principal $Spin_n$ -bundle $P_{Spin}(E)$ together with a double covering

$$\xi : P_{Spin}(E) \rightarrow P_{SO}(E)$$

such that $\xi(p \cdot g) = \xi(p) \cdot \xi_0(g)$ for all $p \in P_{Spin}(E)$ and all $g \in Spin_n$.

For $n = 2$, a Spin structure on E is defined analogously, with $Spin_2$ replaced by SO_2 and $\xi_0 : SO_2 \rightarrow SO_2$ the connected double covering.

For $n = 1$, $P_{SO}(E) \simeq X$ and a spin structure is simply defined to be a double covering of X .

The existence of a spin structure on a vector bundle E is dictated by the vanishing of the first and second Stiefel-Whitney class $w_1(E)$ and $w_2(E)$, respectively.

A *spin manifold* is an oriented Riemannian manifold X with a spin structure on its tangent bundle. Moreover, the inequivalent spin structures on X are in one-to-one correspondence with elements of $H^1(X; \mathbb{Z}_2)$. Given an oriented Riemannian vector bundle, we can construct the associated *Clifford bundle* defined as

$$Cl(E) := P_{SO}(E) \times_{cl(\rho)} Cl_n$$

where $cl(\rho) : SO_n \rightarrow \text{Aut}(Cl_n)$ is induced by lifting orthogonal transformations of \mathbb{R}^n to the Clifford algebra Cl_n . The Clifford bundle $Cl(E)$ is a bundle of Clifford algebras over X , and the fibrewise multiplication in $Cl(E)$ gives an algebra structure to the space of sections of $Cl(E)$. Hence, all the notions regarding Clifford algebras carry over to Clifford bundles. We can then look for bundles of irreducible modules over the Clifford bundle $Cl(E)$. These bundles can be constructed if E has a spin structure. Indeed, a *real spinor bundle* of E is a bundle of the form

$$S(E) := P_{Spin}(E) \times_{\mu} W$$

where W is a left module for Cl_n , and where $Spin_n$ acts on W by left multiplication by elements of $Spin_n \subset Cl_n$. Analogously, a *complex spinor bundle* of E is given by

$$S_{\mathbb{C}}(E) := P_{Spin}(E) \times_{\mu} W^{\mathbb{C}}$$

where $W^{\mathbb{C}}$ is a complex left module for $\mathbb{C}\ell_n$. One can easily prove that the sections of the spinor bundle are a module over the sections of the Clifford bundle. A spinor bundle $S(E)$ is called *irreducible* if the left module $W(W^{\mathbb{C}})$ is the irreducible real(complex) spinor representation of Spin_n . In particular, when the vector bundle E has rank $n = 2m$, the complex spinor bundle $S_{\mathbb{C}}(E)$ associated to the irreducible representation of $\mathbb{C}\ell_{2m}$ decomposes as

$$S_{\mathbb{C}}(E) = S_{\mathbb{C}}^{+}(E) \oplus S_{\mathbb{C}}^{-}(E)$$

where $S_{\mathbb{C}}^{\pm}(E)$ is the ± 1 eigenbundle for Clifford multiplication by the complex volume element $\omega_{\mathbb{C}}$, and can be represented as

$$S_{\mathbb{C}}^{+}(E) \simeq P_{\text{Spin}}(E) \times_{\Delta_{2m}^{\mathbb{C}^{\pm}}} \mathbb{C}^{2m-1}$$

Let E be now equipped with a covariant derivative

$$\nabla : \Gamma(E) \rightarrow \Gamma(E \otimes T^*X)$$

induced by a connection τ on $P_{\text{SO}}(E)$. Since $\text{Cl}(E)$ is an associated vector bundle to $P_{\text{SO}}(E)$, τ induces a covariant derivative ∇_{Cl} on $\text{Cl}(E)$ with the property that

$$\nabla_{\text{Cl}}(\varphi \cdot \psi) = (\nabla_{\text{Cl}}\varphi) \cdot \psi + \varphi \cdot (\nabla_{\text{Cl}}\psi)$$

Moreover, the covariant derivative ∇_{Cl} preserves the subbundles $\text{Cl}^0(E)$ and $\text{Cl}^1(E)$, and the volume element $\omega = e_1 e_2 \cdots e_n$ is globally parallel, i.e. $\nabla_{\text{Cl}}\omega = 0$.

Suppose E is also equipped with a spin structure $\xi : P_{\text{Spin}}(E) \rightarrow P_{\text{SO}}(E)$. We can then lift the connection τ on $P_{\text{SO}}(E)$ to a connection $\tilde{\tau}$ on $P_{\text{Spin}}(E)$. Since a spinor bundle $S(E)$ is an associated vector bundle to $P_{\text{SO}}(E)$, the connection $\tilde{\tau}$ induces a covariant derivative ∇_S on $S(E)$. In particular, the covariant derivative ∇_S is compatible with ∇_{Cl} , in the sense that

$$\nabla_S(\varphi \cdot \sigma) = (\nabla_{\text{Cl}}\varphi) \cdot \sigma + \varphi \cdot (\nabla_S\sigma)$$

for any $\varphi \in \Gamma(\text{Cl}(E))$ and any $\sigma \in \Gamma(S(E))$.

Given an n -dimensional oriented Riemannian manifold X , we denote with $\text{Cl}(X)$ the Clifford bundle associated to TX . The Clifford bundle carries a canonical covariant derivative, which is induced by the Levi-Civita connection on $P_{\text{SO}}(TX)$. Consider now any bundle S of left modules over $\text{Cl}(X)$, not necessarily a spinor bundle, and suppose S is Riemannian and equipped with a Riemannian connection. We can then define a

first-order elliptic differential operator $D : \Gamma(S) \rightarrow \Gamma(S)$ called the *Dirac operator* of S , defined as

$$D\sigma|_x := \sum_{j=1}^n e_j \cdot (\nabla_{e_j}\sigma)|_x$$

at $x \in X$, where e_1, e_2, \dots, e_n is an orthonormal basis of $T_x X$.

If we let $S = Cl(X)$ with its canonical Riemannian connection, and view $Cl(X)$ as a bundle of left modules over itself by Clifford multiplication, then the Dirac operator is a square root of the classical Hodge laplacian.

Another case of major importance is the following. Let X be a spin manifold, and let S be any spinor bundle. The vector bundle S is Riemannian and carries a canonical covariant derivative ∇_{Spin} , called the *spin connection*, induced by the lift of the Levi-Civita connection on $P_{SO}(TX)$. The Dirac operator in this case is called the *Atiyah-Singer operator*, and plays a fundamental role in the Index theorem. Finally, a Dirac operator associated to a bundle S of left modules over $Cl(X)$ can be *twisted* by a Riemannian vector bundle with connection E by considering the Clifford multiplication on $\Gamma(S \otimes E)$ induced by

$$\varphi \cdot (\sigma \otimes e) := (\varphi \cdot \sigma) \otimes e$$

and equipping $S \otimes E$ with the tensor product connection ∇ defined on sections of the form $\sigma \otimes e$ by

$$\nabla(\sigma \otimes e) := (\nabla^S \sigma) \otimes e + \sigma \otimes (\nabla^E e)$$

where ∇^S and ∇^E are the covariant derivatives on S and E , respectively.

Finally, we conclude this appendix by giving the definition of a $Spin^c$ manifold. This involves first constructing the group $Spin^c$. Consider the complex spinor representation

$$\Delta_{\mathbb{C}} : Spin_n \rightarrow GL_{\mathbb{C}}(W^{\mathbb{C}})$$

and let $z : U(1) \rightarrow GL_{\mathbb{C}}(W^{\mathbb{C}})$ denote the multiplication by scalar. We then get the homomorphism $\Delta_{\mathbb{C}} \times z : Spin_n \times U(1) \rightarrow Hom_{\mathbb{C}}(W^{\mathbb{C}}, W^{\mathbb{C}})$, which has the element $(-1, -1)$ in its kernel. Dividing by this element gives the group

$$Spin_n^c := Spin_n \times_{\mathbb{Z}_2} U(1)$$

which satisfies the short exact sequence

$$0 \rightarrow \mathbb{Z}_2 \rightarrow Spin_n^c \xrightarrow{\xi_0} SO_n \times U(1) \rightarrow 1$$

Let E be an oriented Riemannian vector bundle of rank n over a manifold X . A *Spin^c-structure* on E consists of a principal Spin_n^c -bundle $P_{\text{Spin}_n^c}(E)$, and also a principal $U(1)$ -bundle $P_{U(1)}(E)$ over X with a bundle map

$$P_{\text{Spin}_n^c}(E) \xrightarrow{\xi} P_{\text{SO}_n}(E) \times P_{U(1)}(E)$$

such that ξ satisfies $\xi(p \cdot g) = \xi(p)\xi_0(g)$ for all $p \in P_{\text{Spin}_n^c}(E)$ and all $g \in \text{Spin}_n^c$. The first Chern class $d(E)$ of the $U(1)$ -bundle $P_{U(1)}(E)$ is called the *canonical class* of the Spin^c -structure.

A *Spin^c-manifold* is an oriented Riemannian manifold X with a Spin^c -structure on the tangent bundle TX .

Appendix C

Characteristic classes for vector bundles

In this appendix we will briefly recall some basic facts from the theory of characteristic classes, as developed in [70].

All the spaces are assumed to be of the homotopy type of countable CW-complexes. Let G be a Lie group. A *classifying space* for G is a connected topological space BG , together with a principal G -bundle $EG \rightarrow BG$ such that for any compact Hausdorff space X the set of homotopy classes of maps from X to BG is in bijective correspondence with the set of equivalence classes of principal G -bundles over X . In particular, the above correspondence is induced by associating to each map $f : X \rightarrow BG$ the pullback bundle f^*EG over X .

The principal bundle $EG \rightarrow BG$ is called the *universal principal G -bundle*. It can be proven by direct construction that for any Lie group a classifying space does exist. Moreover, by the very properties of a classifying space, it is unique up to homotopy type.

Consider the singular cohomology $H^*(BG; \Lambda)$ with coefficients in a ring Λ . Each non-zero class in $H^*(BG; \Lambda)$ is a *universal characteristic class* for principal G -bundles. Fix a class $c \in H^*(BG; \Lambda)$. For each principal G -bundle $P \rightarrow X$ we define the *c -characteristic class* $c(P) \in H^*(X; \Lambda)$ as

$$c(P) := f_P^*(c)$$

where $f_P : X \rightarrow BG$ is a classifying map for P , and is well defined, since f_P is uniquely defined up to homotopy. Moreover, any such characteristic class is “natural”, in the sense that given a principal G -bundle P over X and a continuous map $\varphi : Y \rightarrow X$ we have

$$c(\varphi^*P) = \varphi^*c(P)$$

We will now specialize to the case $G = \mathrm{BO}_n, \mathrm{BU}_n$. In these particular cases, the universal bundle EG can be obtained as the appropriate bundle of frames of a *universal real (or complex) vector bundle* \mathbb{E}_n over BO_n (or BU_n), which classifies real (or complex) vector bundles. Moreover, despite still difficult to compute, the cohomology rings of the classifying spaces BO_n and BU_n are quite manageable. The cohomology ring $H^*(\mathrm{BO}_n; \mathbb{Z}_2)$ is a \mathbb{Z}_2 -polynomial ring

$$\mathbb{Z}_2[w_1, w_2, \dots, w_n]$$

where w_k is a canonical generators of $H^k(\mathrm{BO}_n; \mathbb{Z}_2)$, and is called the *universal k -th Stiefel-Whitney class*. To any n -dimensional real vector bundle $E \rightarrow X$ classified by a map $f_E : X \rightarrow \mathrm{BO}_n$, we can associate the k -th Stiefel-Whitney class of E , $w_k(E) := f_E^*(w_k)$. In particular, the *total Stiefel-Whitney class* $w := 1 + w_1 + \cdots + w_n$ satisfies

$$w(E \oplus E') = w(E) \cup w(E')$$

If X is a smooth manifold, we can define the k -th Stiefel-Whitney class $w_k(X)$ of X as $w_k(\mathrm{TX})$. An important property of the Stiefel-Whitney classes is that for a compact smooth manifold they are invariants of the homotopy type of the manifold.

Given an n -dimensional real vector bundle E , the first and second Stiefel-Whitney vanish exactly when E is orientable and admits a Spin structure, respectively. This can be seen by using the following equivalent definition of $w_1(E)$ and $w_2(E)$.

Recall that any isomorphism class of principal G -bundles on a space X can be represented as a class in $H^1(X; G)$, the cohomology of X with coefficients in the sheaf of G -valued functions, via the transition functions. When G is not abelian, $H^1(X; G)$ is not a group, but rather a set with a distinguished element given by the trivial G -bundle. However, it still preserves some cohomological properties. Indeed, consider the short exact sequence

$$0 \rightarrow \mathrm{SO}_n \xrightarrow{i} \mathrm{O}_n \xrightarrow{\rho} \mathbb{Z}_2 \rightarrow 0,$$

which induces the exact sequence

$$H^1(X; \mathrm{SO}_n) \xrightarrow{i_*} H^1(X; \mathrm{O}_n) \xrightarrow{\rho_*} H^1(X; \mathbb{Z}_2)$$

Given a rank n vector bundle, we can define $w_1(E) = \rho_*([P_{\mathrm{O}}(E)])$, where $[P_{\mathrm{O}}(E)]$ denotes the class of the orthonormal frame bundle of E . Hence, when $w_1(E) = 0$ we have that the class $[P_{\mathrm{O}}(E)]$ is the image of the class of an SO_n -principal bundle, which is possible if and only if E is orientable.

Similarly, the short exact sequence

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \mathrm{Spin}_n \xrightarrow{\xi_0} \mathrm{SO}_n \rightarrow 0$$

induces the exact sequence

$$H^0(X; \mathrm{SO}_n) \xrightarrow{\delta^0} H^1(X; \mathbb{Z}_2) \rightarrow H^1(X; \mathrm{Spin}_n) \xrightarrow{\xi_{0*}} H^1(X; \mathrm{SO}_n) \xrightarrow{\delta} H^2(X; \mathbb{Z}_2)$$

We can define the second Stiefel-Whitney class by $w_2(E) = \delta([P_{\text{SO}}(E)])$. Hence, $w_2(E) = 0$ if and only if $P_{\text{SO}}(E)$ is equivalent to the \mathbb{Z}_2 -quotient of a principal Spin_n -bundle on X .

For the classifying space BU_n , we have that the cohomology ring $H^*(\text{BU}_n; \mathbb{Z})$ is a \mathbb{Z} -polynomial ring

$$\mathbb{Z}[c_1, c_2, \dots, c_n]$$

where $c_k \in H^k(\text{BU}_n; \mathbb{Z})$ is a canonical generator, and is called the *universal k -th Chern class*. Thus, to any n -dimensional complex vector $E \rightarrow X$ classified by a map $f_E : X \rightarrow \text{BU}_n$ we can associate the k -th Chern class $c_k(E) = f_E^*(c_k)$. We can define the *total Chern class* $c = 1 + c_1 + \dots + c_n$, which satisfies

$$c(E \oplus E') = c(E) \cup c(E') \quad (\text{C.1})$$

For a given complex n -dimensional vector bundle $E \rightarrow X$, we can compute the Chern classes which are *nontorsion* in the following geometric way. Let F be the curvature of an arbitrary covariant derivative on E . Recall that F is a differential form valued in the adjoint representation of $U(n)$, hence in $n \times n$ antihermitian matrices. Then the k -th Chern class of E is given by the deRham class of $\alpha_k(E)$, where

$$\det\left(I + \frac{Ft}{2\pi}\right) = \sum_{k=1}^n \alpha_k(E) t^k$$

The Chern classes defined above can be used as the basic ingredient to define other important characteristic class of vector bundle over a manifold. This is due to the following

Theorem C.1 *Let E be a n -dimensional complex vector bundle over a manifold X . Then there exists a manifold \mathcal{M}_E and a smooth and proper fibration $\pi : \mathcal{M}_E \rightarrow X$ such that*

- i) the homomorphism $\pi^* : H^*(X) \rightarrow H^*(\mathcal{M}_E)$ is injective*
- ii) the bundle π^*E splits into the direct sum of complex line bundles*

$$\pi^*E \simeq \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \dots \oplus \mathcal{L}_n$$

The above theorem “induces” the following *splitting principle*: all polynomial identities in the Chern classes of complex vector bundles can be proven under the assumption that all vector bundles are direct sums of line bundles.

For a real vector bundle E of dimension $2n$ one can prove that the complexification $E \otimes \mathbb{C}$ splits on \mathcal{M}_E as

$$\pi^*(E \otimes \mathbb{C}) \simeq \mathcal{L}_1 \oplus \overline{\mathcal{L}}_1 \oplus \cdots \oplus \mathcal{L}_n \oplus \overline{\mathcal{L}}_n$$

where $\overline{\mathcal{L}}_1$ denotes the complex conjugate of the \mathcal{L}_1 .

Notice that by naturality property, any c -characteristic class of π^*E is in the image of the homomorphism π^* . We can then construct characteristic classes as the unique preimage via the splitting homomorphism π^* of functions of the Chern classes of the splitting line bundles. For instance, by using the property (C.1) of the total Chern class, we have

$$c(E) = \prod_{k=1}^n (1 + x_k)$$

where $x_k = c_1(\mathcal{L}_k)$.

In this way, we can associate to formal power series rational characteristic classes. We can define the *total Todd class* of a complex vector bundle E by

$$\text{Td}(E) := \prod_{k=1}^n \frac{x_k}{1 - e^{-x_k}}$$

Given a real vector bundle E of dimension $2n$, the *total \hat{A} -class* can be defined by

$$\hat{A}(E) := \prod_{k=1}^n \frac{x_k/2}{\sinh(x_k/2)}$$

Finally, we can define the *total Chern character* by

$$\text{ch}(E) := \sum_{k=1}^n e^{x_k}$$

The Chern character can be represented in the deRham cohomology of X as

$$\text{ch}(E) = [\text{Tr}(e^{F/2\pi})] \in H^{ev}(X; \mathbb{R})$$

where F is the curvature of a covariant derivative on E , and the trace Tr is in the adjoint representation of $U(n)$.

Appendix D

Equivariant K-homology

Spectral definition

A natural way to define the equivariant homology theory K_\bullet^G is by means of a *spectrum* for equivariant topological K-theory K_G^\bullet , which is a particular covariant functor $\underline{\text{Vect}}^G(-)$ from the orbit category $\text{Or}(G)$ to the tensor category Spec of spectra [31, 73]. Given any G -complex X , the corresponding pointed G -space is $X_+ = X \amalg \text{pt}$ and one defines the loop spectrum $X_+ \otimes_G \underline{\text{Vect}}^G(-)$ by

$$X_+ \otimes_G \underline{\text{Vect}}^G(-) = \coprod_{G/H \in \text{Or}(G)} (X_+^H \wedge \underline{\text{Vect}}^G(G/H)) / \sim, \quad (\text{D.1})$$

where the equivalence relation \sim is generated by the identifications $f^*(x) \wedge s \sim x \wedge f_*(s)$ with $(f : G/K \rightarrow G/H) \in \text{Mor}(\text{Or}(G))$, $x \in X_+^H$, and $s \in \underline{\text{Vect}}^G(G/K)_*$. One then puts

$$K_*^G(X) := \pi_*(X_+ \otimes_G \underline{\text{Vect}}^G(-)). \quad (\text{D.2})$$

By using various G -homotopy equivalences of the loop spectra (D.1), one shows that this definition of equivariant K-homology comes with a natural induction structure. For the trivial group it reduces to the ordinary K-homology $K_*^e = K_*$ given by the Bott spectrum BU . If G is a finite group, any finite-dimensional representation of G naturally extends to a complex representation of the group ring $\mathbb{C}[G]$. Then there is an analytic assembly map

$$\text{ass} : K_*^G(X) \longrightarrow K_*(\mathbb{C}[G])$$

to the K-theory of the ring $\mathbb{C}[G]$, induced by the collapsing map $X \rightarrow \text{pt}$ and the isomorphisms

$$K_*(\mathbb{C}[H]) \cong \pi_*(\underline{\text{Vect}}^G(G/H)) \cong K_*^G(G/H) \cong R(H)$$

for any subgroup $H \leq G$ [73]. In the following we will give two concrete realizations of the homotopy groups (D.2).

Analytic definition

The simplest realization of the equivariant K-homology group $K_*^G(X)$ is within the framework of an equivariant version of Kasparov's KK-theory KK_*^G . Let \mathcal{A} be a G-algebra, *i.e.*, a C^* -algebra \mathcal{A} together with a group homomorphism

$$\lambda : G \longrightarrow \text{Aut}(\mathcal{A}) .$$

By a Hilbert (G, \mathcal{A}) -module we mean a Hilbert \mathcal{A} -module \mathcal{E} together with a G-action given by a homomorphism $\Lambda : G \rightarrow \text{GL}(\mathcal{E})$ such that

$$\Lambda_g(\varepsilon \cdot a) = \Lambda_g(\varepsilon) \cdot \lambda_g(a) \tag{D.3}$$

for all $g \in G$, $\varepsilon \in \mathcal{E}$ and $a \in \mathcal{A}$. Let $\mathcal{L}(\mathcal{E})$ denote the $*$ -algebra of \mathcal{A} -linear maps $T : \mathcal{E} \rightarrow \mathcal{E}$ admitting an adjoint with respect to the \mathcal{A} -valued inner product on \mathcal{E} . The induced G-action on $\mathcal{L}(\mathcal{E})$ is given by $g \cdot T := \Lambda_g \circ T \circ \Lambda_{g^{-1}}$. Let $\mathcal{K}(\mathcal{E})$ be the subalgebra of $\mathcal{L}(\mathcal{E})$ consisting of generalized compact operators.

Given a pair $(\mathcal{A}, \mathcal{B})$ of G-algebras, let $D^G(\mathcal{A}, \mathcal{B})$ be the set of triples (\mathcal{E}, ϕ, T) where \mathcal{E} is a countably generated Hilbert (G, \mathcal{B}) -module, $\phi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{E})$ is a $*$ -homomorphism which commutes with the G-action,

$$\phi(\lambda_g(a)) = \Lambda_g \circ \phi(a) \circ \Lambda_{g^{-1}} \tag{D.4}$$

for all $g \in G$ and $a \in \mathcal{A}$, and $T \in \mathcal{L}(\mathcal{E})$ such that

- 1) $[T, \phi(a)] \in \mathcal{K}(\mathcal{E})$ for all $a \in \mathcal{A}$; and
- 2) $\phi(a)(T - T^*)$, $\phi(a)(T^2 - 1)$, $\phi(a)(g \cdot T - T) \in \mathcal{K}(\mathcal{E})$ for all $a \in \mathcal{A}$ and $g \in G$.

The standard equivalence relations of KK-theory are now analogously defined. The set of equivalence classes in $D^G(\mathcal{A}, \mathcal{B})$ defines the equivariant KK-theory groups $KK_*^G(\mathcal{A}, \mathcal{B})$.

If X is a smooth proper G-manifold without boundary, and G acts on X by diffeomorphisms, then the algebra $\mathcal{A} = C_0(X)$ of continuous functions on X vanishing at infinity is a G-algebra with automorphism λ_g on \mathcal{A} given by

$$\lambda_g(f)(x) := (g^*f)(x) = f(g^{-1} \cdot x) ,$$

where g^* denotes the pullback of the G-action on X by left translation by $g^{-1} \in G$. We define

$$K_*^G(X) := KK_*^G(C_0(X), \mathbb{C}) \tag{D.5}$$

with G acting trivially on \mathbb{C} . The conditions (D.3) and (D.4) naturally capture the physical requirements that physical orbifold string states are G-invariant and also that the worldvolume fields on a fractional D-brane carry a ‘‘covariant representation’’ of the orbifold group [37].

The equivariant Dirac class

We can determine a canonical class in the abelian group (D.5) as follows. We refer to Appendix B for basic notions of Clifford algebras.

Let $\dim(X) = 2n$, and let G be a finite subgroup of the rotation group $SO(2n)$.¹³ A choice of a complete G -invariant riemannian metric on X naturally lifts the Clifford bundle $Cl(X)$.

The G -manifold X is said to have a G -*spin^c structure* or to be K_G -*oriented* if there is an extension of the orthonormal frame bundle $P_{SO_{2n}}(X)$ to a principal $Spin^c(2n)$ -bundle $P_{Spin^c_{2n}}(X)$ over X which is compatible with the G -action. The extension $P_{Spin^c_{2n}}(X)$ may be regarded as a principal circle bundle over $P_{SO_{2n}}(X)$,

$$\begin{array}{ccccccc}
 & & & & U(1) & & \\
 & & & & \swarrow & \downarrow & \\
 \hat{G} & \longrightarrow & Spin^c(2n) & \longrightarrow & P_{Spin^c_{2n}}(X) & \longrightarrow & X, \\
 \downarrow & & \downarrow & & \downarrow & & \nearrow \\
 G & \longrightarrow & SO(2n) & \longrightarrow & P_{SO_{2n}}(X) & &
 \end{array}$$

where the pullback square on the bottom left defines the required covering of the orbifold group $G < SO(2n)$ by a subgroup of the $spin^c$ group $\hat{G} < Spin^c(2n)$. The kernel of the homomorphism $\hat{G} \rightarrow G$ is identified with the circle group $U(1) < Spin^c(2n)$. We fix a choice of lift and hence assume that G is a discrete subgroup of the $spin^c$ group. \mathbb{Z}_2 -graded Clifford modules are likewise extended to representations of $\mathbb{C}[G] \otimes Cliff(2n)$, with $\mathbb{C}[G]$ the group ring of G , called G -Clifford modules.

Since G lifts to \hat{G} in the $spin^c$ group, the the spinor bundles S^\pm associated to $P_{Spin^c_{2n}}(X)$ are naturally G -bundles. The G -invariant Levi-Civita connection determines a connection on $P_{SO_{2n}}(X)$, and together with a choice of G -invariant connection form on the principal $U(1)$ -bundle $P_{Spin^c_{2n}}(X) \rightarrow P_{SO_{2n}}(X)$, they determine a connection one-form on the principal bundle $P_{Spin^c_{2n}}(X) \rightarrow X$ which is G -invariant. This determines an invariant covariant derivative

$$\nabla^{S \otimes E} : \Gamma(S^+ \otimes E) \rightarrow \Gamma(T_X^* \otimes S^+ \otimes E)$$

where ∇^E is a G -invariant connection on a G -bundle $E \rightarrow X$. The contraction given by Clifford multiplication defines a map

$$Cl : \Gamma(T_X^* \otimes S^+ \otimes E) \rightarrow \Gamma(S^- \otimes E)$$

¹³Throughout the extension to K_1^G or K_G^{-1} and $\dim(X)$ odd can be described in the same way as in degree zero by replacing X with $X \times S^1$.

which graded commutes with the G -action, and the G -invariant spin^c Dirac operator on X with coefficients in E is defined as the composition

$$\mathcal{D}_E^X = \text{Cl} \circ \nabla^{\text{S} \otimes E} . \quad (\text{D.6})$$

We will view the operator (D.6) as an operator on L^2 -spaces

$$\mathcal{D}_E^X : L^2(X, \text{S}^+ \otimes E) \longrightarrow L^2(X, \text{S}^- \otimes E) .$$

It induces a class $[\mathcal{D}_E^X] \in K_0^G(X)$ as follows. The G -algebra $C_0(X)$ acts on the \mathbb{Z}_2 -graded G -Hilbert space $\mathcal{E} := L^2(X, \text{S} \otimes E)$ by multiplication. Define the bounded G -invariant operator $T := \mathcal{D}_E^X ((\mathcal{D}_E^X)^2 + 1)^{-1/2} \in \text{Fred}_G$. Then $[\mathcal{D}_E^X]$ is represented by the G -equivariant Fredholm module (\mathcal{E}, T) .

Geometric definition

Geometric equivariant K -homology can be defined for an arbitrary discrete, countable group G on the category of proper, finite G -complexes X and proven to be isomorphic to analytic equivariant K -homology [15]. Recall that the topological equivariant K -theory $K_G^*(X)$ is defined by applying the Grothendieck functor K^* to the additive category $\text{Vect}_G^{\mathbb{C}}(X)$ whose objects are complex G -vector bundles over X , *i.e.*, $K_G^*(X) := K^*(\text{Vect}_G^{\mathbb{C}}(X))$. In the homological setting, the relevant category is instead the additive category of G -equivariant K -cycles $\text{D}^G(X)$, whose objects are triples (W, E, f) where

- (a) W is a manifold without boundary with a smooth proper cocompact G -action and G - spin^c structure;
- (b) E is an object in $\text{Vect}_G^{\mathbb{C}}(W)$; and
- (c) $f : W \rightarrow X$ is a G -map.

Two G -equivariant K -cycles (W, E, f) and (W', E', f') are said to be *isomorphic* if there is a G -equivariant diffeomorphism $h : W \rightarrow W'$ preserving the G - spin^c structures on W, W' such that $h^*(E') \cong E$ and $f' \circ h = f$.

Define an equivalence relation \sim on the category $\text{D}^G(X)$ generated by the operations of

- i) Bordism: $(W_i, E_i, f_i) \in \text{D}^G(X)$, $i = 0, 1$ are *bordant* if there is a triple (M, E, f) where M is a manifold with boundary ∂M , with a smooth proper cocompact G -action and G - spin^c structure, $E \rightarrow M$ is a complex G -vector bundle, and $f :$

$M \rightarrow X$ is a G -map such that $(\partial M, E|_{\partial M}, f|_{\partial M}) \cong (W_0, E_0, f_0) \amalg (-W_1, E_1, f_1)$. Here $-W_1$ denotes W_1 with the reversed G -spin^c structure;

ii) Direct sum: If $(W, E, f) \in D^G(X)$ and $E = E_0 \oplus E_1$, then

$$(W, E, f) \cong (W, E_0, f) \amalg (W, E_1, f) ;$$

and

iii) Vector bundle modification: Let $(W, E, f) \in D^G(X)$ and H an even-dimensional G -spin^c vector bundle over W . Let $\widehat{W} = S(H \oplus \mathbb{1})$ denote the sphere bundle of $H \oplus \mathbb{1}$, which is canonically a G -spin^c manifold, with G -bundle projection $\pi : \widehat{W} \rightarrow W$. Let

$$S(H) = S(H)^+ \oplus S(H)^-$$

denote the \mathbb{Z}_2 -graded G -bundle over W of spinors on H . Set $\widehat{E} = \pi^*((S(H)^+)^{\vee} \otimes E)$ and $\widehat{f} = f \circ \pi$. Then $(\widehat{W}, \widehat{E}, \widehat{f}) \in D^G(X)$ is the *vector bundle modification* of (W, E, f) by H .

We set

$$K_{0,1}^G(X) = D_{\text{even,odd}}^G(X) / \sim$$

where the parity refers to the dimension of the K -cycle, which is preserved by \sim .

Using the equivariant Dirac class, one can construct a homomorphism from the geometric to the analytic K -homology group. On K -cycles we define $(W, E, f) \mapsto f_*[\mathcal{D}_E^W]$ and extend linearly. This map can be used to express G -index theorems within this homological framework and it extends to give an isomorphism between the two equivariant K -homology groups [15]

Bibliography

- [1] A. Adem and Y. Ruan. Twisted Orbifold K-theory. *Commun. Math. Phys.*, 237:533, 2003.
- [2] O. Alvarez. Topological Quantization and Cohomology. *Commun. Math. Phys.*, 100:279, 1985.
- [3] M. Atiyah. Vector bundles and the Künneth formula. *Topology*, 1:245, 1962.
- [4] M. Atiyah. *K-theory*. W.A.Benjamin, 1967.
- [5] M. Atiyah, R. Bott, and A. Shapiro. Clifford modules. *Topology*, 3:3, 1964.
- [6] M. Atiyah and F. Hirzebruch. Riemann-Roch theorems for differentiable manifolds. *Bull. Am. Math. Soc.*, 65:276, 1959.
- [7] M. Atiyah and F. Hirzebruch. Vector bundles and homogeneous spaces. *Amer. Math. Soc. Symp. in Pure Math.*, 3:7, 1961.
- [8] M. Atiyah and G. Segal. Equivariant K-theory and completion. *J. Differential Geometry*, 3:1–18, 1969.
- [9] M. Atiyah and G. Segal. On equivariant Euler characteristics. *J. Geom. Phys.*, 6:399–405, 1989.
- [10] M. Atiyah and I. Singer. Index theory for skew-adjoint fredholm operators. *Publ. Math. IHES*, 37:5, 1969.
- [11] M. Atiyah and I. Singer. The index of elliptic operators I-IV. *Ann. of Math.*, 87, 1970.
- [12] P. Baum and A. Connes. Chern character for discrete groups. In *A Fete of Topology*, page 163, 1987.

- [13] P. Baum and R. Douglas. Index Theory, Bordism, and K-Homology. *Contemp. Math.*, 10:1, 1982.
- [14] P. Baum and R. Douglas. K-Homology and Index Theory. *Proc. Symp. Pure Math.*, 38:117, 1982.
- [15] P. Baum, N. Higson, and T. Schick. On the Equivalence of Geometric and Analytic K-Homology. *Pure And Applied Mathematics Quarterly*, 3:1, 2007.
- [16] K. Becker, M. Becker, and J. Schwarz. *String Theory and M-theory*. Cambridge University Press, 2007.
- [17] N. Berline, E. Getzler, and M. Vergne. *Heat Kernels and Dirac Operators*. Springer, 1991.
- [18] B. Blackadar. *K-theory for Operator Algebras*. Cambridge University Press, 1998.
- [19] J. Block and E. Getzler. Equivariant cyclic homology and equivariant differential forms. *Ann. Sci. Ecole Norm. Sup.*, 27, 1994.
- [20] R. Bott. Stable homotopy of the classical groups. *Ann. of Math.*, 70:313, 1959.
- [21] R. Bott and L. Tu. *Differential Forms in Algebraic Topology*. Springer-Verlag, 1982.
- [22] G. Bredon. *Equivariant Cohomology Theories*. Springer, 1967.
- [23] J. Brylinski. *Loop spaces, Characteristic classes and Geometric Quantization*. Birkhauser, 1993.
- [24] U. Bunke. A K-theoretic relative index theorem and Callias-type Dirac operators. *Math. Ann.*, 303:241, 1995.
- [25] U. Bunke. Orbifold index and equivariant K-homology. 2007.
- [26] U. Bunke and T. Schick. Smooth K-theory. <http://arxiv.org/abs/0707.0046>.
- [27] A. Carey, S. Johnson, and M. Murray. Holonomy on D-branes. *J. Geom. Phys.*, 52:186, 2004.
- [28] J. Cheeger and J. Simons. Differential characters and geometric invariants. *Geometry and Topology*, page 50, 1983.

- [29] Y.-K. Cheung and Z. Yin. Anomalies, branes, and currents. *Nucl.Phys. B*, 517:69, 1998.
- [30] J. Davis and P. Kirk. *Lecture Notes in Algebraic Topology*. AMS Bookstore, 2001.
- [31] J. Davis and W. Luck. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. *K-theory*, 15:201, 1998.
- [32] J. de Boer, R. Dijkgraaf, K. Hori, A. Keurentjes, J. Morgan, D. Morrison, and S. Sethi. Triples, fluxes and strings. *Adv. Theor. Math. Phys.*, 105:47, 2002.
- [33] P. Deligne and al. *Quantum Fields and Strings: A Course for Mathematicians. Vol 1,2*. American Mathematical Society, 2000.
- [34] P. Dirac. Quantized singularities in the electromagnetic field. *Proc. R. Soc. Lond.*, A133:60, 1931.
- [35] L. Dixon, J. Harvey, C. Vafa, and E. Witten. Strings on Orbifolds. *Nucl. Phys.*, B261:678–686, 1985.
- [36] L. Dixon, J. Harvey, C. Vafa, and E. Witten. Strings on Orbifolds (II). *Nucl. Phys.*, B274:285–314, 1986.
- [37] M. Douglas and G. Moore. D-branes, Quivers, and ALE Instantons. hep-th/9603167.
- [38] E. Dyer. *Cohomology Theories*. W.A.Benjamin, 1969.
- [39] D. Freed. Locality and integration in topological field theory. In M. del Olmo, M. Santander, and J. M. Guilarte, editors, *Group Theoretical methods in Physics*, volume 3, 1993.
- [40] D. Freed. Dirac Charge Quantization and Generalized Differential Cohomology. *Surv. Diff. Geom.*, VII:129, 2000.
- [41] D. Freed. K-theory in Quantum Field Theory. *Current Developments in Mathematics*, math-ph/0206031.
- [42] D. Freed and M. Hopkins. On Ramond-Ramond fields and K-theory. *JHEP*, 0005:044, 2000.

- [43] D. Freed, M. Hopkins, and C. Teleman. Twisted equivariant K-theory with complex coefficients. 2002.
- [44] D. Freed, G. Moore, and G. Segal. Heisenberg Groups and Noncommutative Fluxes. *Annals Phys.*, 322:236, 2007.
- [45] D. Freed and E. Witten. Anomalies in String Theory with D-Branes. *Asian J.Math*, 3:819, 1999.
- [46] H. Garcia-Compean. D-branes in Orbifold Singularities and Equivariant K-Theory. *Nuclear Physics B*, 557:480, 1999.
- [47] K. Goodearl. *Notes on Real and Complex C^* -algebras*. Shiva Publishing, 1982.
- [48] M. Green, J. Harvey, and G. Moore. I-brane inflow and anomalous couplings on D-branes. *Class. Quantum Grav.*, 14:47, 1997.
- [49] M. Green, J. Schwarz, and E. Witten. *Superstring theory, Vol 1,2*. Cambridge University Press, 1987.
- [50] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley Blackwell, 1994.
- [51] F. Harvey, H. Lawson, and J. Zweck. The deRham-Federer theory of differential characters and character duality. *Amer. J. Math*, 125:791, 2003.
- [52] J. Harvey and G. Moore. Noncommutative Tachyons and K-theory. *Journal of Mathematical Physics*, 42:2765, 2001.
- [53] N. Higson and J. Roe. *Analytic K-Homology*. Oxford University Press, 2000.
- [54] M. Hopkins and I. Singer. Quadratic functions in Geometry, Topology, and M-theory. *J.Diff.Geom.*, 70:329, 2005.
- [55] P. Horava. Type IIA D-branes, K-theory, and Matrix Theory. *Advances in Theoretical and Mathematical Physics*, 2:1373, 1999.
- [56] D. Husemoller. *Fibre Bundles*. Springer-Verlag, 1966.
- [57] S. Illmann. Smooth equivariant triangulations of G-manifolds for G a finite group. *Math. Ann.*, 233:199, 1978.
- [58] M. Jakob. A bordism-type description of homology. *Manuscripta Math.*, 96:67, 1998.

- [59] C. Johnson. *D-branes*. Cambridge University Press, 2003.
- [60] A. Kapustin. D-branes in a topologically nontrivial B-field. *Adv.Theor.Math.Phys.*, 4:127, 2000.
- [61] M. Karoubi. *K-Theory: an introduction*. Springer-Verlag, 1978.
- [62] G. Kasparov. The Operator K-Functor and Extensions of C^* -algebras. *Math. USSR Izv.*, 16:513, 1981.
- [63] H. Lawson and M. Michelsohn. *Spin Geometry*. Princeton University Press, 1989.
- [64] J. Lott. \mathbb{R}/\mathbb{Z} index theory. *Comm. Anal. Geom.*, 2:279, 1994.
- [65] W. Luck. Chern characters for proper equivariant homology theories and applications to K- and L-theory. *J.reine angew. Math.*, 543:193–234, 2002.
- [66] W. Luck. Equivariant Cohomological Chern Characters. *International Journal of Algebra and Computation*, 15:1025–1052, 2006.
- [67] W. Luck and R. Oliver. Chern characters for equivariant K-theory of proper G-CW-complexes. In J. Aguade, C. Broto, and C. Casacuberta, editors, *Proceedings of the conference “Cohomological methods in homotopy theory”*, volume 196 of *Progress in Mathematics*. Birkhser, 1998.
- [68] E. Lupercio and B. Uribe. An introduction to gerbes on orbifold. *Annales mathematique Blaise Pascal*, 11:155, 2004.
- [69] Y. Matsuo. Topological Charges of Noncommutative Soliton. *Physics Letters B*, 499:223, 2001.
- [70] J. Milnor and J. Stasheff. *Characteristic Classes*. Princeton University Press, 1974.
- [71] R. Minasian and G. Moore. K-theory and Ramond-Ramond charge. *JHEP*, 9711:002, 1997.
- [72] R. Miranda. *Algebraic Curves and Riemann Surfaces*. AMS Bookstore, 1995.
- [73] G. Mislin and A. Valette. *Proper Group Actions and the Baum-Connes Conjecture*. Birkhauser, 2003.
- [74] G. Moore and E. Witten. Self-Duality, Ramond-Ramond Fields, and K-Theory. *JHEP*, 0005:032, 2000.

- [75] K. Olsen and R. Szabo. Constructing D-Branes from K-theory. *Advances in Theoretical and Mathematical Physics*, 3:889, 1999.
- [76] K. Olsen and R. Szabo. Brane Descent Relations in K-theory. *Nuclear Physics B*, 566:562, 2000.
- [77] V. Periwal. D-brane charges and K-homology. *JHEP*, 7:41, 2000.
- [78] J. Polchinski. Dirichlet Branes and Ramond-Ramond Charges. *Phys. Rev. Lett.*, 75:4724, 1995.
- [79] J. Polchinski. *String Theory, Vol 1,2*. Cambridge University Press, 2005.
- [80] R. Reis and R. Szabo. Geometric K-Homology of Flat D-Branes. *Commun. Math. Phys.*, 266:71, 2006.
- [81] I. Satake. On a generalization of a notion of manifold. *Proc. Nat. Acad. Sci. U.S.A.*, 42:359, 1956.
- [82] G. Segal. Equivariant K-theory. *Publications Mathematiques de l'IHS*, 34:129, 1968.
- [83] A. Sen. $So(32)$ spinors of type i and other solitons on brane-antibrane pair. *JHEP*, 9809:23, 1998.
- [84] A. Sen. Stable non-BPS bound states of BPS D-branes. *JHEP*, 08:010, 1998.
- [85] A. Sen. Stable non-BPS states in string theory. *JHEP*, 06:007, 1998.
- [86] J. Simons and D. Sullivan. Axiomatic Characterization of Ordinary Differential Cohomology. 2007, <http://arxiv.org/abs/math/0701077>.
- [87] J. Slominska. On the equivariant Chern Character homomorphism. *Bull. Acad. Pol. Sci.*, 24:909–913, 1976.
- [88] R. Switzer. *Algebraic topology. Homology and Homotopy*. Springer-Verlag, 1970.
- [89] R. Szabo. D-Branes, Tachyons and K-Homology. *Mod.Phys.Lett.A*, 17:2297, 2002.
- [90] R. Szabo. *An Introduction to String Theory and D-brane Dynamics*. Imperial College Press, 2004.

- [91] R. J. Szabo and A. Valentino. Ramond-Ramond fields, Fractional branes, and orbifold differential K-theory. <http://arxiv.org/abs/0710.2773>.
- [92] T. tom Dieck. *Transformations Groups*. Walter de Gruyter, 1987.
- [93] E. Wigner. The Unreasonable Effectiveness of Mathematics in the Natural Sciences. *Commun. in Pure and Applied Math.*, 13:1, 1960.
- [94] E. Witten. D-branes And K-Theory. *JHEP*, 9812:019, 1998.
- [95] Z. Yosimura. Universal Coefficient Sequences for Cohomology Theories of CW-Spectra. *Osaka J. Math.*, 16:201, 1979.
- [96] B. Zwiebach. *A First Course in String Theory*. Cambridge University Press, 2004.

This document has been entirely
made with 
emacs

using LaTeX on Fedora 7 (Moonshine) Linux distribution.