TRIANGULATED CATEGORIES, SUMMER SEMESTER 2012

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Contents

Triangulated categories are a class of categories which appear in many areas of mathematics. The following graphical representation is based on [1].

$$
\begin{tikzcd}
 & & \text{(tensor) tr. cat.} & & \\
\text{alg. geom.} \arrow[urr] & \text{repr. th.} \arrow[ur] & \text{st. hom. th.} \arrow[u] & \text{motiv. th.} \arrow[ul] & \text{noncomm. top.} \arrow[ull]
\end{tikzcd}
$$

(1) **Algebraic geometry** Here we could take $\mathrm{D}^{\mathrm{b}}(X)$ if X is a smooth and projective variety or the category of perfect complexes or...
(2) **Stable homotopy theory** The stable homotopy category of topological spectra is a triangulated category. The tensor product is the smash product.

(3) **(Modular) representation theory** Derived category of modules over an algebra. Or we could consider a finite group G, a field k with $\mathrm{char}(k) > 0$ and take the stable module category of finitely generated kG-modules. Objects are k-representations of G and morphisms are kG-linear map modulo those factoring through a projective.
(4) **Motivic theory** Voevodsky's derived category of geometric motives.
(5) **Noncommutative topology** G reasonable topological group; can consider the G-equivariant Kasparov category of separable G-C^*-algebras.

The goal of the lecture is to give a gentle introduction to some basic notions in the theory of triangulated categories. In particular, we will consider the examples of the homotopy and derived category of an abelian category, derived functors and t-structures.

We will need the following notions from category theory. Recall that, roughly speaking, an additive category is a category such that the Hom-spaces have the structure of abelian groups and the compositions are bilinear, there exists a 0-object and finite direct sums exist as well. An additive functor is supposed to be compatible with these additional structures. Given a field k, we can consider k-linear additive categories, meaning that we require the Hom-spaces to be k-vector spaces and all compositions to be k-bilinear. Similarly, one defines the notion of a k-linear functor.

1. Triangulated categories and functors

Definition. Let $\mathcal{D}$ be an additive category. The structure of a *triangulated category* on $\mathcal{D}$ is given by an additive autoequivalence $T = [1]\colon \mathcal{D} \longrightarrow \mathcal{D}$, the *shift functor*, and a class of *distinguished triangles*

$$A \longrightarrow B \longrightarrow C \longrightarrow T(A) = A[1]$$

satisfying the following axioms.

TR1 (i) Any triangle of the form

$$A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow A[1]$$

is distinguished.
(ii) Any triangle isomorphic to a distinguished triangle is itself distinguished (a morphism of triangles is a collection of vertical maps such that everything commutes).
(iii) Any morphism $f\colon A \longrightarrow B$ can be completed to a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1].$$

TR2 The triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is distinguished if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1].$$

TR3 Suppose there exists a commutative diagram of distinguished triangles with vertical arrows α and β

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, dashed, "\gamma"] & A[1] \arrow[d, "\alpha[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1], \end{tikzcd}$$

then there exists a γ making this diagram a morphism of triangles.

TR4 For each pair of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "="] & B \arrow[r, "h"] \arrow[d, "g"] & C' \arrow[r] \arrow[d] & A[1] \arrow[d, "="] \\ A \arrow[r, "g\circ f"] & C \arrow[r] \arrow[d] & B' \arrow[r] \arrow[d] & A[1] \arrow[d, "f[1]"] \\ & A' \arrow[r, "="] \arrow[d] & A' \arrow[r] \arrow[d] & B[1] \\ & B[1] \arrow[r, "h[1]"] & C'[1] & \end{tikzcd}$$

where the first two rows and the two central columns are distinguished triangles.

Remark 1.1. A category satisfying TR1-TR3 is called *pre-triangulated*. The last axiom TR4 is usually called the *octahedral axiom* since it can be represented in the form of an octahedron. It is relatively rarely used. There are no easy examples of pre-triangulated categories which are not triangulated.

Remark 1.2. Consider two composable maps f, g as in TR4 and denote the object C from TR1 (iii) by $C(f)$. If one thinks of distinguished triangles as a generalization of short exact sequences, then we get $C(f) = B/A$, $C(g) = C/B$ and $C(g \circ f) = C/A$. Then TR4 says that $C/A\big/B/A \simeq C/B$, since it asserts the existence of a triangle $C(f) \longrightarrow C(g \circ f) \longrightarrow C(g) \longrightarrow C(f)[1]$.

In the following we will derive some easy consequences of the definition.

Proposition 1.3. *Let*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

be a distinguished triangle. Then $g \circ f = 0$.

Proof. Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] & A[1] \arrow[d, "="] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & A[1], \end{tikzcd}$$

where the first row is a triangle by TR1 (i). We are then done by TR3. □

Proposition 1.4. *Let* $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow A[1]$ *be a distinguished triangle in* $\mathcal{D}$*. Then, for any object* A_0*, the sequences*

$$\mathrm{Hom}(A_0, A) \longrightarrow \mathrm{Hom}(A_0, B) \longrightarrow \mathrm{Hom}(A_0, C)$$

and

$$\mathrm{Hom}(C, A_0) \longrightarrow \mathrm{Hom}(B, A_0) \longrightarrow \mathrm{Hom}(A, A_0)$$

are exact.

Proof. Let $h\colon A_0 \longrightarrow B$ be a morphism such that $g \circ h = 0$. Apply TR1 and TR3 to

$$\begin{tikzcd}
A_0 \arrow[r, "\mathrm{id}"] & A_0 \arrow[r] \arrow[d, "h"] & 0 \arrow[r] & A_0[1] \\
A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & A[1]
\end{tikzcd}$$

in order to get a morphism $m\colon A_0 \longrightarrow A$ with $f \circ m = h$. This proves the exactness of the first sequence (im $\subset$ ker follows from the previous proposition) and the second is similar. □

Remark 1.5. We can use TR2 to apply the proposition to the rotated triangles and hence we get long exact sequences.

Proposition 1.6. *Let*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & A[1] \arrow[d, "\alpha[1]"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

be a morphism of triangles. If two out of the three vertical arrows are isomorphisms, then so is the third.

Proof. Without loss of generality assume that α and β are isomorphisms. Apply $\mathrm{Hom}(C', -)$ to both triangles to get

$$\begin{tikzcd}
\mathrm{Hom}(C', A) \arrow[r] \arrow[d, "\alpha_*"] & \mathrm{Hom}(C', B) \arrow[r] \arrow[d, "\beta_*"] & \mathrm{Hom}(C', C) \arrow[r] \arrow[d, "\gamma_*"] & \mathrm{Hom}(C', A[1]) \arrow[r] \arrow[d, "\alpha[1]_*"] & \mathrm{Hom}(C', B[1]) \arrow[d, "\beta[1]_*"] \\
\mathrm{Hom}(C', A') \arrow[r] & \mathrm{Hom}(C', B') \arrow[r] & \mathrm{Hom}(C', C') \arrow[r] & \mathrm{Hom}(C', A'[1]) \arrow[r] & \mathrm{Hom}(C', B'[1]),
\end{tikzcd}$$

a commutative diagram with exact rows. By assumption, α_*, β_*, $\alpha[1]_*$ and $\beta[1]_*$ are isomorphisms, so γ_* is also one by the 5-lemma. Thus, there exists a map $\delta\colon C' \longrightarrow C$ such that $\gamma \circ \delta = \mathrm{id}_{C'}$. Similarly one can use $\mathrm{Hom}(C, -)$ to conclude that there is also a left inverse. Then they have to be equal and γ is an isomorphism as claimed. □

Corollary 1.7. *The distinguished triangle in TR1 (iii) is unique up to isomorphism.* □

Remark 1.8. For a given map $f\colon A \longrightarrow B$ one usually calls C in TR1 (iii) a *cone* of f. We have just seen that a cone is unique up to isomorphism. But this isomorphism is not unique, because of the non-uniqueness of the map in TR3. This seems like a minor issue but has in fact as a consequence that certain natural constructions do not work in the context of triangulated categories.

Proposition 1.9. *Let* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ *be a distinguished triangle. Then f is an isomorphism if and only if $C \simeq 0$.*

Proof. Assume f is an isomorphism and let f^{-1} be the inverse. Then we have the diagram

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}"] \arrow[d, "f^{-1}"] & B \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \arrow[r] & B[1] \\ A \arrow[r, "f"] & B \arrow[r] & C \arrow[r] & A[1]. \end{tikzcd}$$

By TR3 and Proposition 1.6 $C \simeq 0$.

If $C \simeq 0$, use the two rows in the above triangle to get f^{-1}. □

Lemma 1.10. *Let* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ *and* $A' \xrightarrow{f'} B' \xrightarrow{g'} C' \xrightarrow{h'} A'[1]$ *be distinguished triangles. Then their direct sum is also a triangle.*

Proof. Consider the map $f \oplus f'\colon A \oplus A' \longrightarrow B \oplus B'$. By TR1 this can be completed to a triangle $\Delta := A \oplus A' \longrightarrow B \oplus B' \longrightarrow P$. Projecting to the factors we get two morphisms of triangles from Δ to the given ones, so, in particular, we get maps $P \longrightarrow C$ and $P \longrightarrow C'$. Thus we get a map from Δ to the direct sum of the given triangles and Proposition 1.6 gives that the two triangles are isomorphic (to be more precise, we have to use that $\mathrm{Hom}(C \oplus C', -) \simeq \mathrm{Hom}(C, -) \oplus \mathrm{Hom}(C', -)$, that direct sums of exact sequences is exact; we then get $\mathrm{Hom}(C \oplus C', -) \simeq \mathrm{Hom}(P, -)$, hence $P \simeq C \oplus C'$ by Yoneda; we only need that $\mathrm{Hom}(A_0, -)$ resp. $\mathrm{Hom}(-, A_0)$ gives short exact sequences). □

Proposition 1.11. *Let* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ *be a distinguished triangle and assume that $h = 0$. Then $B \simeq A \oplus C$.*

Proof. We know that $0 \longrightarrow C \xrightarrow{\mathrm{id}} C \longrightarrow 0$ is a triangle by TR1 (i) and TR2. There is also $A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow A[1]$. Taking their direct sum gives a triangle by the lemma. Now complete the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\mathrm{id}"] & A \oplus C \arrow[r] & C \arrow[r, "0"] \arrow[d, "\mathrm{id}"] & A[1] \arrow[d, "\mathrm{id}"] \\ A \arrow[r] & B \arrow[r] & C \arrow[r, "0"] & A[1] \end{tikzcd}$$

to a morphism of triangles, which have to be isomorphic. □

Proposition 1.12. *Consider the following diagram*

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] & A[1] \\ A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & A'[1]. \end{tikzcd}$$

If $g' \circ \beta \circ f = 0$, then there are maps $\alpha \colon A \longrightarrow A'$ and $\beta \colon C \longrightarrow C'$ completing the diagram to a morphism of triangles.

Proof. Apply $\mathrm{Hom}(A, -)$ to the lower triangle to get

$$\ldots \longrightarrow \mathrm{Hom}(A, A') \longrightarrow \mathrm{Hom}(A, B') \longrightarrow \mathrm{Hom}(A, C') \longrightarrow \ldots$$

By assumption, the morphism $\beta f \in \mathrm{Hom}(A, B')$ goes to zero in $\mathrm{Hom}(A, C')$, hence we get a morphism $\alpha \in \mathrm{Hom}(A, A')$ such that $f'\alpha = \beta f$. Then we can use TR3 to get γ. $\square$

Remark 1.13. Note that if $\mathrm{Hom}(A, C'[-1]) = 0$, then α and γ are unique. The first statement being obvious, we prove the second. To do this, apply $\mathrm{Hom}(-, C')$ to the upper triangle to get

$$\ldots \longrightarrow \mathrm{Hom}(A[1], C') \longrightarrow \mathrm{Hom}(C, C') \longrightarrow \mathrm{Hom}(B, C') \longrightarrow \ldots$$

Then $g' \circ \beta \in \mathrm{Hom}(B, C')$ goes to $g' \circ \beta \circ f = 0 \in \mathrm{Hom}(A, C')$, hence we get γ, which has to be unique by assumption.

Definition. Let $F \colon \mathcal{D} \longrightarrow \mathcal{D}'$ be an additive functor. Then F is called *exact* if

(1) There exists an isomorphism of functors $F \circ T_{\mathcal{D}} \simeq T_{\mathcal{D}'} \circ F$.
(2) Any distinguished triangle $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ in $\mathcal{D}$ is mapped to a distinguished triangle $F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow F(A)[1]$ in $\mathcal{D}'$, where we identify $F(A'[1])$ with $F(A')[1]$ via the functor isomorphism from (i).

Definition. A *triangulated subcategory* $\mathcal{D}'$ of $\mathcal{D}$ is a full additive subcategory admitting the structure of a triangulated category such that the inclusion functor $\mathcal{D}' \subset \mathcal{D}$ is exact and every object isomorphic to an object of $\mathcal{D}'$ is in $\mathcal{D}'$.

Remark 1.14. The above definition is equivalent to the following one. The category $\mathcal{D}'$ is invariant under shift and if in a triangle two out of three objects are in $\mathcal{D}'$, then so is the third.

Also note that we could equally well define a triangulated subcategory without the assumption that it is closed under isomorphisms.

Example 1.15. Let $F \colon \mathcal{D} \longrightarrow \mathcal{D}'$ be an exact functor. Then $\ker(F) := \{A \in \mathcal{D} \mid F(A) \simeq 0\}$ is a triangulated subcategory of $\mathcal{D}$. Furthermore, if $A \oplus B \in \ker(F)$, then A and B are in $\ker(F)$.

The second assertion being trivial, we prove the first one. Clearly, $\ker(F)$ is invariant under shift. If we take a triangle in $\mathcal{D}$ and two of the three objects have the property that their image is isomorphic to 0, the same holds for the third.

Definition. Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a functor between arbitrary categories. We say that a functor $H\colon \mathcal{B} \longrightarrow \mathcal{A}$ is *right adjoint* to F if there exist functorial isomorphisms

$$\mathrm{Hom}_{\mathcal{B}}(F(A), B) \simeq \mathrm{Hom}_{\mathcal{A}}(A, H(B))$$

for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A functor $G\colon \mathcal{B} \longrightarrow \mathcal{A}$ is *left adjoint* to F if there exist functorial isomorphisms

$$\mathrm{Hom}_{\mathcal{B}}(B, F(A)) \simeq \mathrm{Hom}_{\mathcal{A}}(G(B), A)$$

for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Note that given an $A \in \mathcal{A}$, we can put $B = F(A)$ and then the first equation gives

$$\mathrm{Hom}_{\mathcal{B}}(F(A), F(A)) \simeq \mathrm{Hom}_{\mathcal{A}}(A, HF(A)),$$

so we get a natural transformation $\mathrm{id}_{\mathcal{A}} \longrightarrow HF$. Similarly, we also get a natural transformation $FH \longrightarrow \mathrm{id}_{\mathcal{B}}$.

Proposition 1.16. *Let $F\colon \mathcal{D} \longrightarrow \mathcal{D}'$ be an exact functor between triangulated categories. If a left (or right) adjoint functor exists, then it is exact.*

Proof. We will only consider the case of a right adjoint functor H. Let us first check that it commutes with the respective shifts. Since F is exact, we have a functorial isomorphism $F \circ T_{\mathcal{D}} \simeq T' \circ F$ and similarly for the inverse of the shift. This yields the following functorial isomorphisms

$$\begin{aligned}\mathrm{Hom}(A, H(T'(B))) &= \mathrm{Hom}(F(A), T'(B)) \simeq \mathrm{Hom}(T'^{-1}F(A), B)\\ &= \mathrm{Hom}(FT^{-1}(A), B) \simeq \mathrm{Hom}(T^{-1}(A), H(B))\\ &= \mathrm{Hom}(A, TH(B)).\end{aligned}$$

By the Yoneda lemma we get an isomorphism $HT' \simeq TH$.

Now we have to show that H maps an exact triangle in $\mathcal{D}'$ to an exact triangle in $\mathcal{D}$. Let $A' \longrightarrow B' \longrightarrow C' \longrightarrow A'[1]$ be a triangle in $\mathcal{D}'$ and complete the induced morphism $H(A') \longrightarrow H(B')$ to a triangle in $\mathcal{D}$:

$$H(A') \longrightarrow H(B') \longrightarrow C_0 \longrightarrow H(A')[1],$$

where we tacitly use the isomorphism $HT' \simeq TH$. Using the the assumption that F is exact and the adjunction morphisms $FH(A') \longrightarrow A'$ and $FH(B') \longrightarrow B'$, one obtains a morphism of distinguished triangles

$$\begin{tikzcd} FH(A') \arrow[r] \arrow[d] & FH(B') \arrow[r] \arrow[d] & F(C_0) \arrow[r] \arrow[d, dotted, "\xi"] & FH(A')[1] \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1], \end{tikzcd}$$

where the dotted line exists by TR3.

Applying H to the diagram and using the adjunction $h\colon \mathrm{id} \longrightarrow H\circ F$, yields

$$\begin{tikzcd}
H(A') \arrow[r] \arrow[d] \arrow[dd, bend right=60] & H(B') \arrow[r] \arrow[d] \arrow[dd, bend right=60] & C_0 \arrow[r] \arrow[d, "h_{C_0}"] & H(A')[1] \arrow[d] \\
HFH(A') \arrow[r] \arrow[d] & HFH(B') \arrow[r] \arrow[d] & HF(C_0) \arrow[r] \arrow[d, "H(\xi)"] & HFH(A')[1] \arrow[d] \\
H(A') \arrow[r] & H(B') \arrow[r] & H(C') \arrow[r] & H(A')[1].
\end{tikzcd}$$

It is a fact from category theory that the curved arrows are identity morphisms. Now, for any A_0 we consider the sequence

$$\mathrm{Hom}(A_0, H(B')) \longrightarrow \mathrm{Hom}(A_0, H(C')) \longrightarrow \mathrm{Hom}(A_0, H(A')[1])$$

and note that it is exact due to adjunction and the exactness of F (this sequence is isomorphic to the one we get if we apply $\mathrm{Hom}(F(A_0), -)$ to our original exact triangle). Then $\mathrm{Hom}(A_0, C_0) \simeq \mathrm{Hom}(A_0, H(C'))$ for all A_0 and hence $H(\xi)\circ h_{C_0}\colon C_0 \longrightarrow H(C')$ is an isomorphism. Thus, $H(A') \longrightarrow H(B') \longrightarrow H(C') \longrightarrow H(A')[1]$ is isomorphic to the distinguished triangle $H(A') \longrightarrow H(B') \longrightarrow C_0 \longrightarrow H(A')[1]$ and hence is distinguished. $\square$

Remark 1.17. We will see later (Remark 3.24) that a similar statement does not hold in the abelian setting. Roughly, one uses that the tensor product functor is adjoint to the Hom-functor (on the category of abelian groups, say) and the former can be exact while the latter is only half-exact.

2. A first example: The homotopy category

Definition. Let $\mathcal{A}$ be an additive category. A *complex* in $\mathcal{A}$ is a diagram of objects and morphisms in $\mathcal{A}$

$$\ldots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \ldots$$

such that $d^{k+1}\circ d^k = 0$ for all $k\in\mathbb{Z}$. One often writes a complex as $A^\bullet$.

A complex is *bounded* if $A^k = 0$ for $|k| \gg 0$. It is *bounded below* if $A^k = 0$ for $k \ll 0$ and it is *bounded above* if $A^k = 0$ for $k \gg 0$.

A morphism between two complexes $A^\bullet$ and $B^\bullet$ consists of a collection of maps $f^i\colon A^k \longrightarrow B^k$ for all $k\in\mathbb{Z}$ such that $d^k_{B^\bullet} f^k = f^{k+1} d^k_{A^\bullet}$ for all $k\in\mathbb{Z}$. The category of all complexes in $\mathcal{A}$ will be denoted by $\mathrm{Kom}(\mathcal{A})$, its full subcategory of bounded complexes by $\mathrm{Kom}^b(\mathcal{A})$, the category of bounded below complexes by $\mathrm{Kom}^+(\mathcal{A})$ and the category of bounded above complexes by $\mathrm{Kom}^-(\mathcal{A})$.

Remark 2.1. It is straightforward to check that $\mathrm{Kom}(\mathcal{A})$ is an additive category.

Remark 2.2. Note that complexes naturally arise in, for example, topology. If X is a CW-complex, we can consider singular chains. If X is a differentiable manifold, one can consider the de Rham complex.

Definition. Let $A^\bullet$ be a complex and let n be an integer. We define the complex $A^\bullet[n]$ by setting $A^\bullet[n]^k = A^{k+n}$ and $d^k_{A^\bullet[n]} = (-1)^n d^{k+n}_{A^\bullet}$. For a morphism $f\colon A^\bullet \longrightarrow B^\bullet$ one defines $f[n]\colon A^\bullet[n] \longrightarrow B^\bullet[n]$ by setting $f[n]^k = f^{n+k}$.

Remark 2.3. One checks easily that $[n]$, the n-th shift functor, defines an autoequivalence of $\mathrm{Kom}(\mathcal{A})$ with inverse $[-n]$.

Definition. A morphism $f\colon A^\bullet \longrightarrow B^\bullet$ is called *homotopic to zero* if, for all $k \in \mathbb{Z}$, there exist maps $s^k\colon A^k \longrightarrow B^{k-1}$ such that for any k we have $f^k = d^{k-1}_{B^\bullet} \circ s^k + s^{k+1} \circ d^k_{A^\bullet}$. Two maps f and g are homotopic if $f - g$ is homotopic to zero.

Proposition 2.4. *Being homotopic is an equivalence relation. The elements homotopic to zero form an "ideal".*

Proof. The first statement is trivial. As for the second: Take, for example, a morphism $f\colon A^\bullet \longrightarrow B^\bullet$ and a morphism $g\colon B^\bullet \longrightarrow C^\bullet$ which is homotopic to zero. We have to check that $g \circ f$ is homotopic to zero. By definition, $g^k = d^{k-1}_{C^\bullet} \circ s^k + s^{k+1} \circ d^k_{B^\bullet}$ for all k. But then

$$g^k \circ f^k = d^{k-1}_{C^\bullet} \circ s^k \circ f^k + s^{k+1} \circ d^k_{B^\bullet} \circ f^k = d^{k-1}_{C^\bullet} \circ s^k \circ f^k + s^{k+1} f^{k+1} d^k_{C^\bullet}.$$

Setting $(s')^k = s^k \circ f^k$ for all $k \in \mathbb{Z}$ shows that $g \circ f$ is homotopic to zero. The proof that $g \circ f$ is homotopic to zero if f is such is completely analogous and left to the reader. $\square$

This result allows us to define a new category as follows.

Definition. Let $\mathcal{A}$ be an additive category and $\mathrm{Kom}(\mathcal{A})$ the category of complexes over $\mathcal{A}$. The *homotopy category* $K(\mathcal{A})$ has the same objects as $\mathrm{Kom}(\mathcal{A})$ and for any two objects $A^\bullet$ and $B^\bullet$ the space of morphisms is $\mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A^\bullet, B^\bullet)/\sim$, where $\sim$ is the equivalence relation of being homotopic. We denote the appropriate bounded subcategories by $K^*(\mathcal{A})$, where $* \in \{-, +, b\}$.

Remark 2.5. The shift functor clearly gives an autoequivalence of $K(\mathcal{A})$.

Definition. Let $f\colon A^\bullet \longrightarrow B^\bullet$ be a morphism in $\mathrm{Kom}(\mathcal{A})$. The *mapping cone* of f, denoted by $C(f)$, is the following object of $\mathrm{Kom}(\mathcal{A})$:

$$C(f)^k = A^{k+1} \oplus B^k$$

$$d^k_{C(f)} = \begin{pmatrix} -d^{k+1}_{A^\bullet} & 0 \\ f^{k+1} & d^k_{B^\bullet} \end{pmatrix}$$

One can easily check that this is well-defined, that is, $C(f)$ is really a complex.

We have morphisms $\alpha(f)\colon B^\bullet \longrightarrow C(f)$ and $\beta(f)\colon C(f) \longrightarrow A^\bullet[1]$, where $\alpha(f)$ is the natural injection and $\beta(f)$ is the natural projection. It is again easily checked that these are morphisms of complexes.

Lemma 2.6. *For any $f\colon A^\bullet \longrightarrow B^\bullet$ in* $\mathrm{Kom}(\mathcal{A})$ *there exists a $\phi\colon A^\bullet[1] \longrightarrow C(\alpha(f))$ such that*

(1) *ϕ is an isomorphism in $K(\mathcal{A})$.*

(2) *The following diagram commutes in $K(\mathcal{A})$:*

$$\begin{tikzcd}
B^\bullet \arrow[r, "\alpha(f)"] \arrow[d, "\mathrm{id}"] & C(f) \arrow[r, "\beta(f)"] \arrow[d, "\mathrm{id}"] & A^\bullet[1] \arrow[r, "-f[1]"] \arrow[d, "\phi"] & B^\bullet[1] \arrow[d, "\mathrm{id}"] \\
B^\bullet \arrow[r, "\alpha(f)"] & C(f) \arrow[r, "\alpha(\alpha(f))"] & C(\alpha(f)) \arrow[r, "\beta(\alpha(f))"] & B^\bullet[1]
\end{tikzcd}$$

Proof. First note that

$$C(\alpha(f))^k = B^{k+1} \oplus C(f)^k = B^{k+1} \oplus A^{k+1} \oplus B^k.$$

Define $\phi^k \colon A[1]^k \longrightarrow C(\alpha(f))^k$ and $\psi^k \colon C(\alpha(f))^k \longrightarrow A[1]^k$ by

$$\phi^k = \begin{pmatrix} -f^{k+1} \\ \mathrm{id}_{A^{k+1}} \\ 0 \end{pmatrix}$$

and

$$\psi^k = (0, \mathrm{id}_{A^{k+1}}, 0).$$

Then we have the following

(1) ϕ and ψ are morphisms of complexes.
(2) $\psi \circ \phi = \mathrm{id}_{A^\bullet[1]}$.
(3) $\phi \circ \psi$ is homotopic to $\mathrm{id}_{C(\alpha(f))}$.
(4) $\psi \circ \alpha(\alpha(f)) = \beta(f)$.
(5) $\beta(\alpha(f)) \circ \phi = -f[1]$.

For (1) note that the differential of $C(\alpha(f))$ can be written as

$$d^k_{C(\alpha(f))} = \begin{pmatrix} -d^{k+1}_{B^\bullet} & 0 & 0 \\ 0 & -d^{k+1}_{A^\bullet} & 0 \\ \mathrm{id}^{k+1}_{B^\bullet} & f^{k+1} & d^k_{B^\bullet} \end{pmatrix}$$

Items (2), (4) and (5) are trivial. To get (3) one defines $s^k \colon C(\alpha(f))^k \longrightarrow C(\alpha(f))^{k-1}$ by

$$s^k = \begin{pmatrix} 0 & 0 & \mathrm{id}_{B^k} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and checks

$$\mathrm{id}_{C(\alpha(f))^k} - \phi^k \circ \psi^k(b^{k+1}, a^{k+1}, b^k) = (b^{k+1}, a^{k+1}, b^k) - (-f^{k+1}(a^{k+1}), a^{k+1}, 0)$$

which is equal to

$$s^{k+1} \circ d^k_{C(\alpha(f))}(b^{k+1}, a^{k+1}, b^k) + d^{k-1}_{C(\alpha(f))} \circ s^k(b^{k+1}, a^{k+1}, b^k),$$

since the first term is $(b^{k+1} + f^{k+1}(a^{k+1}) + d^k(b^k), 0, 0)$ and the second is $(-d^k b^k, 0, b^k)$. $\square$

Definition. A sequence $A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow A^\bullet[1]$ in $K(\mathcal{A})$ is a *distinguished triangle* if it is isomorphic to a sequence of the form $A'^\bullet \xrightarrow{f} B'^\bullet \xrightarrow{\alpha(f)} C(f) \xrightarrow{\beta(f)} A'^\bullet[1]$ for some $f \in \mathrm{Kom}(\mathcal{A})$.

Theorem 2.7. *The category $K(\mathcal{A})$ with the shift functor $[1]$ and distinguished triangles as in the previous definition is a triangulated category.*

Proof. TR1 (ii) and (iii) are obvious and TR2 follows from Lemma 2.6. Since the cone of $0 \longrightarrow X$ is X, we can apply TR2 to get TR1 (i).

Let us prove TR3. We can assume that we are in the following situation

$$\begin{tikzcd} A^\bullet \arrow[r, "f"] \arrow[d, "\phi"] & B^\bullet \arrow[r, "\alpha(f)"] \arrow[d, "\psi"] & C(f) \arrow[r, "\beta(f)"] & A^\bullet[1] \arrow[d, "\phi[1]"] \\ A'^\bullet \arrow[r, "f'"] & B'^\bullet \arrow[r, "\alpha(f')"] & C(f') \arrow[r, "\beta(f')"] & A'^\bullet[1]. \end{tikzcd}$$

We have to construct a map $\omega\colon C(f) \longrightarrow C(f')$ such that

$$\omega \circ \alpha(f) = \alpha(f') \circ \psi \ \text{ and } \ \phi[1] \circ \beta(f) = \beta(f') \circ \gamma. \tag{2.1}$$

We know that $\psi \circ f$ is homotopic to $f' \circ \phi$, thus, by definition, there exist maps $s^k\colon A^k \longrightarrow B'^{k-1}$ such that $\psi^k \circ f^k - f'^k \circ \phi^k = s^{k+1} \circ d^k_{A^\bullet} + d^{k-1}_{B'^\bullet} \circ s^k$. Define $\omega^k\colon C(f)^k = A^{k+1} \oplus B^k \longrightarrow C(f')^k = A'^{k+1} \oplus B'^k$ by

$$\omega^k = \begin{pmatrix} \phi^{k+1} & 0 \\ s^{k+1} & \psi^k \end{pmatrix}.$$

It is straightforward to check that this is a morphism of complexes and that it satisfies Equation 2.1.

Let us now prove TR4. Recall that we are given maps $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$. We may assume that $C' = C(f)$, $B' = C(g \circ f)$ and $A' = C(g)$. We, in particular, have to construct a distinguished triangle $C(f) \longrightarrow C(g \circ f) \longrightarrow C(g)$. We define $u\colon C(f) \longrightarrow C(g \circ f)$ and $v\colon C(g \circ f) \longrightarrow C(g)$ by

$$u^k\colon A^{k+1} \oplus B^k \longrightarrow A^{k+1} \oplus C^k, \ \ u^k = \begin{pmatrix} \mathrm{id}_{A^{k+1}} & 0 \\ 0 & g^k \end{pmatrix},$$

$$v^k\colon A^{k+1} \oplus C^k \longrightarrow B^{k+1} \oplus C^k, \ \ v^k = \begin{pmatrix} f^{k+1} & 0 \\ 0 & \mathrm{id}_{C^k} \end{pmatrix}.$$

The map $w\colon A' \longrightarrow C'[1]$ is defined as the composition $A' \longrightarrow B[1] \longrightarrow C'[1]$, thus it corresponds to the matrix $\begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0 \end{pmatrix}$. It is then straightforward that the diagram commutes (tedious but doable exercise!) and we are left with checking that $C(f) \xrightarrow{u} C(g \circ f) \xrightarrow{v} C(g) \xrightarrow{w} C(f)[1]$ is a distinguished triangle. We will construct an isomorphism $\phi\colon C(u) \longrightarrow C(g)$ and its inverse $\psi\colon C(g) \longrightarrow C(u)$ such that $\phi \circ \alpha(u) = v$ and $\beta(u) \circ \psi = w$. Note that

$$C(u)^k = C(f)^{k+1} \oplus C(g \circ f)^k = A^{k+2} \oplus B^{k+1} \oplus A^{k+1} \oplus C^k$$

and

$$C(g)^k = B^{k+1} \oplus C^k.$$

We define ϕ and ψ by

$$\phi^k = \begin{pmatrix} 0 & \mathrm{id}_{B^{k+1}} & f^{k+1} & 0 \\ 0 & 0 & 0 & \mathrm{id}_{C^k} \end{pmatrix}$$

and

$$\psi^k = \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{B^{k+1}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{C^k} \end{pmatrix}.$$

It is straightforward to check that ϕ and ψ are morphisms of complexes using that

$$d^k_{C(u)} = \begin{pmatrix} d^{k+2}_{A^\bullet} & 0 & 0 & 0 \\ -f^{k+2} & -d^{k+1}_{B^\bullet} & 0 & 0 \\ \mathrm{id}_{A^{k+2}} & 0 & -d^{k+1}_{A^\bullet} & 0 \\ 0 & g^{k+1} & g^{k+1} \circ f^{k+1} & d^k_{C^\bullet} \end{pmatrix}.$$

It is also immediate that the wanted commutativity holds. Furthermore, $\phi \circ \psi = \mathrm{id}_{C(g)}$. Lastly, if one defines

$$s^k \colon C(u)^k \longrightarrow C(u)^{k-1}\ , \quad s^k = \begin{pmatrix} 0 & 0 & \mathrm{id}_{A^{k+1}} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

one computes

$$(\mathrm{id}_{C(u)} - \psi \circ \phi)^k = s^{k+1} \circ d^k_{C(u)} + d^{k-1}_{C(u)} \circ s^k.$$

Hence, $\psi \circ \phi = \mathrm{id}_{C(u)}$ in $K(\mathcal{A})$. □

3. Localization and the derived category

In this section we will assume that the category $\mathcal{A}$ is abelian. Recall that an additive category is abelian if kernels and cokernels exist and the cokernel of a kernel is isomorphic to the kernel of the cokernel. Recall that the kernel of a map $f \colon A \longrightarrow B$ in a category is defined to be a pair (K, k) where K is an object and $k \colon K \longrightarrow A$ is a map with $f \circ k = 0$ and with the following property: Whenever there exists a map $h \colon C \longrightarrow A$ such that $f \circ h = 0$, then there exists a unique map $l \colon C \longrightarrow K$ such that $k \circ l = h$. Note that this immediately implies that the kernel is unique up to a unique isomorphism. The cokernel is defined by reverting all arrows.

As an example one can consider the category of abelian groups, where the kernel and cokernel are defined in the usual fashion and the last condition can be reformulated as follows. Given $f \colon A \longrightarrow B$, we have $A/\ker(f) \simeq \mathrm{im}(f)$. An example of a category which is additive but not abelian is the category of free abelian groups of finite rank.

Remark 3.1. Most notions one uses in abelian categories, such as kernels or cokernels, are defined via diagrams and universal properties and properties of these constructions can be verified in this abstract fashion. However, after a certain point the verifications in this fashion become fairly cumbersome. Therefore, one usually uses the Freyd–Mitchell embedding theorem which roughly states that any abelian category can be embedded into the abelian category of modules over some ring. Hence we are allowed to use elements and will do so in the following.

Note that $\mathrm{Kom}(\mathcal{A})$ is an abelian category if $\mathcal{A}$ is one.

Definition. Let $A^\bullet$ be a complex in $\mathrm{Kom}(\mathcal{A})$. The k-th *cohomology* of $A^\bullet$ is defined to be $H^k(A^\bullet) = \ker(d^k)/\operatorname{im}(d^{k-1})$. A complex is *acyclic* if $H^k(A^\bullet) = 0$ for all $k \in \mathbb{Z}$. Noting that any morphism of complexes $f\colon A^\bullet \to B^\bullet$ defines a map $H^k(f)\colon H^k(A^\bullet) \to H^k(B^\bullet)$ (exercise!), we call f a *quasi-isomorphism* (short: qis) if $H^k(f)$ is an isomorphism for all $k \in \mathbb{Z}$.

Proposition 3.2. *Let $f\colon A^\bullet \to B^\bullet$ be a morphism of complexes which is homotopic to zero. Then $H^k(f) = 0$ for all $k \in \mathbb{Z}$. In particular, if f and g are homotopic, then $H^k(f) = H^k(g)$ for all $k \in \mathbb{Z}$.*

Proof. By assumption, $f = ds + sd$. Take a representative of an element $x \in H^k(A^\bullet)$, then $d(x) = 0$, hence $f(x) = ds(x)$ which is zero in $H^k(B^\bullet)$. □

Lemma 3.3. *Let $\mathcal{A}$ be an abelian category and assume that there is the following commutative diagram with exact rows*

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C'. &
\end{tikzcd}$$

Then there exists a natural exact sequence

$$\ker(\alpha) \longrightarrow \ker(\beta) \longrightarrow \ker(\gamma) \xrightarrow{\phi} \operatorname{coker}(\alpha) \longrightarrow \operatorname{coker}(\beta) \longrightarrow \operatorname{coker}(\gamma)$$

so that the following diagram commutes

$$\begin{tikzcd}
 & B \arrow[r, "g"] & C \\
B \arrow[d, "\beta"] & \ker(\gamma \circ g) \arrow[l] \arrow[u] \arrow[r, "g"] \arrow[d, "\beta"] & \ker(\gamma) \arrow[u] \arrow[d, "\phi"] \\
B' & A' \arrow[l, "f'"'] \arrow[r] & \operatorname{coker}(\alpha).
\end{tikzcd}$$

Proof. The existence of the exact sequence is proved by diagram chasing. For example, the first two maps are just induced by f resp. g and the exactness is clear. We have $A' \to B' \to B'/\operatorname{im}(\beta)$ and it is straightforward to check that $\operatorname{im}(\alpha) \subset A'$ is in the kernel of this map. This argument shows the existence of the last two maps. For the existence of ϕ one starts as follows. Take

$c \in C$ with $\gamma(c) = 0$ and an element $b \in B$ such that $g(b) = c$. Then $g'\beta(b) = 0$, hence $\beta(b) \in \ker(g') = \operatorname{im}(f')$, thus $\beta(b) = f'(a')$ for some $a' \in A'$. So we send c to a'. It remains to check that this procedure is well-defined and this is left to the reader. The commutativity of the diagram is straightforward. $\square$

Proposition 3.4. *Let* $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$ *be an exact sequence in* $\mathrm{Kom}(\mathcal{A})$. *Then there exists a canonical long exact sequence in* $\mathcal{A}$*:*

$$\ldots \longrightarrow H^k(A^\bullet) \longrightarrow H^k(B^\bullet) \longrightarrow H^k(C^\bullet) \longrightarrow H^{k+1}(A^\bullet) \longrightarrow \ldots$$

Proof. There are exact sequences

$$A^{k-1} \xrightarrow{d^{k-1}} \ker(d^k_{A^\bullet}) \longrightarrow H^k(A^\bullet) \longrightarrow 0\ ,$$

$$0 \longrightarrow H^k(A^\bullet) \longrightarrow \operatorname{coker}(d^{k-1}_{A^\bullet}) = A^k/\operatorname{im}(d^{k-1}) \longrightarrow A^{k+1}\ ,$$

$$0 \longrightarrow \ker(d^{k-1}_{A^\bullet}) \longrightarrow A^{k-1} \longrightarrow \operatorname{im}(d^{k-1})\ ,$$

$$0 \longrightarrow H^k(A^\bullet) \longrightarrow \operatorname{coker}(d^{k-1}_{A^\bullet}) \xrightarrow{d^k_{A^\bullet}} \ker(d^{k+1}_{A^\bullet}) \longrightarrow H^{k+1}(A^\bullet) \longrightarrow 0\ . \tag{3.1}$$

Now consider the following commutative diagram with exact rows

$$\begin{tikzcd}
 & \operatorname{coker}(d^{k-1}_{A^\bullet}) \arrow[r] \arrow[d, "d^k_{A^\bullet}"] & \operatorname{coker}(d^{k-1}_{B^\bullet}) \arrow[r] \arrow[d, "d^k_{B^\bullet}"] & \operatorname{coker}(d^{k-1}_{C^\bullet}) \arrow[r] \arrow[d, "d^k_{C^\bullet}"] & 0 \\
0 \arrow[r] & \ker(d^{k+1}_{A^\bullet}) \arrow[r] & \ker(d^{k+1}_{B^\bullet}) \arrow[r] & \ker(d^{k+1}_{C^\bullet}) &
\end{tikzcd}$$

Applying Lemma 3.3 to the diagram we get a short exact sequence for the H^k and applying (3.1) we get the connection homomorphism. $\square$

Definition. Let $\mathcal{D}$ be a triangulated and $\mathcal{A}$ be an abelian category. An additive functor $F\colon \mathcal{D} \longrightarrow \mathcal{A}$ is called *cohomological* if for any distinguished triangle $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ the sequence $F(A) \longrightarrow F(B) \longrightarrow F(C)$ is exact.

Writing F^k for $F \circ T^k$ we obtain a long exact sequence.

Example 3.5. By Proposition 1.4 the functors $\operatorname{Hom}(A_0, -)$ and $\operatorname{Hom}(-, A_0)$ are cohomological for any object $A_0 \in \mathcal{D}$.

Lemma 3.6. *Let* $\mathcal{A}$ *be an abelian category. Then* $H^0\colon K(\mathcal{A}) \longrightarrow \mathcal{A}$ *is a cohomological functor.*

Proof. It suffices to show that if $f\colon A^\bullet \longrightarrow B^\bullet$ is a morphism in $\mathrm{Kom}(\mathcal{A})$, then the sequence

$$H^0(B^\bullet) \longrightarrow H^0(C(f)) \longrightarrow H^0(A^\bullet[1])$$

is exact. Since the sequence $0 \longrightarrow B^\bullet \longrightarrow C(f) \longrightarrow A^\bullet[1] \longrightarrow 0$ is exact in $\mathrm{Kom}(\mathcal{A})$ the result follows from Proposition 3.4. $\square$

Proposition 3.7. *A morphism $f\colon A^\bullet \longrightarrow B^\bullet$ is a quasi-isomorphism if and only if $C(f)$ is an acyclic complex.*

Proof. Follows from the exact sequence associated to the cohomological functor H^0 and the fact that $H^0(A^\bullet[k]) = H^k(A^\bullet)$. $\square$

We now want to have a category where all the quasi-isomorphisms become invertible. This is achieved by a process called localization.

Let $\mathcal{C}$ be a category and let S be a family of morphisms in $\mathcal{C}$.

Definition. The family S is a *multiplicative system* if it satisfies the following conditions.

(S1) For any $X \in \mathcal{C}$, id_X is in S.
(S2) If f and g are in S and $g \circ f$ exists, then also $g \circ f$ in S.
(S3) Any diagram

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with $g \in S$ can be completed to a commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d, "h"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with $h \in S$. Similarly, with all arrows reversed.
(S4) If $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ then the following conditions are equivalent
  (a) There exists $t\colon Y \longrightarrow Y'$, $t \in S$, such that $t \circ f = t \circ g$.
  (b) There exists $s\colon X' \longrightarrow X$, $s \in S$, such that $g \circ s = f \circ s$.

Definition. Let $\mathcal{C}$ be a category and S a multiplicative system. We define the localization of $\mathcal{C}$, denoted by $\mathcal{C}_S$, as the category having the same objects and where the morphisms are given as follows. Let X, Y be objects of $\mathcal{C}$, then

$$\mathrm{Hom}_{\mathcal{C}_S}(X, Y) = \big\{(X', s, f) \mid X' \in \mathcal{C}, s\colon X' \longrightarrow X, f\colon X' \longrightarrow Y, s \in S\big\} / \sim,$$

where $\sim$ is the following equivalence relation

$$(X', s, f) \sim (X'', t, g)$$

if and only if there exists a commutative diagram

$$\begin{tikzcd} & X & \\ X' \arrow[ur, "s"] \arrow[dr, "f"'] & X''' \arrow[l] \arrow[r] \arrow[u, "u"] & X'' \arrow[ul, "t"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

with $u \in S$.

The composition of $(X', s, f) \in \mathrm{Hom}_{\mathcal{C}_S}(X, Y)$ and $(Y', t, g) \in \mathrm{Hom}_{\mathcal{C}_S}(Y, Z)$ is defined as follows. We use (S3) to find a commutative diagram

$$\begin{tikzcd} & & X'' \arrow[dl, "t'"'] \arrow[dr, "h"] & & \\ & X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

and set $(Y', t, g) \circ (X', s, f) = (X'', s \circ t', g \circ h)$.

Remark 3.8. It is sometimes convenient to work with the equivalent definition where one reverses all the arrows. Since in the definition of a multiplicative system both directions were required, all proofs done below in one case also work in the other case.

It is easy but tedious to check that $\mathcal{C}_S$ is indeed a category. We define a functor $Q\colon \mathcal{C} \longrightarrow \mathcal{C}_S$ to be the identity on objects and to send a morphism $f\colon X \longrightarrow Y$ to (X, id_X, f). If $\mathcal{C}$ is an additive category, then so is $\mathcal{C}_S$. Indeed, the zero object is $Q(0)$, the product of two objects X and Y is $Q(X \oplus Y)$ (exercise!). Morphisms are added as follows. Let (X', s, f) and (X'', s', f') be two maps from X to Y in $\mathcal{C}_S$. Considering the diagram

$$\begin{tikzcd} & X'' \arrow[d, "s'"] \\ X' \arrow[r, "s"] & X, \end{tikzcd}$$

one finds an object U and maps $r\colon U \longrightarrow X'$ and $r'\colon U \longrightarrow X''$ completing it to a commutative diagram. Note that both r, r' are in S. One then readily checks that (X', s, f) is equivalent to $(U, s \circ r, f \circ r)$ and that (X'', s', f') is equivalent to $(U, s' \circ r', f' \circ r')$. Noting that $s \circ r = s' \circ r'$, one defines the addition of the maps as the roof $(U, s \circ r, f \circ r + f' \circ r')$. We leave it to the reader to check that this is well-defined.

Proposition 3.9. *For any $s \in S$, $Q(s)$ is an isomorphism in $\mathcal{C}_S$. If $\mathcal{C}'$ is a category and $F\colon \mathcal{C} \longrightarrow \mathcal{C}'$ is a functor such that $F(s)$ is an isomorphism for any $s \in S$, then F factors through $\mathcal{C}_S$.*

Proof. By definition, $Q(s)$ is (X, id_X, s). The inverse is the diagram

$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "\mathrm{id}"] & \\ Y & & X. \end{tikzcd}$$

To see the second assertion, we first show that if a functor $G\colon \mathcal{C}_S \longrightarrow \mathcal{C}'$ with $G \circ Q = F$ exists, then it must be unique. Indeed, the objects of $\mathcal{C}$ and $\mathcal{C}_S$ are identical, so on objects it has to be equal to F. Furthermore, let (Z, s, f) be a map in $\mathcal{C}_S$. It is immediate that $(Z, s, f) \circ Q(s) = Q(f)$. Applying G to this we get $G(Z, s, f) \circ F(s) = F(f)$ and, since $F(s)$ is invertible in $\mathcal{C}'$, we have $G(Z, s, f) = F(f) \circ (F(s))^{-1}$. We can define G by these properties. It is straightforward to check that this is well-defined. □

Proposition 3.10. *Let $\mathcal{C}$ be a category, $\mathcal{C}'$ a full subcategory, S be a multiplicative system and S' the family of morphisms of $\mathcal{C}'$ which belong to S. Assume that S' is a multiplicative system in $\mathcal{C}'$ and that, moreover, one of the following conditions holds:*

1. *Whenever $s\colon X \longrightarrow Y$ is a morphism in S with $Y \in \mathcal{C}'$, there exists a $g\colon W \longrightarrow X$ with $W \in \mathcal{C}'$ and $s \circ g \in S$,*
2. *The same as in (i) but with arrows reversed.*

Then $\mathcal{C}'_{S'}$ is a full subcategory of $\mathcal{C}_S$.

Proof. Let us show that the inclusion functor is full. Let (X', s, f) be a map in $\mathcal{C}_S$ from X to Y with $X, Y \in \mathcal{C}'$. By (1), there exists $g\colon W \longrightarrow X'$ such that $s \circ g$ is in S'. Consider the morphism $(W, s \circ g, f \circ g)$ from X to Y. Note that since $\mathcal{C}' \subset \mathcal{C}$ is full, the map $f \circ g$ is a map in $\mathcal{C}'$. But these morphisms are equivalent in $\mathcal{C}_S$ because of the following diagram

$$\begin{tikzcd}
& X & \\
X' \arrow[ur, "s"] \arrow[dr, "f"'] & W \arrow[l, "g"'] \arrow[u, "s\circ g"'] \arrow[r, "\mathrm{id}"] & W \arrow[ul, "s\circ g"'] \arrow[dl, "f\circ g"] \\
& Y &
\end{tikzcd}$$

The faithfulness of the inclusion functor is similarly straightforward. □

Definition. Let $\mathcal{D}$ be a triangulated category and let $\mathcal{N}$ be a family of objects in $\mathcal{D}$. Then $\mathcal{N}$ is called a *null system* if it satisfies the following conditions

(N1) $0 \in \mathcal{N}$.
(N2) $A \in \mathcal{N}$ if and only if $A[1] \in \mathcal{N}$.
(N3) If $A \longrightarrow B \longrightarrow C \longrightarrow A[1]$ is a distinguished triangle and $A, B \in \mathcal{N}$, then also $C \in \mathcal{N}$.

Note that (N3) combined with (N2) (and rotation of triangles) says that if two out of three objects in a triangle are in $\mathcal{N}$ then so is the third.

We now set

$$S(\mathcal{N}) = \{f\colon A \longrightarrow B \mid C(f) \in \mathcal{N}\}$$

Proposition 3.11. *If $\mathcal{N}$ is a null system, then $S(\mathcal{N})$ is a multiplicative system.*

Proof. (S1) is clear, since the cone of the identity map of an element is the zero object and hence in $\mathcal{N}$. (S2) follows immediately from the octahedral axiom and (N3). Let us prove (S3).

Let

$$\begin{tikzcd} & C \arrow[d, "g"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

be a diagram with $g \in S(\mathcal{N})$, thus we have a triangle

$$C \longrightarrow B \longrightarrow C(g) \longrightarrow C[1].$$

Consider the map $\alpha(g) \circ f \colon A \longrightarrow C(g)$ and take its cone. We have a diagram

$$\begin{tikzcd} D \arrow[r] & A \arrow[r, "\pi\circ f"] \arrow[d, "f"] & C(g) \arrow[r] \arrow[d, "\mathrm{id}"] & D[1] \\ C \arrow[r, "g"] & B \arrow[r, "\pi"] & C(g) \arrow[r] & C[1]. \end{tikzcd}$$

By (TR3), this diagram can be completed to a morphism of triangles and the left square gives us (S3). The same proof works when one reverses all the arrows.

Finally, let us consider (S4). Assume that $f \colon A \longrightarrow B$ is a morphism such that there exists $t \colon B \longrightarrow B'$ with $t \circ f = 0$ and $t \in S$. We have to show that there exists a $s \colon A' \longrightarrow A$ with $f \circ s = 0$. Let

$$C \xrightarrow{g} B \xrightarrow{t} B' \longrightarrow C[1],$$

with $C \in \mathcal{N}$. Then consider

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & A[1] \\ C \arrow[r, "g"] & B \arrow[r, "t"] & B' \arrow[r] & C[1], \end{tikzcd}$$

complete it to a morphism of triangles, so, in particular, we get a map $h \colon A \longrightarrow C$ such that $f = g \circ h$. Then look at the distinguished triangle

$$A' \xrightarrow{s} A \xrightarrow{h} C \longrightarrow A'[1].$$

We have $f \circ s = g \circ h \circ s = 0$. The other direction is proved similarly. □

Given a null system $\mathcal{N}$, we will write $\mathcal{C}/\mathcal{N}$ for the localization of $\mathcal{C}$ in $S(\mathcal{N})$.

Proposition 3.12. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{N}$ a null system in $\mathcal{D}$, $\mathcal{D}'$ a full triangulated subcategory of $\mathcal{D}$. Let $\mathcal{N}' = \mathcal{N} \cap \mathcal{D}'$. Then*

(1) *$\mathcal{N}'$ is a null system in $\mathcal{D}'$.*
(2) *Assume moreover that any morphism $B \longrightarrow C$ in $\mathcal{D}$ with $B \in \mathcal{D}'$, $C \in \mathcal{N}$ factors through an object of $\mathcal{N}'$. Then $\mathcal{D}'/\mathcal{N}'$ is a full subcategory of $\mathcal{D}/\mathcal{N}$.*

Proof. Item (i) being clear, we only consider (ii). We will verify condition (1) of Proposition 3.10. So, let $s\colon A \longrightarrow B$ be a morphism such that $B \in \mathcal{D}'$ and $s \in S(\mathcal{N})$. We need to find $t\colon D \longrightarrow A$ such that $s \circ t$ is in $S(\mathcal{N})$. By definition, we have a triangle $A \longrightarrow B \longrightarrow C(s) \longrightarrow A[1]$ with $C(s) \in \mathcal{N}$. The map $f\colon B \longrightarrow C(s)$ factors through an object C' of $\mathcal{N}'$, so $f = \beta \circ \alpha$. Then we use TR4 to get a triangle $C(\alpha) \longrightarrow A[1] \longrightarrow C(\beta)$. Rotate this triangle and set $D = C(\alpha)[-1]$. Then $C(s \circ t)$ fits into a triangle with $C(s)$ and $C(t) \simeq C(\beta)[-1]$ and is hence in $\mathcal{N}$ because the latter two are. Condition (2) is similar. $\square$

We can apply the localization procedure to produce the derived category of an abelian category $\mathcal{A}$.

Consider the null system

$$\mathcal{N} = \left\{ A^\bullet \in K(\mathcal{A}) \mid H^k(A^\bullet) = 0 \; \forall \, k \in \mathbb{Z} \right\}.$$

By definition, this is the triangulated subcategory of acyclic complexes. Clearly, this is a null system: (N1) and (N2) are obvious and (N3) follows from Lemma 3.6. By Proposition 3.7 $S(\mathcal{N})$ are the quasi-isomorphisms.

Definition. Let $\mathcal{A}$ be an abelian category and $K(\mathcal{A})$ its homotopy category. The *derived category* $D(\mathcal{A})$ is defined to be $K(\mathcal{A})/\mathcal{N}$.

In other words, the derived category of $\mathcal{A}$ is obtained by inverting all quasi-isomorphisms. If we start with $K^*(\mathcal{A})$, where $* \in \{+, -, b\}$, we get the corresponding categories $D^*(\mathcal{A})$. The cohomology functor $H^k\colon K^*(\mathcal{A}) \longrightarrow \mathcal{A}$ factors through the derived category for any $k \in \mathbb{Z}$ because it takes quasi-isomorphisms to isomorphisms. Hence, we get cohomology functors $H^k\colon D^*(\mathcal{A}) \longrightarrow \mathcal{A}$.

Remark 3.13. Let $\mathcal{D}$ is a triangulated category. Note that the notion of a null system is compatible with the triangulated structure. We can define the structure of a triangulated category on $\mathcal{D}_{S(\mathcal{N})}$ for a null system $\mathcal{N}$ by defining the shift functor in the natural way and by taking for distinguished triangles those isomorphic to the image of a distinguished triangle in $\mathcal{D}$. It can then be checked that the localization functor $Q\colon \mathcal{D} \longrightarrow \mathcal{D}_{S(\mathcal{N})}$ becomes exact. In particular, this reasoning applies to the derived category, which is therefore triangulated.

Proposition 3.14. *The category $D^*(\mathcal{A})$, $* = b, +, -$, is equivalent to the full subcategory of $D(\mathcal{A})$ consisting of objects $A^\bullet$ such that $H^k(A^\bullet) = 0$ for $|k| \gg 0$ resp. $k \ll 0$ resp. $k \gg 0$. The category $\mathcal{A}$ is equivalent to the full subcategory of $D(\mathcal{A})$ consisting of objects $A^\bullet$ such that $H^k(A^\bullet) = 0$ for $k \neq 0$.*

Proof. Let $A^\bullet$ be a complex and define the truncated complex $\tau^{\leq k}(A^\bullet)$ resp. $\tau^{\geq k}(A^\bullet)$ by

$$\tau^{\leq k}(A^\bullet) = [\, \dots \longrightarrow A^{k-2} \longrightarrow A^{k-1} \longrightarrow \ker(d^k) \longrightarrow 0 \longrightarrow \dots \,],$$

$$\tau^{\geq k}(A^\bullet) = [\, \dots \longrightarrow 0 \longrightarrow \operatorname{coker}(d^{k-1}) \longrightarrow A^{k+1} \longrightarrow A^{k+2} \longrightarrow \dots \,].$$

It is straightforward to check that for an object $A^\bullet$ with $H^j(A^\bullet) = 0$ for $j < k$ resp. $H^j(A^\bullet) = 0$ for $j > k$ the maps $A^\bullet \longrightarrow \tau^{\geq k}(A^\bullet)$ resp. $\tau^{\leq k}(A^\bullet) \longrightarrow A^\bullet$ are quasi-isomorphisms. The result follows from this and Proposition 3.12. $\square$

Remark 3.15. A complex $A^\bullet$ becomes isomorphic to 0 in $D(\mathcal{A})$ if and only if it is acyclic. By definition, a morphism of complexes $f\colon A^\bullet \longrightarrow B^\bullet$ is 0 in $D(\mathcal{A})$ if any only if there exists a quasi-isomorphism $s\colon C^\bullet \longrightarrow A^\bullet$ such that $f \circ s$ is homotopic to zero. It then follows immediately that $f = 0 \in D(\mathcal{A}) \Longrightarrow H^k(f) = 0 \;\forall\; k \in \mathbb{Z}$.

Remark 3.16. Any null-homotopic morphism f has the property that there exists a qis s such that $f \circ s$ is homotopic to zero. The converse is false. To see this, consider the complex $A^\bullet = \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$, where the first map is multipication by 2 and the second is the projection. Take f to be the identity and $s\colon 0 \longrightarrow A^\bullet$. It is easily seen that id is not homotopic to zero.

Remark 3.17. A map can induce the zero map in cohomology but not be 0 in $D(\mathcal{A})$. For example, we can consider f

$$\begin{tikzcd}
A^\bullet = 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d, "\mathrm{id}"] & \mathbb{Z} \arrow[r] \arrow[d, "2"] & 0 \\
B^\bullet = 0 \arrow[r] & \mathbb{Z} \arrow[r, "\pi"] & \mathbb{Z}/3\mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

The cohomology of the upper complex is 0 in degree zero and $\mathbb{Z}/2\mathbb{Z}$ in degree 1, while that of the lower complex is $\mathbb{Z}$ in degree 0 and 0 in degree 1. One can show that there does not exist a qis $s\colon C^\bullet \longrightarrow A^\bullet$ such that $f \circ s \sim 0$ as follows. One takes a cycle in $C^\bullet$ such that its image generates the cohomology of $A^\bullet$. If k is the homotopy of fs, then $2k(x) = 1$, which is impossible.

Example 3.18. Let us now consider the derived category in a very special case. Recall that an abelian category $\mathcal{A}$ is *semisimple* if any exact sequence in $\mathcal{A}$ splits. An example of such a category is the category of vector spaces over a field. The category of abelian groups on the other hand is not semisimple.

We say that a complex in $\mathrm{Kom}(\mathcal{A})$ is *cyclic* if all its differentials are zero. The structure of the category of cyclic complexes, denoted by $\mathrm{Kom}_0(\mathcal{A})$, is clear: It is isomorphic to the category $\prod_{i=-\infty}^{\infty} \mathcal{A}[i]$. We have the inclusion functor $\mathrm{Kom}_0(\mathcal{A}) \longrightarrow \mathrm{Kom}(\mathcal{A})$. In the other direction, we consider the cohomology functor $\mathrm{Kom}(\mathcal{A}) \longrightarrow \mathrm{Kom}_0(\mathcal{A})$ which sends a complex to the (cyclic) complex of its cohomologies. Since this functor transforms quasi-isomorphisms to isomorphisms, we get an induced functor

$$\kappa\colon D(\mathcal{A}) \longrightarrow \mathrm{Kom}_0(\mathcal{A})$$

and we will check that κ is an equivalence if $\mathcal{A}$ is semisimple. To see this, let $A^\bullet$ be an arbitrary complex. Let $B^k = \mathrm{im}(d^{k-1})$, $Z^k = \ker(d^k)$ and $H^k = H^k(A^\bullet)$. We have the following two exact sequences in $\mathcal{A}$:

$$0 \longrightarrow Z^k \longrightarrow A^k \longrightarrow B^{k+1} \longrightarrow 0,$$

$$0 \longrightarrow B^k \longrightarrow Z^k \longrightarrow H^k \longrightarrow 0.$$

Since $\mathcal{A}$ is semisimple, we have $A^k = B^k \oplus H^k \oplus B^{k+1}$ and the map

$$d^k\colon B^k \oplus H^k \oplus B^{k+1} \longrightarrow B^{k+1} \oplus H^{k+1} \oplus B^{k+1}$$

is given by

$$d^k(b^k, h^k, b^{k+1}) = (b^{k+1}, 0, 0).$$

We now define

$$f_A \colon A^\bullet \longrightarrow \oplus_i H^i(A^\bullet)$$
$$f_A^k \colon (b^k, h^k, b^{k+1}) \longmapsto h^k$$

and

$$g_A \colon \oplus_i H^i(A^\bullet) \longrightarrow A^\bullet$$
$$g_A^k \colon h^k \longmapsto (0, h^k, 0).$$

Define $l\colon \mathrm{Kom}_0(\mathcal{A}) \longrightarrow D(\mathcal{A})$ as the composition of the embedding $\mathrm{Kom}_0(\mathcal{A}) \longrightarrow \mathrm{Kom}(\mathcal{A})$ with the localization functor. We will now check that κ and l are inverse to each other. Clearly, $\kappa \circ l$ is isomorphic to the identity functor in $\mathrm{Kom}_0(\mathcal{A})$. On the other hand, $l \circ \kappa$ maps a complex $A^\bullet$ in $D(\mathcal{A})$ to the complex $\oplus_i H^i(A^\bullet)$. The above morphisms f_A and g_A provide isomorphisms between $l \circ \kappa(A^\bullet)$ and $A^\bullet$ in $D(\mathcal{A})$ proving the claim.

Of course, the derived category is usually more complicated than in the above case. Let us continue with the investigation of some of its properties.

Proposition 3.19. *Let $\mathcal{A}$ be an abelian category and let $0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$ be an exact sequence in $\mathrm{Kom}(\mathcal{A})$. Let $C(f)$ be the mapping cone of f and let $\phi^k\colon C(f)^k = A^{k+1} \oplus B^k \longrightarrow C^k$ be the morphism $(0, g^k)$. Then $\phi = \{\phi^k\}_{k\in\mathbb{Z}} : C(f) \longrightarrow C^\bullet$ is a morphism of complexes, $\phi \circ \alpha(f) = g$ and ϕ is a quasi-isomorphism.*

Proof. It is straightforward to check that ϕ is indeed a morphism of complexes. Now consider the objects $C(\mathrm{id}_{A^\bullet})$ and $C(f)$. There is a map $\gamma\colon C(\mathrm{id}_{A^\bullet}) \longrightarrow C(f)$, sending (x^{k+1}, x^k) to $(x^{k+1}, f^k(x^k))$ (an easy check confirms that this is compatible with the differentials). It is clear that we have the following exact sequence of complexes

$$0 \longrightarrow C(\mathrm{id}_{A^\bullet}) \xrightarrow{\gamma} C(f) \xrightarrow{\phi} C^\bullet \longrightarrow 0.$$

To see that $\phi\colon C(f) \longrightarrow C^\bullet$ is a quasi-isomorphism it suffices to check that $H^k(C(\mathrm{id}_{A^\bullet})) = 0$ for all $k \in \mathbb{Z}$, see Proposition 3.4. But since $C(\mathrm{id}_{A^\bullet})$ is zero in $K(\mathcal{A})$, this is obvious. □

Corollary 3.20. *Given an exact sequence $0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0$ in $\mathrm{Kom}(\mathcal{A})$, there exists a distinguished triangle $A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow A[1]$ in $D(\mathcal{A})$, where the map $h\colon C^\bullet \longrightarrow A[1]$ is $\beta(f) \circ \phi^{-1}$ with notation from the previous proposition.* □

Remark 3.21. The above distinguished triangle gives rise to a long exact cohomology sequence with connection homomorphisms being, up to sign, the connection homomorphisms from Proposition 3.4. Also note that if $A^\bullet$, $B^\bullet$ and $C^\bullet$ are concentrated in degree zero, then h is zero in $D(\mathcal{A})$ if and only if the exact sequence splits (see Proposition 1.11).

Remark 3.22. Recall the filtrations $\tau^{\geq k}$ and $\tau^{\leq k}$. They transform a morphism homotopic to zero into a morphism homotopic to zero and a quasi-isomorphism into a quasi-isomorphism. Hence, we get functors $\tau^{\geq k}\colon D(\mathcal{A}) \longrightarrow D^{+}(\mathcal{A})$ and $\tau^{\leq k}\colon D(\mathcal{A}) \longrightarrow D^{-}(\mathcal{A})$. Using Proposition 3.19 we get distinguished triangles in $D(\mathcal{A})$:

$$\tau^{\leq k}(A^\bullet) \longrightarrow A^\bullet \longrightarrow \tau^{\geq k+1}(A^\bullet) \longrightarrow \tau^{\leq k}(A^\bullet)[1]$$

$$\tau^{\leq k-1}(A^\bullet) \longrightarrow \tau^{\leq k}(A^\bullet) \longrightarrow H^k(A^\bullet)[-k] \longrightarrow \tau^{\leq k-1}(A^\bullet)[1]$$

$$H^k(A^\bullet)[-k] \longrightarrow \tau^{\geq k}(A^\bullet) \longrightarrow \tau^{\geq k+1}(A^\bullet) \longrightarrow H^k(A^\bullet)[-k+1].$$

Note that $A^\bullet/\tau^{\leq k}(A^\bullet)$ is the complex $A^k/\ker d^k \longrightarrow A^{k+1} \longrightarrow \dots$, which is clearly quasi-isomorphic to $\tau^{\geq k+1}(A^\bullet)$, the latter being $0 \longrightarrow \operatorname{coker} d^k \longrightarrow A^{k+2} \longrightarrow \dots$ Since we have the short exact sequence

$$0 \longrightarrow \tau^{\leq k}(A^\bullet) \longrightarrow A^\bullet \longrightarrow A^\bullet/\tau^{\leq k}(A^\bullet) \longrightarrow 0,$$

the existence of the first sequence follows from the previous corollary. The other two sequences are also clear.

Our next goal will be to give an equivalence of $D(\mathcal{A})$ with a certain homotopy category. In order to do this we need to recall some notions. Let $\mathcal{A}$ be an abelian category and let $A \in \mathcal{A}$ be an object. It is a standard fact that the functors $\operatorname{Hom}(-, A)$ and $\operatorname{Hom}(A, -)$ are left-exact.

Definition. Let $\mathcal{A}$ be an abelian category. An object $A \in \mathcal{A}$ is called *projective* if $\operatorname{Hom}(A, -)$ is an exact functor and A is *injective* if the functor $\operatorname{Hom}(-, A)$ is exact. Given an arbitrary object $A \in \mathcal{A}$, a *projective resolution* is an exact sequence

$$\dots \longrightarrow P^2 \longrightarrow P^1 \longrightarrow A \longrightarrow 0$$

with P^i projective for all i. Dually, an *injective resolution* is an exact sequence

$$0 \longrightarrow A \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

with I^i injective for all i.

Since the functors in question are in any case left-exact, the conditions boil down to the following. An object P is projective if, given a diagram

$$\begin{tikzcd} & P \arrow[d] & \\ X \arrow[r] & Y \arrow[r] & 0, \end{tikzcd}$$

there exists a map $P \longrightarrow X$ making it commutative. An object I is injective if, given a diagram

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & X \arrow[u] \arrow[r] & Y, \end{tikzcd}$$

there exists a map $Y \longrightarrow I$ making the diagram commutative.

Example 3.23. Any free module over a commutative ring (with identity) R is projective in the category of R-modules. More generally, a module is projective iff it is a direct summand of a free module. To see this, set $F(A)$ to be the free module on the set of an R-module A. Clearly, there is a surjection $\pi\colon F(A) \longrightarrow A$. If A is projective, then we get a map $i\colon A \longrightarrow F(A)$ such that $\pi \circ i = \mathrm{id}_A$, proving the claim.

In fact, over some rings ($\mathbb{Z}$, fields,...), the projectives are precisely the free modules. But this is not always the case. As an example take $R = R_1 \times R_2$ for two rings R_i. Then $R_1 \times 0$ and $0 \times R_2$ are projective modules because their sum is R. But they are not free, because, for example, $(0,1)(R_1 \times 0) = 0$.

The category of finite abelian groups is a category having *no* projective modules (for example, let us check that $\mathbb{Z}/2\mathbb{Z}$ is not projective. Namely, consider the surjection $\mathbb{Z}/8\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z}$ mapping 1 to 3. Then the map $\mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/4\mathbb{Z}$ which maps 1 to 2 (the only choice), cannot be lifted).

Remark 3.24. The $\mathbb{Z}$-module $\mathbb{Q}$ has the property that the functor $\mathbb{Q} \otimes -$ is exact. In other words, $\mathbb{Q}$ is flat. On the other hand, $\mathbb{Q}$ is not a projective module, because it then would have to be a direct summand of a free module, but any two elements in $\mathbb{Q}$ are linearly dependent over $\mathbb{Z}$. It can be checked that the functors $\mathbb{Q} \otimes -$ and $\mathrm{Hom}(\mathbb{Q}, -)$ are adjoint to each other. Therefore, the adjoint of an exact functor need not be exact (only half-exact).

Example 3.25. It can be proved that a module A over a principal ideal domain R is injective iff it is divisible, that is, for every $r \neq 0$ in R and for every $a \in A$ we have $a = br$ for some $b \in A$. In other words, the multiplication map $A \xrightarrow{\cdot r} A$ is surjective.

Definition. Let $\mathcal{A}$ be an abelian category. One says that $\mathcal{A}$ has *enough injectives* if for any $A \in \mathcal{A}$ there exists an exact sequence $0 \longrightarrow A \longrightarrow I$ with I injective. Similarly, $\mathcal{A}$ is said to have *enough projectives* if for any A there exists a projective P and a short exact sequence $P \longrightarrow A \longrightarrow 0$. The full subcategory of all injective objects in $\mathcal{A}$ will be denoted by $\mathcal{I}$ and the subcategory of projective objects by $\mathcal{P}$.

Remark 3.26. If $\mathcal{A}$ has enough injectives, then any object A has an injective resolution. Indeed, pick an embedding $0 \longrightarrow A \xrightarrow{f} I^1$ into an injective and consider $\mathrm{coker}(f)$. Then consider an embedding $0 \longrightarrow \mathrm{coker}(f) \xrightarrow{g} I^2$ and define $d^1\colon I^1 \longrightarrow I^2$ as the composition. Clearly, the kernel of d^1 is isomorphic to A and hence the sequence

$$0 \longrightarrow A \xrightarrow{f} I^1 \xrightarrow{d^1} I^2$$

is exact. We then proceed inductively.

The same argument shows that if $\mathcal{A}$ has enough projectives, then any object has a projective resolution.

Having recalled the necessary notions we proceed to the next larger result. We will need the following.

Lemma 3.27. *Let $s\colon I^\bullet \longrightarrow A^\bullet$ be a quasi-isomorphism with $I^\bullet \in K^+(\mathcal{I})$ and $A^\bullet \in K^+(\mathcal{A})$. Then there exists a morphism of complexes $t\colon A^\bullet \longrightarrow I^\bullet$ such that $t \circ s$ is homotopic to $\mathrm{id}_{I^\bullet}$.*

Proof. Consider the triangle

$$I^\bullet \xrightarrow{s} A^\bullet \longrightarrow C(s) \xrightarrow{\delta} I^\bullet[1].$$

By Proposition 3.7 the complex $C(s)$ is acyclic. We will now show that any morphism from an acyclic complex to a left-bounded complex of injectives is homotopic to zero. Wlog we may assume that we are in the following situation

$$\begin{tikzcd} \dots \arrow[r] & 0 \arrow[r] & C^0 \arrow[r, "d^0"] \arrow[d, "\delta^0"] & C^1 \arrow[r, "d^1"] \arrow[d, "\delta^1"] & \dots \\ \dots \arrow[r] & 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & \dots \end{tikzcd}$$

Since I^0 is injective we get a map $k^0\colon C^1 \longrightarrow I^0$ with $\delta^0 = k^0 \circ d^0$.

Now consider the diagram

$$\begin{tikzcd} C^0 \arrow[r, "d^0"] & C^1 \arrow[r, "d^1"] \arrow[d, "\delta^1 - d^0_{I^\bullet} k^0"] & C^2 \\ & I^1 & \end{tikzcd}$$

and note that $(\delta^1 - d^0_{I^\bullet} k^0) \circ d^0 = 0$. Therefore, we can consider the following diagram instead of the previous one

$$\begin{tikzcd} & C^1 \arrow[d] \arrow[dr, "d^1"] & \\ 0 \arrow[r] & \mathrm{coker}(d^0) \arrow[r, "d'"] \arrow[d] & C^2 \\ & I^1 & \end{tikzcd}$$

where the composition $C^1 \longrightarrow I^1$ is equal to $\delta^1 - d^0_{I^\bullet} k^0$. Using the injectivity of I^1 we get $k^1\colon C^2 \longrightarrow I^1$ making the diagram commutative. By induction, we see that δ is homotopic to zero. By Proposition 1.11 this means that $A^\bullet \simeq I^\bullet \oplus C(s)$ and we can take t to be the projection to $I^\bullet$. □

Proposition 3.28. *The natural functor $K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{A})$ is fully faithful.*

Proof. It follows from the proof of Proposition 3.10 that the localization of $K^+(\mathcal{I})$ with respect to quasi-isomorphisms it a full subcategory of $D^+(\mathcal{A})$. By the previous lemma, any quasi-isomorphism between objects of $K^+(\mathcal{I})$ is already invertible in $K^+(\mathcal{I})$, hence the result. □

Before we can state the next result, we have to recall an important notion. Given maps $f\colon Z \longrightarrow X$ and $g\colon Z \longrightarrow Y$, the *pushout* is the fourth object and the two new maps in the following diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "a"] \\ X \arrow[r, "b"] & P. \end{tikzcd}$$

This has a universal property, namely, given maps $h\colon X \longrightarrow Q$ and $i\colon Y \longrightarrow Q$ such that $i \circ g = h \circ f$, then there exists a unique map $u\colon P \longrightarrow Q$ making everything commutative.

The above notion makes sense for any category $\mathcal{A}$. If $\mathcal{A}$ is abelian, the object P has a simple description. Namely, the above diagram is a pushout if and only if the following sequence is exact

$$Z \xrightarrow{\phi} X \oplus Y \xrightarrow{\psi} P \longrightarrow 0, \tag{3.2}$$

where $\phi = (f, g)$ and $\psi = \begin{pmatrix} b \\ -a \end{pmatrix}$. Thus, P is the quotient of $X \oplus Y$ by elements of the form $(f(z), -g(z))$. We will denote the pushout by $X \oplus_Z Y$.

Theorem 3.29. *If $\mathcal{A}$ has enough injectives, then the embedding $K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{A})$ is an equivalence, thus $K^+(\mathcal{I}) \simeq D^+(\mathcal{A})$.*

Proof. We have to show that for any complex $A^\bullet \in K^+(\mathcal{A})$ there exists a complex $I^\bullet \in K^+(\mathcal{I})$ with a quasi-isomorphism $t\colon A^\bullet \longrightarrow I^\bullet$. Wlog we can assume that $A^k = 0$ for $k < 0$. We will construct I^k and $d^k_{I^\bullet}$ and the t^k inductively.

To begin, consider the following diagram

$$\begin{tikzcd} 0 \arrow[r] & A^0 \arrow[rr, "d^0_{A^\bullet}"] \arrow[d, dashed, "t^0"] & & A^1 \arrow[d, dashed, "a"] \\ 0 \arrow[r] & I^0 \arrow[r, dashed, "b"] & I^0 \oplus_{C^0} C^1 \arrow[r, dashed, "c"] & I^1. \end{tikzcd}$$

Here we first construct an injection $t^0\colon A^0 \longrightarrow I^0$ with $I^0 \in \mathcal{I}$, which exists because $\mathcal{A}$ has enough injectives, then the pushout and then an injection of the pushout into an injective object I^1. Define $d^0_{I^\bullet} = c \circ b$ and $t^1 = c \circ a$.

Now assume that we have constructed everything up to the k-th step and consider the following diagram

$$\begin{tikzcd} & A^k \arrow[rr, "d^k_{A^\bullet}"] \arrow[dl, "t^k"'] \arrow[d, "\tilde{t}^k"] & & A^{k+1} \arrow[d, "a"] \\ I^k \arrow[r, "p"] & \operatorname{coker}(d^{k-1}_{I^\bullet}) \arrow[r, dashed, "b"] & \operatorname{coker}(d^{k-1}_{I^\bullet}) \oplus_{A^k} A^{k+1} \arrow[r, dashed, "c"] & I^{k+1}. \end{tikzcd}$$

Define $d^{k+1}_{I^\bullet} = c \circ b \circ p$ and $t^{k+1} = c \circ a$. It is immediate to check that I^k is a complex (basically we factorize over the coker in every step) and that t is indeed a morphism of complexes.

Note that $H^0(A^\bullet) = \ker(d^0_{A^\bullet})$. Let $x \in \ker(d^0_{I^\bullet})$. Since d^0 is the composition of the map to the pushout and an injection, its kernel is the kernel of the map to the pushout, but if x maps to zero, then it is of the form $x = t^0(a) = 0 = \pm d^0_{A^\bullet}(a)$ and vice versa, so $H^0(I^\bullet) \simeq H^0(A^\bullet)$ and t^0 induces the isomorphism.

We will now check that $H^k(t)$ is an epimorphism and $H^{k+1}(t)$ is a monomorphism. So, let $x \in H^k(I^\bullet)$, thus $d^k_{I^\bullet}(x) = 0$. This means that $b \circ p(x) = 0$ (since c is mono), thus the map ψ from Equation 3.2 maps $(p(x), 0)$ to zero, hence $(p(x), 0) = \phi(\widetilde{x})$ and unravelling the definition of ϕ we see that $H^k(t)$ is surjective.

Now let us check that $H^{k+1}(t)$ is a monomorphism. Let $x \in H^{k+1}(A^\bullet)$ such that $H^{k+1}(t)(x) = 0$. Since $t^{k+1} = c \circ a$ and c is mono, this means that $a(x) = 0$, so $x \in \operatorname{im}(d^k_{A^\bullet})$, thus it is 0 in $H^{k+1}(A^\bullet)$ and hence we have proved the claim. □

Remark 3.30. In a similar fashion one proves that the natural functor $K^-(\mathcal{P}) \longrightarrow D^-(\mathcal{A})$ is fully faithful. It is an equivalence if $\mathcal{A}$ has enough projectives.

Now let $\mathcal{A}$ be an abelian category and $\mathcal{A}'$ an abelian subcategory. Denote by $D^+_{\mathcal{A}'}(\mathcal{A})$ the full triangulated subcategory of $D^+(\mathcal{A})$ consisting of complexes whose cohomology objects belong to $\mathcal{A}'$. There is a natural functor

$$\delta\colon D^+(\mathcal{A}') \longrightarrow D^+_{\mathcal{A}'}(\mathcal{A}).$$

We say that $\mathcal{A}'$ is *thick* in $\mathcal{A}$ if for any exact sequence

$$B \longrightarrow B' \longrightarrow C \longrightarrow A \longrightarrow A'$$

with $B, B', A, A' \in \mathcal{A}'$ also $C \in \mathcal{A}'$.

Proposition 3.31. *Let $\mathcal{A}$ be an abelian category and $\mathcal{A}'$ a thick full abelian subcategory. Assume that for any monomorphism $f\colon A' \longrightarrow A$ with $A' \in \mathcal{A}'$ there exists a morphism $g\colon A \longrightarrow B$ with $B \in \mathcal{A}'$ such that $g \circ f$ is a monomorphism. Then the functor δ defined above is an equivalence of categories.*

Proof. By Proposition 3.10 we have to check that for any object $A^\bullet \in D^+_{\mathcal{A}'}(\mathcal{A})$ there exists an object $B \in D^+(\mathcal{A}')$ and a quasi-isomorphism $A \simeq B$. This is done with similar techniques as in Theorem 3.29. □

4. Derived functors

Let $\mathcal{A}$ and $\mathcal{A}'$ be abelian categories and let $F\colon \mathcal{A} \longrightarrow \mathcal{A}'$ be an additive functor. Note that F induces a functor on the level of chain complexes and also respects homotopies, hence it induces a functor $K(F)\colon K(\mathcal{A}) \longrightarrow K(\mathcal{A}')$. We would like to extend this functor to the level of derived categories. This is sometimes straightforward, for example, we have the following result.

Proposition 4.1. *Let $F\colon \mathcal{A} \longrightarrow \mathcal{A}'$ be an exact functor. Then the functor*

$$K^*(F)\colon K^*(\mathcal{A}) \longrightarrow K^*(\mathcal{A}')$$

transforms quasi-isomorphisms into quasi-isomorphisms (here $* = +, -, b$). Therefore, it induces a functor

$$D^*(F)\colon D^*(\mathcal{A}) \longrightarrow D^*(\mathcal{A}').$$

Proof. Let $A^\bullet$ be an acyclic complex and set $B^k = \operatorname{im}(d^k) = \ker(d^{k+1})$. The functor F maps the exact sequence

$$0 \longrightarrow B^k \xrightarrow{e^k} A^{k+1} \xrightarrow{p^k} B^{k+1} \longrightarrow 0$$

to the exact sequence

$$0 \longrightarrow F(B^k) \longrightarrow F(A^{k+1}) \longrightarrow F(B^{k+1}) \longrightarrow 0.$$

Since $d^k = e^k \circ p^{k-1}$ we get $F(d^k) = F(e^k) \circ F(p^{k-1})$. Moreover, $F(e^k)$ is injective and $F(p^k)$ is surjective. Therefore, $F(B^k)$ is isomorphic to the image of $F(d^k)$ and $F(B^{k+1})$ is isomorphic to the cokernel of $F(d^{k+1})$, hence $F(A^\bullet)$ is an acyclic complex.

Now if $s\colon A^\bullet \longrightarrow B^\bullet$ be a quasi-isomorphism, then its cone is acyclic. Since $F(C(s)) \simeq C(F(s))$, the complex $C(F(s))$ is acyclic by what we just proved and hence $F(s)$ is a quasi-isomorphism.

Having established this, the functor $D(F)$ is induced by the universal property of localization. □

If F is not exact, then it will in general not preserve quasi-isomorphisms and it is not clear how to extend it to the derived categories. The basic idea of derived functors is that if one wants to apply F to a complex, then we should apply it to a quasi-isomorphic complex belonging to a class of complexes adapted to F. One needs half-exactness for this and from now on we assume that $F\colon \mathcal{A} \longrightarrow \mathcal{A}'$ is a left-exact functor. Assume furthermore that $\mathcal{A}$ has enough injectives. In particular, we have an equivalence $T\colon K^+(\mathcal{I}) \longrightarrow D^+(\mathcal{A})$, see Theorem 3.29. Denote the quasi-inverse of this by T^{-1} and consider the following diagram.

$$\begin{tikzcd} K^+(\mathcal{I}) \arrow[r] \arrow[dr, bend right, "T"'] & K^+(\mathcal{A}) \arrow[r, "K(F)"] \arrow[d, "Q_{\mathcal{A}}"] & K^+(\mathcal{A}') \arrow[d, "Q_{\mathcal{A}'}"] \\ & D^+(\mathcal{A}) \arrow[ul, bend right, "T^{-1}"'] & D^+(\mathcal{A}'). \end{tikzcd}$$

Definition. The *right derived functor* RF of F is the functor

$$RF = Q_{\mathcal{A}'} \circ K(F) \circ T^{-1}.$$

Theorem 4.2. *Under the above assumptions, the following holds.*

(1) *There exists a natural morphism of functors*

$$Q_{\mathcal{A}'} \circ K(F) \longrightarrow RF \circ Q_{\mathcal{A}}.$$

(2) *The functor $RF\colon D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A}')$ is exact.*

(3) *Suppose $G\colon D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A}')$ is an exact functor. Then any functor morphism $Q_{\mathcal{A}'} \circ K(F) \longrightarrow G \circ Q_{\mathcal{A}}$ factorizes through a unique morphism $RF \longrightarrow G$.*

Proof. Let $A^\bullet \in D^+(\mathcal{A})$ and set $I^\bullet = T^{-1}(A^\bullet)$. The natural transformation $\mathrm{id} \longrightarrow T \circ T^{-1}$ yields a functorial morphism $A^\bullet \longrightarrow I^\bullet$ in $D^+(\mathcal{A})$ which is represented by a roof

$$A^\bullet \xleftarrow{\text{qis}} C^\bullet \longrightarrow I^\bullet.$$

Now recall from the proof of Lemma 3.27 that $\mathrm{Hom}_{K(\mathcal{A})}(C^\bullet, I^\bullet) = 0$ for any acyclic complex $C^\bullet$. This implies that if we are given a quasi-isomorphism $s\colon B^\bullet \longrightarrow \widetilde{B}^\bullet$, the induced map in $K(\mathcal{A})$

$$\mathrm{Hom}(B^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}(\widetilde{B}^\bullet, I^\bullet)$$

is bijective (complete the map s to a triangle and apply $\mathrm{Hom}(-, I^\bullet)$). Hence, from the above roof we get a unique map $A^\bullet \longrightarrow I^\bullet$ in $K(\mathcal{A})$, which is independent of the choice of $C^\bullet$. Combining everything, we get a functorial map $K(F)(A^\bullet) \longrightarrow K(F)(I^\bullet) = RF(I^\bullet)$.

We now need to check that RF as defined above is an exact functor. Clearly, T is an exact functor, hence also its inverse is exact (see Proposition 1.16). Therefore, RF is the composition of three exact functors and hence exact. The last property is left to the reader. □

These properties determine the right derived functor RF of a left exact functor F up to isomorphism.

Definition. Let $RF\colon D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A}')$ be the right derived functor of a left exact functor $F\colon \mathcal{A} \longrightarrow \mathcal{A}'$. Then, for any complex $A^\bullet \in D^+(\mathcal{A})$, one defines

$$R^iF(A^\bullet) := H^i(RF(A^\bullet)).$$

The induced additive functors $R^iF\colon \mathcal{A} \longrightarrow \mathcal{A}'$ are the *higher derived functors* of F. Note that $R^iF(A) = 0$ for $i < 0$ and $R^0F(A) = F(A)$ for any $A \in \mathcal{A}$. Indeed, if

$$A \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \dots$$

is an injective resolution, then $R^iF(A) = H^i(\dots \longrightarrow F(I^0) \longrightarrow F(I^1) \longrightarrow \dots)$ and, in particular,

$$R^0F(A) = \ker(F(I^0) \longrightarrow F(I^1)) = F(A),$$

since F is left-exact.

An object $A \in \mathcal{A}$ is called F-acyclic if $R^iF(A) \simeq 0$ for $i \neq 0$.

Corollary 4.3. *Under the above assumptions any short exact sequence*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$ gives rise to a long exact sequence

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow R^1F(A) \longrightarrow \dots$$

Proof. The short exact sequence gives a triangle in $D^+(\mathcal{A})$ (Corollary 3.20) and applying RF to it we get a triangle in $D^+(\mathcal{B})$. Lemma 3.6 then gives the result. □

Remark 4.4. The hypothesis can be weakened and this is important in some applications.

Firstly, there is the following general approach. Let $F\colon K^+(\mathcal{A}) \longrightarrow K^+(\mathcal{A}')$ be an exact functor. Then the right derived functor $RF\colon D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A}')$ satisfying conditions (1)-(3) of Theorem 4.2 exists whenever there exists a triangulated subcategory $K_F \subset K^+(\mathcal{A})$ adapted to F, meaning that it satisfies the following conditions.

(i) If $A^\bullet \in K_F$ is acyclic, then $F(A^\bullet)$ is also acyclic.
(ii) Any $A^\bullet \in K^+(\mathcal{A})$ is quasi-isomorphic to a complex in K_F.

A slightly less general approach is the following. Let $F\colon \mathcal{A} \longrightarrow \mathcal{A}'$ be a left-exact functor. In this situation one defines 'adapted' on the level of abelian categories. Namely, a class of objects $I_F \subset \mathcal{A}$ is adapted to F if the following conditions hold.

(i) If $A^\bullet \in K^+(\mathcal{A})$ is acyclic with $A^k \in I_F$ for all k, then $F(A^\bullet)$ is acyclic.
(ii) Any object $A \in \mathcal{A}$ can be embedded into an object of I_F.

Under these conditions, the localization of $K^+(I_F)$ by quasi-isomorphisms is equivalent to $D^+(\mathcal{A})$, compare the proof of Theorem 3.29 (we did not use the injectivity of objects). Condition (i) ensures that F transforms quasi-isomorphism to quasi-isomorphisms. One then defines RF as before. The procedure in the more general case above is similar.

Note that if $\mathcal{A}$ has enough injectives, then the class of injective objects is adapted to any left-exact functor F. Indeed, condition (ii) is clearly satisfied. Next, let $I^\bullet$ be a bounded below acyclic complex of injectives. Then $0\colon I^\bullet \longrightarrow I^\bullet$ is a quasi-isomorphism. On the other hand, it is homotopic to $\mathrm{id}_{I^\bullet}$ by Lemma 3.27. Therefore, the zero morphism of $F(I^\bullet)$ is homotopic to the identity morphism so that $F(I^\bullet)$ is indeed acyclic.

Remark 4.5. If one starts with a right-exact functor $F\colon \mathcal{A} \longrightarrow \mathcal{A}'$, then one constructs the left derived functor LF. This case works, for example, under the assumption that $\mathcal{A}$ has enough projectives, so any object is a quotient of a projective, and the functor we get will be defined on $D^-(-)$.

Example 4.6. Let $\mathcal{A}$ be an abelian category with sufficiently many injective objects and let A, B be objects in $\mathcal{A}$. Then there exist natural isomorphisms of functors

$$\mathrm{Ext}^i(A,B) = H^i R\mathrm{Hom}(A,B) \simeq \mathrm{Hom}_{D(\mathcal{A})}(A, B[i]).$$

Indeed, to compute $R\mathrm{Hom}(A,B)$ one replaces B by its injective resolution $B \longrightarrow I^\bullet$ so that $\mathrm{Ext}^i(A,B)$ is the i-th cohomology of the complex $\mathrm{Hom}(A, I^i)$. Now we use the following general construction. Given two complexes $A^\bullet$ and $B^\bullet$ in $\mathrm{Kom}(\mathcal{A})$, we define the "inner Hom"-complex $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ with objects

$$\mathrm{Hom}^n(A^\bullet, B^\bullet) = \prod_{i\in\mathbb{Z}} \mathrm{Hom}(A^i, B^{i+1})$$

and differential given by

$$df = d_B \circ f - (-1)^n f \circ d_A, \quad f \in \mathrm{Hom}^n(A^\bullet, B^\bullet).$$

Then, clearly,

$$\ker(d^i_{\mathrm{Hom}^\bullet}) = \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A^\bullet, B^\bullet[i]),$$

and the morphisms in the image are those homotopic to the zero morphism. Hence,

$$H^i\mathrm{Hom}^\bullet(A^\bullet, B^\bullet) = \mathrm{Hom}_{K(\mathcal{A})}(A^\bullet, B^\bullet[i]).$$

Hence, $\mathrm{Ext}^i(A,B) \simeq \mathrm{Hom}_{K(\mathcal{A})}(A, I^\bullet[i])$. Since $I^\bullet$ is a complex of injectives, we have

$$\mathrm{Hom}_{D(\mathcal{A})}(A, I^\bullet[i]) \simeq \mathrm{Hom}_{K(\mathcal{A})}(A, I^\bullet[i]),$$

and $B \simeq I^\bullet$ in $D(\mathcal{A})$, proving the assertion.

Definition. Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ is of *finite homological dimension* l if $\mathrm{Hom}_{D(\mathcal{A})}(A, B[i]) = 0$ for all $i > l$.

If $\mathcal{A}$ has enough injectives, then the above definition is equivalent, by the previous example, to requiring that $\mathrm{Ext}^i(A,B) = 0$ for all $i > l$. An abelian category is semisimple if and only if its homological dimension is zero.

Proposition 4.7. *Let $\mathcal{A}$ be an abelian category of homological dimension ≤ 1. Then any object $A^\bullet$ in $\mathrm{D^b}(\mathcal{A})$ is isomorphic to the direct sum $\oplus_i H^i(A^\bullet)[i]$.*

Proof. We use induction on the length of the complex $A^\bullet$. Suppose $A^\bullet$ is a complex of length k with $H^i(A^\bullet) = 0$ for $i < i_0$. By Remark 3.22 there exists a triangle

$$H^{i_0}(A^\bullet)[-i_0] \longrightarrow \tau^{\geq i_0}(A^\bullet) \simeq A^\bullet \longrightarrow \tau^{\geq i_0+1}(A^\bullet) \longrightarrow H^{i_0}(A^\bullet)[-i_0+1].$$

Here, $\tau^{\geq i_0+1}(A^\bullet) = A'^\bullet$ is a complex of length $k-1$. By the induction hypothesis, we have $A'^\bullet \simeq \oplus_{i>i_0} H^i(A^\bullet)[i]$. Now we compute

$$\mathrm{Hom}(A'^\bullet, H^{i_0}(A^\bullet)[-i_0+1]) \simeq \oplus_{i>i_0}\mathrm{Ext}^{1+i-i_0}(H^i(A'^\bullet), H^{i_0}(A^\bullet)) \simeq 0.$$

Hence, the last map is zero and the triangle splits by Proposition 1.11. □

Example 4.8. Let R be a commutative ring with identity. For any R-module M the functor $T(-) = M \otimes_R -$ is a right exact functor from $R-\mathrm{Mod}$, the category of R-modules, to the category of abelian groups. Since $R-\mathrm{Mod}$ has enough projectives, the left derived functor LT exists. To be more precise, given an R-module N, take a projective resolution of it and apply T to it. The cohomology objects of the resulting complex are the usual Tor-functors.

Example 4.9. Let R be as in the previous example. The functors $\mathrm{Hom}(-, M)$ resp. $\mathrm{Hom}(M, -)$ are left-exact for any R-module M. For the latter covariant functor one uses injective resolutions and for the contravariant one, one uses projective resolutions. The cohomology objects of the resulting complex are then the usual Ext-functors. In fact, it can be checked that the two constructions yield isomorphic results.

Example 4.10. Let X be a topological space. A *presheaf* $\mathcal{F}$ of abelian groups on X consists of the following data. For any open subset $U \subset X$, an abelian group $\mathcal{F}(U)$, called *sections* of $\mathcal{F}$ over U, and for every inclusion $V \subset U$ of open subsets a group homomorphism $\rho_{UV} \colon \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$, the restriction morphism. This data has to satisfy the following conditions: (0) $\mathcal{F}(\emptyset) = 0$, (1) $\rho_{UU} = \mathrm{id}_{\mathcal{F}(U)}$ and (2) if $W \subset V \subset U$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$. A presheaf is a *sheaf* if, given an open covering V_i of any open subset U and sections $s_i \in \mathcal{F}(V_i)$

such that $s_i = s_j$ on all intersections $V_i \cap V_j$, there exists a unique element $s \in \mathcal{F}(U)$ restricting to the s_i.

As an example one can consider the sheaf of continuous functions on a topological space or the sheaf of holomorphic functions on a complex manifold.

Now, the global sections functor Γ from sheaves of abelian groups on X to abelian groups can be checked to be left-exact. Note that if one has a continuous map $f\colon X \longrightarrow Y$, one can define the *pushforward* $f_*(\mathcal{F})$ of a sheaf $\mathcal{F}$ on X to be the sheaf on Y (check this!) defined by $f_*(\mathcal{F}) = \mathcal{F}(f^{-1}(U))$ for any open subset $U \subset Y$, and the global section functor of a sheaf can be seen as $f_*(\mathcal{F})$ for $f\colon X \longrightarrow \{\text{point}\}$. Furthermore, the category of sheaves has enough injectives and, therefore, its right derived functor $R\Gamma$ exists. The higher derived functors are denoted by $H^i(X, \mathcal{F})$ and are the usual sheaf cohomology functors. The usual cohomology groups like $H^i(X, \mathbb{Z})$ for, say, a smooth manifold X, can be considered in this context.

Proposition 4.11. *Let $F_1\colon \mathcal{A} \longrightarrow \mathcal{A}'$ and $F_2\colon \mathcal{A}' \longrightarrow \mathcal{A}''$ be two left exact functors of abelian categories. Assume that there exist adapted classes $\mathcal{I}_{F_1} \subset \mathcal{A}$ and $\mathcal{I}_{F_2} \subset \mathcal{A}'$ for F_1, respectively F_2, such that $F_1(\mathcal{I}_{F_1}) \subset \mathcal{I}_{F_2}$.*

Then the derived functors $RF_1\colon D+(\mathcal{A}) \longrightarrow D^+(\mathcal{A}')$, $RF_2\colon D+(\mathcal{A}') \longrightarrow D^+(\mathcal{A}'')$ and $R(F_2 \circ F_1)\colon D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{A}'')$ exist and there is a natural isomorphism $R(F_2 \circ F_1) \simeq RF_2 \circ RF_1$.

Proof. The existence of RF_1 and RF_2 follows from the assumptions. Moreover, since $F_1(\mathcal{I}_{F_1}) \subset \mathcal{I}_{F_2}$, the class $\mathcal{I}_{F_1}$ is also adapted to the composition $F_2 \circ F_1$ and, therefore, $R(F_2 \circ F_1)$ exists as well.

A natural morphism $R(F_2 \circ F_1) \longrightarrow RF_2 \circ RF_1$ exists by the universal property of the derived functor $R(F_2 \circ F_1)$.

If $A^\bullet \in D^+(\mathcal{A})$ is isomorphic to a complex $I^\bullet \in K^+(\mathcal{I}_{F_1})$, then the morphism

$$R(F_2 \circ F_1)(A^\bullet) \longrightarrow R(F_2)(R(F_1)(A^\bullet)),$$

is an isomorphism, because the left hand side is isomorphic to $(K(F_2) \circ K(F_1))(I^\bullet)$, but so is the right hand side

$$R(F_2)(R(F_1)(A^\bullet)) \simeq RF_2(K(F_1)(I^\bullet)) \simeq K(F_2)K(F_1)(I^\bullet).$$

□

5. t-structures

There are many examples of abelian categories $\mathcal{A}$ and $\mathcal{B}$ whose derived categories are equivalent, but the abelian categories themselves are not. The concept of a t-structure gives an abstract way of finding abelian categories in a given triangulated category $\mathcal{D}$.

Definition. Let $\mathcal{D}$ be a triangulated category. Two full subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ of $\mathcal{D}$ are called a *t-structure* on $\mathcal{D}$ if the following conditions are satisfied. We will use the notation $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$.

(1) $\mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}$ (for complexes: closed under shifts to the left) and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$ (for complexes: closed under shifts to the right).

(2) $\mathrm{Hom}_{\mathcal{D}}(X,Y)=0$ for $X\in\mathcal{D}^{\leq 0}$ and $Y\in\mathcal{D}^{\geq 1}$.
(3) For any object X in $\mathcal{D}$ there exists a distinguished triangle
$$X_0 \longrightarrow X \longrightarrow X_1 \longrightarrow X_0[1],$$
with $X_0\in\mathcal{D}^{\leq 0}$ and $X_1\in\mathcal{D}^{\geq 1}$.

The full subcategory $\mathcal{A}:=\mathcal{D}^{\leq 0}\cap\mathcal{D}^{\geq 0}$ is called the *heart* of the t-structure.

Remark 5.1. An equivalent way of defining a t-structure is the following. A t-structure on a triangulated category $\mathcal{D}$ is a full subcategory $\mathcal{F}\subset\mathcal{D}$, such that $\mathcal{F}[1]\subset\mathcal{F}$ and with the property that if one defines
$$\mathcal{F}^{\perp}=\{E\in\mathcal{D}\mid \mathrm{Hom}_{\mathcal{D}}(F,E)=0 \text{ for all } F\in\mathcal{F}\},$$
then for every object $X\in\mathcal{D}$ there exists a triangle $F\longrightarrow X\longrightarrow E$ with $F\in\mathcal{F}$ and $E\in\mathcal{F}^{\perp}$. The connection with the definition is given by the identification $\mathcal{D}^{\leq 0}=\mathcal{F}$ and $\mathcal{D}^{\geq 1}=\mathcal{F}^{\perp}$.

Example 5.2. Let $\mathcal{D}=D(\mathcal{A})$ for an abelian category $\mathcal{A}$. Denote by $\mathcal{D}^{\leq 0}(\mathcal{A})$ resp. by $\mathcal{D}^{\geq 0}(\mathcal{A})$ the full subcategory of complexes $A^\bullet$ satisfying $H^k(A^\bullet)=0$ for $k>0$ resp. $k<0$. Clearly, condition (1) is satisfied. Furthermore, (3) is fulfilled by Remark 3.22. To check (2), consider a morphism $\phi\colon A^\bullet\longrightarrow B^\bullet$ with $A^\bullet\in\mathcal{D}^{\leq 0}(\mathcal{A})$ and $B^\bullet\in\mathcal{D}^{\geq 1}(\mathcal{A})$. By the latter condition, we can assume that $B^k=0$ for $k<0$ and that $d^0_{B^\bullet}\colon B^0\longrightarrow B^1$ is injective. Represent ϕ by a roof of the form $A^\bullet \xleftarrow{s} C^\bullet \xrightarrow{f} B^\bullet$ with s a quasi-isomorphism. Since $A^\bullet\in\mathcal{D}^{\leq 0}$, also $C^\bullet\in\mathcal{D}^{\leq 0}$ and, therefore, the map $r\colon \tau_{\leq 0}(C^\bullet)\longrightarrow C^\bullet$ is a quasi-isomorphism. It is then readily checked that the roof $A^\bullet \xleftarrow{sr} \tau_{\leq 0}(C^\bullet) \xrightarrow{fr} B^\bullet$ also represents ϕ. But $fr=0$, since for $i\neq 0$ either $B^i=0$ or $\tau_{\leq 0}(C^\bullet)^i=0$ and for $i=0$ we have $d^0_{B^\bullet}\circ(fr)^0=(fr)^1\circ d^0_{\tau_{\leq 0}(C^\bullet)}=0$ so that $(fr)^0=$ is zero because $d^0_{B^\bullet}$ is a monomorphism. In this example, the heart is equivalent to $\mathcal{A}$ by Proposition 3.14.

Lemma 5.3. *Let $\mathcal{D}$ be a triangulated category and assume we are given two distinguished triangles* $X\xrightarrow{f} Y\xrightarrow{g} Z\xrightarrow{h_i} X[1]$, *for $i=1,2$. If $\mathrm{Hom}_{\mathcal{D}}(X[1],Z)=0$, then $h_1=h_2$.*

Proof. By (TR3) we have a morphism of triangles
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & Y \arrow[r, "g"] \arrow[d, "\mathrm{id}"] & Z \arrow[r, "h_1"] \arrow[d, "\phi"] & X[1] \arrow[d, "\mathrm{id}"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h_2"] & X[1], \end{tikzcd}$$
thus $\phi\circ g=g$ and $h_1=h_2\circ\phi$. Now $(\mathrm{id}_Z-\phi)\circ g=0$ and applying $\mathrm{Hom}(-,Z)$ to the upper triangle, we get a map $\psi\colon X[1]\longrightarrow Z$ such that $\mathrm{id}_Z-\phi=\psi\circ h_1$. By hypothesis, $\psi=0$, hence $\phi=\mathrm{id}_Z$ and, therefore, $h_1=h_2$. ☐

Proposition 5.4. (i) *The inclusion $\mathcal{D}^{\leq n}\longrightarrow\mathcal{D}$ (resp. $\mathcal{D}^{\geq n}\longrightarrow\mathcal{D}$) admits a right adjoint functor $\tau^{\leq n}\colon\mathcal{D}\longrightarrow\mathcal{D}^{\leq n}$ (resp. a left adjoint $\tau^{\geq n}\colon\mathcal{D}\longrightarrow\mathcal{D}^{\geq n}$).*

(ii) *There exists a unique morphism* $d\colon \tau^{\geq n+1}(X) \longrightarrow \tau^{\leq n}(X)[1]$ *such that*

$$\tau^{\leq n}(X) \longrightarrow X \longrightarrow \tau^{\geq n+1}(X) \xrightarrow{d} \tau^{\leq n}(X)[1]$$

is a distinguished triangle. Moreover, d is a morphism of functors.

Proof. We may assume $n = 0$. By definition of a t-structure, there exists a distinguished triangle

$$X_0 \longrightarrow X \longrightarrow X_1 \longrightarrow X_0[1],$$

and we define $\tau^{\leq 0}(X) := X_0$ and $\tau^{\geq 1}(X) := X_1$ on objects. To define it on morphisms, let $f\colon X \longrightarrow Y$ be a map in $\mathcal{D}$. Considering the decomposition of Y with respect to the t-structure and applying $\mathrm{Hom}(X_0, -)$ to this decomposition, we get a morphism $\tau^{\leq 0}(f)\colon X_0 \longrightarrow Y_0$. Similarly, one establishes that $\tau^{\geq 1}$ is a functor. Now if $A \in \mathcal{D}^{\leq 0}$, then $\mathrm{Hom}(A, X) \simeq \mathrm{Hom}(A, X_0)$ resp. $\mathrm{Hom}(X, B) \simeq \mathrm{Hom}(X_1, B)$ for $B \in \mathcal{D}^{\geq 1}$. This follows from the long exact sequence associated to $\mathrm{Hom}(A, -)$ resp. $\mathrm{Hom}(-, B)$. Thus, we proved (i). The first part of (ii) follows from the definition and the previous lemma. To prove that d is a natural transformation, let $f\colon X \longrightarrow Y$ and consider the associated triangles. By (TR3) we get a morphism between them, and hence d is a morphism of functors. □

Corollary 5.5. *We have $X \in \mathcal{D}^{\leq n}$ if and only if $\tau^{\geq n+1}(X) \simeq 0$, and similarly $X \in \mathcal{D}^{\geq n}$ if and only if $\tau^{\leq n-1}(X) \simeq 0$.*

Proof. Follows immediately from the exact triangle $\tau^{\leq n}X \longrightarrow X \longrightarrow \tau^{\geq n+1}X \xrightarrow{+}$. □

Proposition 5.6. *Let $X' \longrightarrow X \longrightarrow X'' \longrightarrow X'[1]$ be a distinguished triangle in $\mathcal{D}$. If X' and X'' belong to $\mathcal{D}^{\geq 0}$ resp. $\mathcal{D}^{\leq 0}$, then so does X.*

Proof. We will consider the case $\mathcal{D}^{\geq 0}$. We have $\mathrm{Hom}(\tau^{\leq -1}(X), X') = \mathrm{Hom}(\tau^{\leq -1}(X), X'') = 0$ by definition. Hence, also $\mathrm{Hom}(\tau^{\leq -1}(X), X) = 0$. Adjunction gives $\mathrm{Hom}(\tau^{\leq -1}(X), \tau^{\leq -1}(X)) = 0$, so $\tau^{\leq -1}(X) \simeq 0$. The result now follows from the previous corollary. □

Proposition 5.7. *The heart $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ of a t-structure is an abelian category.*

Proof. Firstly, note that if $X' \longrightarrow X \longrightarrow X'' \longrightarrow X[1]$ is a a distinguished triangle with X' and X'' in $\mathcal{A}$, then the same holds for X.

Considering the triangle $X \longrightarrow X \oplus Y \longrightarrow Y \longrightarrow X[1]$, we see that $\mathcal{A}$ is an additive category. Now let $f\colon X \longrightarrow Y$ be a morphism in $\mathcal{A}$ and embed it into a distinguished triangle $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$. Rotating the triangle and using Proposition 5.6, we see that $Z \in \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq -1}$. We will prove that

$$\tau^{\geq 0}(Z) \simeq \mathrm{coker}(f)$$

and

$$\tau^{\leq 0}(Z[-1]) \simeq \ker(f).$$

To do this, take $W \in \mathcal{A}$ and consider the long exact sequences

$$\mathrm{Hom}(X[1], W) \longrightarrow \mathrm{Hom}(Z, W) \longrightarrow \mathrm{Hom}(Y, W) \longrightarrow \mathrm{Hom}(X, W)$$

and

$$\mathrm{Hom}(W, Y[-1]) \longrightarrow \mathrm{Hom}(W, Z[-1]) \longrightarrow \mathrm{Hom}(W, X) \longrightarrow \mathrm{Hom}(W, Y).$$

Note that $\mathrm{Hom}(X[1], W) \simeq \mathrm{Hom}(W, Y[-1]) \simeq 0$. Furthermore, $\mathrm{Hom}(Z, W) \simeq \mathrm{Hom}(\tau^{\geq 0}Z, W)$ and $\mathrm{Hom}(W, Z[-1]) \simeq \mathrm{Hom}(W, \tau^{\leq 0}(Z[-1]))$. Hence, the claimed equalities hold because the universal properties of ker and coker are satisfied.

Finally, let is prove that $\mathrm{coim}(f) \simeq \mathrm{im}(f)$. Embed $Y \longrightarrow \tau^{\geq 0}Z$ into a distinguished triangle $I \longrightarrow Y \longrightarrow \tau^{\geq 0}Z$. By Proposition 5.6, $I \in \mathcal{D}^{\geq 0}$. We will now apply the octahedral axiom to $Y \longrightarrow Z \longrightarrow \tau^{\geq 0}Z$. To begin, note that a cone of $Y \longrightarrow Z$ is $X[1]$, a cone of $Y \longrightarrow \tau^{\geq 0}Z$ is $I[1]$ and a cone of $Z \longrightarrow \tau^{\geq 0}Z$ is $\tau^{\leq 0}Z$. Hence, we get a triangle

$$\tau^{\leq 0}(Z[-1]) \longrightarrow X \longrightarrow I \longrightarrow \tau^{\leq 0}Z.$$

Therefore, $I \in \mathcal{D}^{\leq 0}$, and so $I \in \mathcal{A}$. Now, $\tau^{\leq 0}(Z[-1]) \simeq \ker(f)$ and thus $I \simeq \mathrm{im}(f)$. Similarly, the triangle $I \longrightarrow Y \longrightarrow \tau^{\geq 0}Z$ gives that $I \simeq \mathrm{coim}(f)$. $\square$

Remark 5.8. For $m \leq n$ there exist natural isomorphisms

$$\tau^{\leq m}X \longrightarrow \tau^{\leq m}\tau^{\leq n}X$$

and

$$\tau^{\geq n}X \longrightarrow \tau^{\geq n}\tau^{\geq m}X.$$

Indeed, $\mathcal{D}^{\leq m} \subset \mathcal{D}^{\leq n}$, hence there exists a canonical morphism of functors that are adjoint to embeddings of these subcategories, and after one more application of $\tau^{\leq m}$ this morphism becomes an isomorphism. The second assertion is proved similarly. It is also true, but more difficult to check, that there exists a natural isomorphism $\tau^{\geq m}\tau^{\leq n}X \longrightarrow \tau^{\leq n}\tau^{\geq m}X$.

Definition. Define a functor $H^0 \colon \mathcal{D} \longrightarrow \mathcal{A}$ by

$$H^0(X) := \tau^{\leq 0}\tau^{\geq 0}X \simeq \tau^{\geq 0}\tau^{\leq 0}X.$$

We also set $H^n(X) = H^0(X[n])$.

Definition. Let $\mathcal{D}$ be a triangulated category. The *Grothendieck group* $K(\mathcal{D})$ is defined to be the quotient of the free abelian group on the set of isomorphism classes of $\mathcal{D}$ (we need $\mathcal{D}$ to be small for this) by the following relations: [B]=[A]+[C] whenever there exists a triangle in $\mathcal{D}$:

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1].$$

From the exact triangle $A \longrightarrow A \oplus B \longrightarrow B \longrightarrow T(A)$ one gets $[A] + [B] = [A \oplus B]$ and from the triangle $A \longrightarrow 0 \longrightarrow T(A) \longrightarrow T(A)$ we have $[T(A)] = -[A]$. Furthermore $[0] = 0$.

It is easily checked that this group has the following universal property. Whenever there is a function f from the set of isomorphism classes of objects in $\mathcal{D}$ to an abelian group G such that the Euler relations (the above relations) hold, then f factors through $K(\mathcal{D})$, that is, there is a unique group homomorphism $\overline{f}$ so that we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "f"] \arrow[dr, "{[\,]}"'] & & G \\ & K(\mathcal{D}) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

If $F : \mathcal{D} \longrightarrow \mathcal{D}'$ is a triangulated functor, then F induces a group homomorphism between $K(\mathcal{D})$ and $K(\mathcal{D}')$ by sending $[A]$ to $[F(A)]$.

If we consider a (small) abelian category $\mathcal{A}$, then the Grothendieck group of $\mathcal{A}$ it usually defined to be the free abelian group $K(\mathcal{A})$ generated by all objects where we factor out the subgroup generated by relations: $[F] = [F'] + [F'']$, whenever there is an exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0 \ .$$

Denote the image of an object F in $K(\mathcal{A})$ by $\psi(F)$. As in the above case there is a universal property, namely: Every additive function λ, i.e. $\lambda(F) = \lambda(F') + \lambda(F'')$ whenever there is an exact sequence as above, from $\mathcal{A}$ to an abelian group G factors through $K(\mathcal{A})$. Now considering an object of $\mathcal{A}$ as a 0-complex in $\mathrm{D^b}(\mathcal{A})$ and setting $\lambda = [\ ] : \mathcal{A} \longrightarrow K(\mathcal{D})$, we see that λ is an additive function because of the Euler relations (see Corollary 3.20) and therefore we get a homomorphism

$$\begin{aligned} \Phi : K(\mathcal{A}) &\longrightarrow K(\mathcal{D}) \\ \psi(F) &\longmapsto [F]. \end{aligned}$$

Now considering a complex $A^\bullet \in \mathrm{D^b}(\mathcal{A})$, we can use the truncation functors to write it as an alternating sum of its cohomology objects. It is then easily checked that these two constructions are inverse to each other. Hence $K(\mathcal{A}) \simeq K(\mathrm{D^b}(\mathcal{A}))$.

This last statement holds more generally for the heart $\mathcal{A}$ of a *bounded* t-structure on a triangulated category $\mathcal{D}$. Here, a t-structure is called bounded if for each $X \in \mathcal{D}$ there exist integers $m \leq n$ such that $X \in \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m}$. This last condition is needed to ensure that the alternating sum of the cohomology objects of X with respect to the t-structure is finite. We summarize this discussion in the following.

Proposition 5.9. *Let $\mathcal{D}$ be a triangulated category and $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a bounded t-structure with heart $\mathcal{A}$. Then $K(\mathcal{A}) \simeq K(\mathcal{D})$.* □

Our next goal is to produce new t-structures from given ones. This is done via torsion pairs.

Definition. Let $\mathcal{A}$ be an abelian category. A *torsion pair* in $\mathcal{A}$ are two full additive subcategories $(\mathcal{T}, \mathcal{F})$ such that for any $T \in \mathcal{T}$ and $F \in \mathcal{F}$ we have $\mathrm{Hom}(T, F) = 0$ and, furthermore, for any object $A \in \mathcal{A}$ there exists an exact sequence

$$0 \longrightarrow T \longrightarrow A \longrightarrow F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Example 5.10. Consider $\mathcal{A} = \mathrm{Ab}$, the category of abelian groups and take $\mathcal{T}$ to be the torsion groups and $\mathcal{F}$ to be the torsion free groups.

Remark 5.11. The exact sequence above is unique up to isomorphism. Indeed, assume there exists another one with objects T' and F'. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r, "i"] & A \arrow[r, "j"] \arrow[d, "\mathrm{id}"] & F \arrow[r] & 0 \\ 0 \arrow[r] & T' \arrow[r, "k"] & A \arrow[r, "l"] & F' \arrow[r] & 0 \end{tikzcd}$$

By definition of torsion pair we have $l \circ i = 0$ and since T is the kernel of l, we have a map $\phi : T \longrightarrow T'$. Similarly, we get a map $\psi : T' \longrightarrow T$, which by uniqueness has to be an inverse to ϕ. Therefore $T \simeq T'$ and similarly $F \simeq F'$.

Lemma 5.12. *For a torsion pair $(\mathcal{T}, \mathcal{F})$ we have $\mathcal{T}^{\perp} = \mathcal{F}$ and ${}^{\perp}\mathcal{F} = \mathcal{T}$.*

Proof. First recall that

$$\mathcal{T}^{\perp} = \{A \in \mathcal{A} \mid \mathrm{Hom}(T, A) = 0 \; \forall \; T \in \mathcal{T}\}$$

and

$${}^{\perp}\mathcal{F} = \{A \in \mathcal{A} \mid \mathrm{Hom}(A, F) = 0 \; \forall \; F \in \mathcal{F}\}$$

Let us prove the first claimed equality, the second is similar. Clearly any $F \in \mathcal{F}$ is in $\mathcal{T}^{\perp}$. Now take a $C \in \mathcal{T}^{\perp}$ and consider its short exact sequence

$$0 \longrightarrow T \longrightarrow C \longrightarrow F \longrightarrow 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Since the first map has to be zero, C is isomorphic to F. □

Lemma 5.13. *The category $\mathcal{F}$ is closed under subobjects and the category $\mathcal{T}$ is closed under quotients.*

Proof. Let us prove the first statement: consider a monomorphism $C \hookrightarrow F'$ with $F' \in \mathcal{F}$. By definition of torsion pair there is a sequence

$$0 \longrightarrow T \longrightarrow C \longrightarrow F \longrightarrow 0$$

The composition $f \colon T \longrightarrow C \longrightarrow F'$ has to be zero. On the other hand it is a monomorphism being a composition of monomorphisms. Therefore $T = 0$ ($f \circ \mathrm{id}_T = 0$, so $\mathrm{id}_T = 0$, since f is a monomorphism) and $C \simeq F \in \mathcal{F}$. The proof of the second statement is analogous. □

Remark 5.14. In general $\mathcal{F}$ is not closed under taking quotients and $\mathcal{T}$ is not closed under subobjects.

Now consider an abelian category $\mathcal{A}$, its bounded derived category $\mathrm{D^b}(\mathcal{A})$ and a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$. Then we have the following result.

Theorem 5.15. *The pair*

$$\mathcal{D}^{\leq 0} := \Big\{ A^\bullet \in \mathrm{D}^{\mathrm{b}}(\mathcal{A}) \mid H^i(A^\bullet) = 0\ \forall i > 0;\ H^0(A^\bullet) \in \mathcal{T} \Big\}$$

$$\mathcal{D}^{\geq 0} := \Big\{ A^\bullet \in \mathrm{D}^{\mathrm{b}}(\mathcal{A}) \mid H^i(A^\bullet) = 0\ \forall i < -1;\ H^{-1}(A^\bullet) \in \mathcal{F} \Big\}$$

is a t-structure on $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$.

Proof. Property (1) from the definition of a t-structure is clear (just note that $0 \in \mathcal{F}$ and $0 \in \mathcal{T}$). We will now check condition (2). Assume there exists a $0 \neq f \in \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathcal{A})}(A^\bullet, B^\bullet)$ with $A^\bullet \in \mathcal{D}^{\leq 0}$ and $B^\bullet \in \mathcal{D}^{\geq 1}$. We can represent f as

$$\begin{tikzcd} & C^\bullet & \\ A^\bullet \arrow[ur, "\phi"] & & B^\bullet, \arrow[ul, "s"'] \end{tikzcd}$$

where s is a quasi-isomorphism. Therefore, $C^\bullet \in \mathcal{D}^{\geq 1}$ and $0 \neq \phi \in \mathrm{Hom}_{\mathrm{K}^b(\mathcal{A})}(A^\bullet, C^\bullet)$. Thus, ϕ is a morphism of complexes which is not homotopic to zero.

Using Remark 3.22 and (TR3) we obtain a commutative triangle of distinguished triangles

$$\begin{tikzcd} \tau^{\leq 0}A^\bullet \arrow[r, "\mu"] \arrow[d, "\tau^{\leq 0}\phi"] & A^\bullet \arrow[r] \arrow[d, "\phi"] & \tau^{\geq 1}A^\bullet \arrow[r] \arrow[d, "h"] & \tau^{\leq 0}A^\bullet[1] \arrow[d] \\ \tau^{\leq 0}C^\bullet \arrow[r] & C^\bullet \arrow[r] & \tau^{\geq 1}C^\bullet \arrow[r] & \tau^{\leq 0}C^\bullet[1] \end{tikzcd}$$

By assumption, $\tau^{\geq 1}A^\bullet$ is acyclic and hence μ is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$. In particular, $\tau^{\leq 0}\phi \neq 0 \in \mathrm{K}^b(\mathcal{A})$.

Now, consider $\tau^{\leq 0}A^\bullet$ and define a subcomplex $\sigma(\tau^{\leq 0}A^\bullet)$ of it as follows, for $i < 0$ it has the same objects, the object in degree 0 is $\mathrm{im}(d^{-1})$ and the maps are clear. The quotient complex is clearly isomorphic to $H^0(A^\bullet)$. Thus, we obtain the following commutative diagram of triangles

$$\begin{tikzcd} \sigma(\tau^{\leq 0}A^\bullet) \arrow[r, "\mu"] \arrow[d, "\sigma(\tau^{\leq 0}\phi)"] & \tau^{\leq 0}A^\bullet \arrow[r] \arrow[d, "\tau^{\leq 0}\phi"] & H^0(A^\bullet) \arrow[r] \arrow[d, "h'"] & \sigma(\tau^{\leq 0}A^\bullet)[1] \arrow[d] \\ \sigma(\tau^{\leq 0}C^\bullet) \arrow[r] & \tau^{\leq 0}C^\bullet \arrow[r, "\rho"] & H^0(C^\bullet) \arrow[r] & \sigma(\tau^{\leq 0}C^\bullet)[1]. \end{tikzcd}$$

Now $\sigma(\tau^{\leq 0}C^\bullet)$ is acyclic, so ρ is an isomorphism in $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$. Since $H^0(A^\bullet) \in \mathcal{T}$ and $H^0(C^\bullet) \simeq H^0(B^\bullet) \in \mathcal{F}$, we get $h' = 0$. Hence also $\tau^{\leq 0}\phi = 0$, which is a contradiction.

Lastly, we will prove that condition (3) is also satisfied. Let $A^\bullet$ be a complex. Since $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$, we have a short exact sequence $0 \longrightarrow T \xrightarrow{\mu} H^0(A^\bullet) \xrightarrow{\pi} F \longrightarrow 0$. Now consider the following commutative diagram of exact sequences in $\mathcal{A}$ obtained by pullback along μ from the lower exact sequence.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{im}(d^{-1}) \arrow[r, "\mu''"] \arrow[d, "="] & E \arrow[r] \arrow[d, "\mu'"] & T \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & \mathrm{im}(d^{-1}) \arrow[r] & \ker(d^0) \arrow[r] \arrow[d] & H^0(A^\bullet) \arrow[r] \arrow[d, "\pi"] & 0 \\
 & & F \arrow[r, "="] \arrow[d] & F \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Factor $d^{-1}\colon A^{-1} \longrightarrow A^0$ through $\mathrm{im}(d^{-1})$ and write $d^{-1} = i\rho$. Let $\widetilde{d}^{-1} = \mu''\rho\colon A^{-1} \longrightarrow E$. Define a subcomplex $A'^\bullet$ of $A^\bullet$ by $A'^k = A^k$ for $k \leq -1$, $A'^0 = E$, $A'^k = 0$ for $k > 0$ and $d^k_{A'^\bullet} = d^k_{A^\bullet}$ for $k < -1$ and $d^{-1}_{A'^\bullet} = \widetilde{d}^{-1}$. By construction, $A'^\bullet \in \mathcal{D}^{\leq 0}$, since H^0 of this complex is $E/\,\mathrm{im}(d^{-1})$. Let $A''^\bullet$ be the quotient complex. We obtain a triangle $A'^\bullet \longrightarrow A^\bullet \longrightarrow A''^\bullet \longrightarrow A'^\bullet[1]$. Clearly, $H^k(A''^\bullet) = 0$ for $k < 0$. Now $A''^0 = A^0/E$, $A''^1 = A^1$ and we have a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d] & A^0 \arrow[r] \arrow[d, "d^0"] & A^0/E \arrow[r] \arrow[d, "\widetilde{d}^0"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & A^1 \arrow[r] & A^1 \arrow[r] & 0.
\end{tikzcd}$$

Thus, $H^0(A''^\bullet) = \ker(\widetilde{d}^0) \simeq \ker(d^0)/E \simeq F \in \mathcal{F}$. □

See the appendix for an application of the above to stability conditions.

6. Bondal's theorem

The goal of this section is to prove the following result due to Bondal.

Theorem 6.1. *let X be a smooth projective variety over an algebraically closed field k (if $k = \mathbb{C}$ one can think of a complex manifold). The category of coherent sheaves on X is an abelian category and we write $\mathrm{D}^{\mathrm{b}}(X)$ for its bounded derived category. Assume that $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(X)$ admits a strongly full exceptional sequence. Then there exists an equivalence $\mathcal{D} \simeq \mathrm{D}^{\mathrm{b}}(A - mod)$ for an explicitly described k-algebra A.*

Remark 6.2. The above statement holds in a more general setting. For instance, $\mathcal{D}$ could be the derived category of an abelian category which has enough injectives or projectives and satisfies the condition that the vector space $\oplus_i \mathrm{Hom}_{\mathcal{D}}(A, B[i])$ is finite-dimensional for any two objects $A, B \in \mathcal{D}$.

Of course, at this point even the statement of this result is not entirely clear. So, let us start with introducing all the necessary notions.

Definition. Let $\mathcal{D}$ be a k-linear triangulated category. An object E in $\mathcal{D}$ is called *exceptional* if $\mathrm{Hom}_{\mathcal{D}}(E,E)=k$ and $\mathrm{Hom}_{\mathcal{D}}(E,E[l])=0$ for $l\neq 0$. An *exceptional sequence* is a sequence $E_1,\ldots,E_n$ such that any E_i is an exceptional object and, furthermore, $\mathrm{Hom}_{\mathcal{D}}(E_i,E_j[l])=0$ for $i>j$ and all l. An exceptional sequence is *full* if the smallest triangulated subcategory of $\mathcal{D}$ containing all the E_i is equivalent to $\mathcal{D}$. Finally, a strongly exceptional sequence is an exceptional sequence such that $\mathrm{Hom}_{\mathcal{D}}(E_i,E_j[l])=0$ for $l\neq 0$.

Definition. Let $\mathcal{D}$ be a triangulated category and $\mathcal{D}'$ a full triangulated subcategory. The subcategory $\mathcal{D}'$ is called *right admissible* if the inclusion functor admits a right adjoint $\pi\colon \mathcal{D}\longrightarrow\mathcal{D}'$, that is, we have functorial isomorphisms $\mathrm{Hom}_{\mathcal{D}}(A,B)\simeq \mathrm{Hom}_{\mathcal{D}'}(A,\pi(B))$ for all $A\in\mathcal{D}'$ and all $B\in\mathcal{D}$.

Note that the right adjoint is exact by Proposition 1.16.

Proposition 6.3. *The following conditions are equivalent.*

(i) *A full triangulated subcategory $\mathcal{D}'\subset\mathcal{D}$ is right admissible.*

(ii) *For any object $A\in\mathcal{D}$ there exists a distinguished triangle*

$$B\longrightarrow A\longrightarrow C\longrightarrow B[1]$$

with $B\in\mathcal{D}'$ and $C\in\mathcal{D}'^{\perp}$, where $\mathcal{D}'^{\perp}=\{D\in\mathcal{D}\mid \mathrm{Hom}_{\mathcal{D}}(D',D)=0\;\forall D'\in\mathcal{D}'\}$.

(iii) *The smallest triangulated subcategory of $\mathcal{D}$ containing $\mathcal{D}'$ and $\mathcal{D}'^{\perp}$ is equivalent to $\mathcal{D}$.*

Proof. First of all, note that $\mathcal{D}'^{\perp}$ is a triangulated category. Indeed, taking a triangle where two out of three objects are in $\mathcal{D}'^{\perp}$, one applies $\mathrm{Hom}(B,-)$ with $B\in\mathcal{D}'$ to see this.

Now, suppose $\mathcal{D}'$ is admissible. Set $B:=\pi(A)$ and use the adjunction property to associate a map $B\longrightarrow A$ to the identity map of B ($\mathrm{Hom}_{\mathcal{D}'}(\pi(A),\pi(A))\simeq\mathrm{Hom}_{\mathcal{D}}(\pi(A),A)$). Completing this map to a triangle as in the statement of the proposition one checks that a cone is indeed in $\mathcal{D}'^{\perp}$ by applying $\mathrm{Hom}(B',-)$ for $B'\in\mathcal{D}'$ to the triangle:

$$\mathrm{Hom}(B',B)\simeq\mathrm{Hom}(B',\pi(A))\simeq\mathrm{Hom}(B',A).$$

Conversely, if such a triangle is given for any $A\in\mathcal{D}$, we can define the functor π on objects by sending A to B. To see that this is well-defined and functorial, consider a second triangle $B'\longrightarrow A'\longrightarrow C'\longrightarrow B'[1]$ and a map $A'\longrightarrow A$. Applying $\mathrm{Hom}(B',-)$ to the first triangle, we get $\mathrm{Hom}(B',B)\simeq\mathrm{Hom}(B',A)$ and hence we get a map $B'\longrightarrow B$ for any f. If $A'=A$, then this map is an isomorphism and this finishes the proof of the equivalence between (i) and (ii).

It is obvious that (ii) implies (iii). Suppose the converse holds. We have to check that the category of object fitting into the middle of a diagram as in (ii) is closed under shifts and taking cones. For shifts this is obvious, since $\mathcal{D}'$ and $\mathcal{D}'^{\perp}$ are triangulated subcategories and so we simply can shift the triangle. Let $f\colon A \longrightarrow A'$ be a map. We have checked above that we have a commutative diagram which is unique up to a unique isomorphism

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "\psi"] & A \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "\phi"] & B[1] \arrow[d] \\
B' \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & B'[1] \arrow[d] \\
C(\psi) & C(f) & C(\phi) & C(\phi)[1].
\end{tikzcd}$$

It is now a cumbersome but doable exercise to check that there exist maps completing the lower row to a triangle, which is what we wanted. $\square$

Lemma 6.4. *Let $\mathcal{D}$ be a k-linear triangulated category such that the vector space $\oplus_i \mathrm{Hom}_{\mathcal{D}}(A, B[i])$ is finite-dimensional for any two objects $A, B \in \mathcal{D}$. If $E \in \mathcal{D}$ is exceptional, then the objects $\oplus_i E[i]^{\oplus i}$ form an admissible subcategory of $\mathcal{D}$.*

Proof. The collection of the objects is a triangulated subcategory because it is equivalent to $\langle E \rangle$, the smallest triangulated subcategory of $\mathcal{D}$ containing E. Indeed, $\langle E \rangle$ is built from E by taking direct sums, shifts and cones of E and has the given explicit description because E is exceptional. Now, given any object $A \in \mathcal{D}$ consider the canonical map

$$\bigoplus \mathrm{Hom}_{\mathcal{D}}(E, A[i]) \otimes E[-i] \longrightarrow A$$

and complete it to a distinguished triangle

$$\bigoplus \mathrm{Hom}_{\mathcal{D}}(E, A[i]) \otimes E[-i] \longrightarrow A \longrightarrow B \longrightarrow (\mathrm{Hom}_{\mathcal{D}}(E, A[i]) \otimes E[-i])[1].$$

Applying $\mathrm{Hom}(E, -)$ to the triangle we conclude that $\mathrm{Hom}(E, B[k]) = 0$ for all k. We are now done by Proposition 6.3. $\square$

Note that for any exceptional object E the category $\langle E \rangle$ is equivalent to the bounded derived category of k-vector spaces by sending E to k (in degree 0).

Definition. A sequence of full admissible triangulated subcategories $\mathcal{D}_1, \ldots, \mathcal{D}_n$ of $\mathcal{D}$ is *semi-orthogonal* if for all $i > j$ we have $\mathcal{D}_j \subset \mathcal{D}_i^{\perp}$, that is, for objects $D_i \in \mathcal{D}_i$ the equality $\mathrm{Hom}(D_i, D_j) = 0$ holds. Such a sequence is full if $\mathcal{D}$ is generated by the $\mathcal{D}_i$ (in the same sense as for exceptional sequences).

Remark 6.5. An equivalent definition is the following. A semiorthogonal decomposition of a triangulated category $\mathcal{D}$ is a sequence of full triangulated subcategories $\mathcal{D}_1, \ldots, \mathcal{D}_n$ such that $\mathrm{Hom}(\mathcal{D}_i, \mathcal{D}_j) = 0$ for $i > j$ and for every object $D \in \mathcal{D}$ there exists a chain of morphisms

$$0 = D_n \longrightarrow D_{n-1} \longrightarrow \cdots \longrightarrow D_1 \longrightarrow D_0 = D$$

such that the cone of $D_k \longrightarrow D_{k-1}$ is contained in $\mathcal{D}_k$ for each $k = 1, \ldots, n$.

Note that the condition on the homomorphisms implies that the decompositions are unique and functorial. This shows that the inclusion functors admit right adjoints and hence this second description implies the first ("full" is obvious from the existence of the chain of morphisms). The converse is obvious.

Example 6.6. Let $E_1, \dots, E_n$ be a full exceptional sequence in $\mathcal{D}$. Then $\mathcal{D}_i := \langle E_i \rangle$ form a semi-orthogonal decomposition.

We also need some recollections on quivers.

Definition. Let k is an algebraically closed field.

(1) A *quiver* $Q = (Q_0, Q_1, s, t\colon Q_1 \longrightarrow Q_0)$ is given by a finite set of vertices Q_0 and a finite set of arrows Q_1. An arrow starts at $s(\rho)$ and terminates at $t(\rho)$, where $\rho \in Q_1$.
(2) A non-trivial *path* in Q is a sequence $\rho_1 \cdots \rho_m$ of arrows satisfying $t(\rho_{i+1}) = s(\rho_i)$ for all i (thus, in this example the path starts in $s(\rho_m)$ and terminates at $t(\rho_1)$). For each element $x \in Q_0$ e_x is the *trivial path* starting and ending in x.
(3) The *path algebra* $A = kQ$ is the k-algebra having the paths in Q as basis and with the product of two paths being the obvious composition if the paths are composable and 0 otherwise. The multplication is associative.

For example, the path algebra of the quiver with one vertex and one loop is isomorphic to $k[t]$, with t corresponding to the loop. Another example is the quiver $1 \xrightarrow{\rho} 2$ which is 3-dimensional as a k-vector space and the multiplication rules are, for example, $\rho e_1 = \rho$, $\rho e_2 = 0$ etc.

It is a basic fact that the path algebra kQ of a finite quiver without closed loops is hereditary, that is, the subobject of a projective kQ-module is projective. Thus, for any kQ-modules M, N $\mathrm{Ext}^i(M, N) = 0$ whenever $i \geq 2$.

A *representation* of a quiver associates to each vertex i a vector space V_i and to each edge $\rho\colon i \longrightarrow j$ a linear map $V_i \longrightarrow V_j$. It is clear what a map of representations is and it is easily checked that the category of representations of a quiver is an abelian category. In fact, it is equivalent to the category of kQ-modules as follows. Given a kQ-module M define the vector spaces V_i by $e_i M$ and the linear maps are given by the action of the corresponding arrows. Conversely, given a representation we get a module $M = \oplus_i V_i$.

Denoting by Ae_i the space having as base the paths starting at i, we clearly have $A = \oplus_i Ae_i$, thus the Ae_i are projective modules, since they are direct summands of the projective A-module A. In fact, these modules are the unique up to isomorphism indecomposable projective modules. Clearly, the representation V corresponding to the module Ae_i is $e_i Ae_i$, so $V_j = k$ for $j = i$ and 0 else, since $e_i Ae_i$ has as base the paths starting and ending in i and we have assumed that our quiver has no loops.

Lastly, a quiver with relations is the quotient of the path algebra of a quiver by an ideal generated by paths.

Proof of Theorem 6.1. Let $E_0, \dots, E_n$ be a strongly full exceptional sequence, set $E = \oplus_i E_i$ and $A = \mathrm{Hom}(E, E)$. Clearly, A is the path algebra of a quiver with relations which has

$n+1$ vertices and for the indecomposable projectives $P_0, \ldots, P_n$ of A we have isomorphisms $\mathrm{Hom}_{\mathcal{D}}(E_i, E_j) \simeq \mathrm{Hom}_A(P_i, P_j)$.

Let $A^\bullet \in \mathrm{D^b}(X)$ and define $\Phi(A^\bullet)$ to be the complex

$$\mathrm{Hom}^\bullet(E, A^\bullet) = \oplus_k \mathrm{Hom}(E, A^\bullet[k])[-k]$$

with zero differential. The action of A on this complex is obvious.

Clearly, $\Phi(E_i)$ is a complex of A-modules which has trivial cohomology for $i \neq 0$ and $H^0(\Phi(E_i)) = P_i$. Hence, $\Phi(E_i) \simeq P_i$ in the derived category.

We will now check that this functor is fully faithful, that is

$$\mathrm{Hom}(X, Y) \simeq \mathrm{Hom}(\Phi(X), \Phi(Y)). \tag{6.1}$$

First of all, note that the fully faithfulness is clear for objects E_i and E_j by the beginning of the proof. Since the E_i are a strongly full exceptional sequence, (6.1) also holds for shifts of the E_i. The set of all these objects has the property that any other object can be reached from it by taking one cone. So, let us consider triangles $A \longrightarrow X \longrightarrow B$ and $C \longrightarrow Y \longrightarrow D$ where (6.1) holds for the pairs (A, B), (A, C), (A, D) and (B, D). We get a commutative diagram

$$\begin{tikzcd}
 & {} & {} & {} & \\
\ldots \arrow[r] & \underline{\mathrm{Hom}(A, C)} \arrow[r] \arrow[u] & \mathrm{Hom}(A, Y) \arrow[r] \arrow[u] & \underline{\mathrm{Hom}(A, D)} \arrow[r] \arrow[u] & \ldots \\
\ldots \arrow[r] & \underline{\mathrm{Hom}(X, C)} \arrow[r] \arrow[u] & \mathrm{Hom}(X, Y) \arrow[r] \arrow[u] & \underline{\mathrm{Hom}(X, D)} \arrow[r] \arrow[u] & \ldots \\
\ldots \arrow[r] & \underline{\mathrm{Hom}(B, C)} \arrow[r] \arrow[u] & \mathrm{Hom}(B, Y) \arrow[r] \arrow[u] & \underline{\mathrm{Hom}(B, D)} \arrow[r] \arrow[u] & \ldots \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}$$

and applying Φ we get another commutative diagram where Φ provides an isomorphism for all underlined Hom-spaces. Therefore, Φ is fully faithful.

Finally, the functor Φ is essentially surjective, because $\mathrm{D^b}(A - \mathrm{mod}) \simeq K^b(\mathcal{P})$, where $\mathcal{P}$ is the category of projective A-modules and the latter is generated by the P_i which are in the image of Φ. $\square$

Example 6.7. Let us study one specific example. Consider the objects $\mathcal{O}$ and $\mathcal{O}(1)$ on $\mathbb{P}^1$. We know that $\mathrm{Hom}(\mathcal{O}, \mathcal{O}) = \mathbb{C} = \mathrm{Hom}(\mathcal{O}(1), \mathcal{O}(1))$. Furthermore, $\mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)) = \mathbb{C}^2$, $\mathrm{Hom}(\mathcal{O}(1), \mathcal{O}) = 0 = \mathrm{Hom}(\mathcal{O}, \mathcal{O}(1)[1]) = \mathrm{Hom}(\mathcal{O}(1), \mathcal{O}[1])$ and all the higher Ext-groups vanish anyway since the global dimension is 1. Thus, $\mathcal{O}, \mathcal{O}(1)$ is a strongly exceptional sequence. The algebra $A = \mathrm{Hom}(E, E)$, where $E = \mathcal{O} \oplus \mathcal{O}(1)$, is equal to the path algebra of the Kronecker quiver $\bullet \rightrightarrows \bullet$. Let us check that the colection $\mathcal{O}, \mathcal{O}(1)$ generates the category. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}(n-2) \xrightarrow{(y,-x)} \mathcal{O}(n-1)^{\oplus 2} \xrightarrow{(x,y)^t} \mathcal{O}(n) \longrightarrow 0.$$

Setting $n = 1$, we see that $\mathcal{O}(-1)$ is contained in the triangulated category generated by $\mathcal{O}$ and $\mathcal{O}(1)$, and setting $n = 2$ proves the same assetion for $\mathcal{O}(2)$. Iterating the argument we get $\mathcal{O}(k)$ for all $k \in \mathbb{Z}$. Since every sheaf on $\mathbb{P}^1$ can be resolved by sums of copies of the sheaves $\mathcal{O}(k)$, we conclude that our collection indeed generates the category. Thus, we have an equivalence $\mathrm{D^b}(\mathbb{P}^1) \simeq \mathrm{D^b}(A-\mathrm{mod})$.

7. Some remarks about homological mirror symmetry

In this section we very briefly recall some notions from algebraic and symplectic geometry and give a vague introduction to the homological mirror symmetry conjecture.

7.1. **Some notions from algebraic geometry.** The following is a very brief and fairly imprecise collection of some basic notions in algebraic geometry.

Classically, the objects of study in algebraic geometry were so-called *algebraic sets* $X \subset \mathbb{C}^n$ given as zero sets of polynomials, that is, $X = \{p \in \mathbb{C}^n \mid f_i(p) = 0, i = 1, \ldots, k\}$ for some $f_i \in \mathbb{C}[x_1, \ldots, x_n]$. Clearly, we can also just take the ideal I generated by the f_i and the zero set will not change. We denote it by $V(I) = X$. On the other hand, given an algebraic set X, we can consider $I(X) = \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(p) = 0\ \forall p \in X\}$. Note that this ideal is radical, that is, if a power of a polynomial is in $I(X)$, then so is the polynomial itself. Obviously, if $X = V(I)$, then $I(X)$ contains I and, furthermore, Hilbert's Nullstellensatz states that $I(V(I)) = \sqrt{I}$, the radical of I. In fact, there is a 1-1 correspondence between algebraic sets in $\mathbb{C}^n$ and radical ideals in $\mathbb{C}[x_1, \ldots, x_n]$.

Now, if one interprets elements of $\mathbb{C}[x_1, \ldots, x_n]$ as functions on $\mathbb{C}^n$, then functions on an algebraic set X are given by $\mathbb{C}[x_1, \ldots, x_n]/I(X)$. So we are actually studying finitely generated $\mathbb{C}$-algebras without nilpotent elements (since $I(X)$ is radical).

Clearly, this is a fairly restrictive class of objects. Instead one considers an arbitrary commutative ring with identity A, noetherian for simplicity, and associates to it a topological space, its *spectrum*, denoted by $\mathrm{Spec}(A)$. As a set, these are just all prime ideals. The topology is defined by declaring sets of the form $V(I) := \{P \in \mathrm{Spec}(A) \mid I \subset P\}$ for $I \subset A$ (clearly, we can also just take the ideal generated by I) to be the closed ones. Furthermore, one can define a sheaf of rings on $\mathrm{Spec}(A)$, denoted by $\mathcal{O}_{\mathrm{Spec}(A)}$. While we will not recall its definition, the important properties are the following. The stalk of $\mathcal{O}_{\mathrm{Spec}(A)}$ at a point $P \in \mathrm{Spec}(A)$ is A_P, the localisation of A at P. Given an element $f \in A$, we have an open set $D(f)$ in $\mathrm{Spec}(A)$ defined by $D(f) := \{P \in \mathrm{Spec}(A) \mid f \notin P\}$. The important thing is that the sections of $\mathcal{O}_{\mathrm{Spec}(A)}$ over $D(f)$ are A_f, the localisation of A in the multiplicative set $\left\{f^k,\ k \in \mathbb{N}_{\geq 0}\right\}$. More generally, given an A-module M, one has a sheaf over $\mathrm{Spec}(A)$, denoted by $\widetilde{M}$, which is a module over the sheaf $\mathcal{O}_{\mathrm{Spec}(A)}$. Its important properties are $\widetilde{M}(D(f)) = M_f$ and $\widetilde{M}_P = M_P$. Furthermore, the functor from A-modules to $\mathcal{O}_{\mathrm{Spec}(A)}$-modules given by $M \longrightarrow \widetilde{M}$ is fully-faithful, exact, commutes with direct sums and tensor products.

An *affine scheme* is, by definition the pair $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$. This is an example of a *locally ringed space*, that is, of a topological space equipped with a sheaf of rings whose stalks are local rings. A morphism between locally ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ of a continuous map $f\colon X \longrightarrow Y$ and a map $f^\sharp\colon \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ of sheaves of rings on Y. Furthermore, we assume that the induced map $f^\sharp_P$ on the stalks is a local homomorphism of local rings, that is, the inverse image of the maximal ideal is the maximal ideal. A *scheme* is now defined to be a localy ringed space which locally looks like an affine scheme. So, a scheme is, in particular, a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ a sheaf of rings on X. It then makes sense to say when a sheaf of abelian groups on X is an $\mathcal{O}_X$-module. An $\mathcal{O}_X$-module $\mathcal{F}$ is *quasi-coherent* if, roughly speaking, over an affine subspace $\mathrm{Spec}(A) \subset X$ it looks like $\widetilde{M}$ for an A-module M. It is *coherent* if M is a finitely generated module. The categories of (quasi-)coherent sheaves $(Q)\,\mathbf{Coh}(X)$ on a scheme are abelian. In fact, it determines the scheme. Namely, a result of Gabriel states that two schemes X and Y are isomorphic if and only if the categories $\mathbf{Coh}(X)$ and $\mathbf{Coh}(Y)$ are equivalent.

On the other hand, the bounded derived category $\mathrm{D^b}(X) := \mathrm{D^b}(\mathbf{Coh}(X))$ does not determine the geometry. In fact, Mukai proved that, given an abelian variety A and its dual abelian variety $\widehat{A}$, there is always an equivalence $\mathrm{D^b}(A) \simeq \mathrm{D^b}(\widehat{A})$ of triangulated categories, although in general A and $\widehat{A}$ are not isomorphic.

We, of course, know how $\mathrm{D^b}(X)$ looks like but under additional assumptions we can understand it even better. The assumptions will be that our scheme X is a complex smooth projective variety. Here, "complex" roughly means that all the rings X is glued from are finitely generated $\mathbb{C}$-algebras, "variety" means that these algebras are integral domains, "projective" means that X can embedded into projective space and "smooth" algebraically means that all stalks of the structure sheaf $\mathcal{O}_X$ are regular local rings. In this case we can associate to X a smooth complex manifold X^{an} and the sheaf $\mathcal{O}_X$ corresponds to the sheaf of holomorphic functions on X^{an}. Furthermore, given any coherent sheaf on X there is an associated sheaf on X^{an} and the sheaf cohomology groups agree.

A *locally free sheaf* on a scheme X is a sheaf which locally is isomorphic to the n-th direct sum of the structure sheaf. Any such sheaf gives a vector bundle over X and vice versa. If X is smooth and projective, then the full triangulated subcategory of complexes of locally free sheaves is actually equivalent to $\mathrm{D^b}(X)$. Combining this with the previous explanations, we could see objects in $\mathrm{D^b}(X)$ as complexes of holomorphic vector bundles.

7.2. **Symplectic geometry and homological mirror symmetry.** A *symplectic manifold* is a smooth manifold M equipped with a *symplectic form* ω, a closed $(d\omega = 0)$ differential 2-form, which is assumed to be non-degenerate (it gives an isomorphism between the tangent and cotangent bundle). Any such manifold has even dimension over $\mathbb{R}$. An *isotropic submanifold* N is a submanifold of M such that the symplectic form restricts to zero (tangent space of N is isotropic subspace of the tangent space of M). An isotropic submanifold is *Lagrangian* if it is of maximal dimension, that is, $\dim(N) = \frac{1}{2}\dim(M)$.

Now, let X be a Calabi–Yau variety, that is, a smooth projective variety with trivial canonical bundle. Alternatively see this as the data of a complex structure on a manifold and a Ricci flat Kähler form ω. This data (along with a "B-field") describes a sypersymmetric nonlinear sigma model which is believed to provide a (2,2) superconformal field theory (SCFT), which depends on the complex and the symplectic structure. A procedure called topological twisting is supposed to isolate the two structures. Kontsevich proposed that the two parts obtained via the topological twisting should be the derived category of coherent sheaves on the complex side and the Fukaya category of Lagrangian submanifolds on the symplectic side. The definition of the latter category is not as easy as that of the former one, but, very roughly speaking, objects are Lagrangian submanifolds and morphisms are defined in terms of Floer homology. One can enlarge this category to get something triangulated, denoted by $\mathrm{D}^{\mathrm{b}}\mathcal{F}(-)$.

The homological mirror symmetry (HMS) conjecture can now be stated as follows. If two Calabi–Yau manifolds X and X' define mirror symmetric SCFTs, then $\mathrm{D}^{\mathrm{b}}(X) \simeq \mathrm{D}^{\mathrm{b}}\mathcal{F}(X')$ and vice versa. The important thing here is that one side depends only on the complex structure and the other on the symplectic structure.

Assuming this conjecture does hold, the groups of exact autoequivalences on both sides should coincide. In particular, $\mathrm{D}^{\mathrm{b}}\mathcal{F}(X)$ comes with an action of symplectomorphisms modulo those homotopic to the identity. Denote this by $\pi_0(\mathrm{Sympl}(X))$. Hence, we get a group homomorphism $\pi_0(\mathrm{Sympl}(X)) \longrightarrow \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}\mathcal{F}(X'))$ and, by HMS, a group homomorhism

$$\pi_0(\mathrm{Sympl}(X)) \longrightarrow \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X)).$$

Now, in symplectic geometry there is the notion of a *Dehn twist*, a symplectomorphism which is a local construction that is performed along a Lagrangian sphere. The spherical twist functors found by Seidel and Thomas correspond to these Dehn twists via HMS, and were in some sense the first "honest" autoequivalences of $\mathrm{D}^{\mathrm{b}}(X)$, since they, in general, tear apart the standard t-structure, which is not the case for the autoequivalences known before.

8. Fourier–Mukai functors

The goal of this section is to unterstand how objects on the product of two smooth projective varieties give rise to exact functors between the derived categories of these varieties. The basic idea is fairly simple.

Definition. Let X and Y be two smooth projective complex varieties, write $q\colon X \times Y \longrightarrow X$ and $p\colon X \times Y \longrightarrow Y$ for the projections and let $\mathcal{P} \in \mathrm{D}^{\mathrm{b}}(X \times Y)$ be an object. The *Fourier–Mukai functor* $\Phi_{\mathcal{P}}$ with kernel $\mathcal{P}$ is defined to be the exact functor

$$\Phi_{\mathcal{P}}\colon \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(Y), \quad \mathcal{F}^\bullet \longmapsto p_*(q^*\mathcal{F}^\bullet \otimes \mathcal{P}).$$

Of course, all the functors in the above definition are required to be derived. Let us have a closer look at the background necessary to be able to write the formula down.

8.1. **Direct image.** Let X and Y be two schemes and $f\colon X \longrightarrow Y$ be a morphism. If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module. Using the map $f^\sharp\colon \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$, we see that $f_*\mathcal{F}$ is an $\mathcal{O}_Y$-module. This is the *direct image* of $\mathcal{F}$, which turns out to be a left-exact functor from $\mathbf{QCoh}(X)$ to $\mathbf{QCoh}(Y)$. To get a feeling for this, let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$

be two affine schemes and let $f\colon X \longrightarrow Y$ be a morphism (which corresponds to $\phi\colon B \longrightarrow A$). If N is an A-module and $\widetilde{N}$ the associated sheaf on X, then $f_*(\widetilde{N}) = (N_B)^{\sim}$, where N_B is N considered as a B-module.

Since $\mathbf{QCoh}(X)$ has enough injectives, the right derived functor

$$Rf_*\colon D^+(\mathbf{QCoh}(X)) \longrightarrow D^+(\mathbf{QCoh}(Y))$$

exists.

The *higher direct images* $R^i f_*(\mathcal{F}^\bullet)$ are, by definition, the cohomology sheaves of $Rf_*(\mathcal{F}^\bullet)$. The global section functor is a special case of the direct image functor, namely when $Y = \operatorname{Spec}(k)$.

It is a fact that for f as above $R^i f_*(\mathcal{F})$ are trivial for any quasi-coherent sheaf $\mathcal{F}$ and $i > \dim(X)$. Therefore, we have an exact functor

$$Rf_*\colon \mathrm{D^b}(\mathbf{QCoh}(X)) \longrightarrow \mathrm{D^b}(\mathbf{QCoh}(Y)).$$

One has to assume more if one wants a functor between $\mathrm{D^b}(\mathbf{Coh}(X))$ and $\mathrm{D^b}(\mathbf{Coh}(Y))$. Namely, if f is a proper morphism of noetherian schemes, then the higher direct images of a coherent sheaf $\mathcal{F}$ are again coherent. Thus, if f is proper, we get an exact functor

$$Rf_*\colon \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(Y)$$

defined as the composition of the embedding of $\mathrm{D^b}(X)$ into $\mathrm{D^b}(\mathbf{QCoh}(X))$ and the functor Rf_* for quasi-coherent sheaves. If X and Y are smooth and projective over $\mathbb{C}$, then any map f between them is in fact proper, hence we always get the derived functor.

8.2. **Inverse image.** If $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$-modules, there is an *inverse image* $f^*\mathcal{G}$ which is an $\mathcal{O}_X$-module. This gives a right-exact functor from $\mathcal{O}_Y$-modules to $\mathcal{O}_X$-modules.

We will not give the precise definition, but again just consider the affine case. So, let X, Y and $f\colon X \longrightarrow Y$ (corresponding to a ring homomorhism ϕ) be as in the previous subsection and let M be a B-module. Then $f^*(\widetilde{M}) = (M \otimes_B A)^{\sim}$. In particular, if ϕ is a flat map, then f^* is an exact functor.

Given a map between two schemes $f\colon X \longrightarrow Y$, one says that X is flat over Y if $\mathcal{O}_X$ is, which means that the stalk of $\mathcal{O}_X$ at any point x is a flat module over the stalk of $\mathcal{O}_Y$ at $y = f(x)$ (we can consider $\mathcal{O}_{X,x}$ as a module over $\mathcal{O}_{Y,y}$ via the map $f^\sharp$). The above translates to the fact that the functor f^* is exact if f is a flat map. It is a fact that the projection maps from $X \times Y$ to X and Y are flat if X and Y are smooth and projective. Hence, we do not need to derive q^* in the above definition.

8.3. **Tensor product.** Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on a smooth projective complex variety X. We can naturally define a presheaf on X by setting $(\mathcal{F} \otimes \mathcal{G})(U) := \mathcal{F}(U) \otimes \mathcal{G}(U)$ for any $U \subset X$ open. It is a fact that this construction does not give a sheaf in general, but through the construction of sheafification, the *tensor product sheaf* $\mathcal{F} \otimes \mathcal{G}$ exists and is coherent. In fact,

$$\mathcal{F} \otimes (-)\colon \mathbf{Coh}(X) \longrightarrow \mathbf{Coh}(X)$$

is a right exact functor, similar to what one expects from modules.

For any coherent sheaf $\mathcal{G}$ there exists a locally free sheaf $\mathcal{E}$ and a surjection from $\mathcal{E}$ to $\mathcal{G}$. In other words, $\mathbf{Coh}(X)$ has enough locally free sheaves and hence every sheaf has a locally free resolution. Furthermore, if $\mathcal{E}^\bullet$ is an acyclic bounded above complex of locally free sheaves, then $\mathcal{F} \otimes \mathcal{E}^\bullet$ is still acyclic (this follows from the local situation, as an acyclic complex of free modules remains acyclic if tensored by any module). Thus, the class of locally free sheaves is adapted to the functor $\mathcal{F} \otimes (-)$ and the left-derived functor

$$\mathcal{F} \otimes^L (-) \colon D^-(X) \longrightarrow D^-(X)$$

exists.

Since X is smooth, there always exists a resolution of length $n = \dim(X)$, so the above functor is actually defined on the bounded derived category $\mathrm{D^b}(X)$.

In order to define the derived tensor product of complexes, one has to work a bit harder. Let $\mathcal{F}^\bullet$ be a bounded above complex of coherent sheaves. This gives the exact functor

$$\mathcal{F}^\bullet \otimes (-) \colon K^-(\mathbf{Coh}(X)) \longrightarrow K^-(\mathbf{Coh}(X)),$$

$$(\mathcal{F}^\bullet \otimes \mathcal{E}^\bullet)^i := \bigoplus_{k+l=i} \mathcal{F}^k \otimes \mathcal{E}^l, \quad d = d_{\mathcal{F}} \otimes 1 + (-1)^i 1 \otimes d_{\mathcal{E}}.$$

One now has to verify that the category of complexes of locally free sheaves is adapted to $\mathcal{F}^\bullet \otimes (-)$. This boils down to checking that the image of an acyclic complex of locally free sheaves is again acyclic. Hence, one gets a functor on the bounded above complexes and since everything is smooth, in fact on $\mathrm{D^b}(X)$. By construction, the tensor product in the definition need not be derived, if $\mathcal{P}$ is a complex of locally free sheaves.

For later use we also review

8.4. **Derived dual.** If M is a finite rank module over a ring R, we can define the dual module to be $\mathrm{Hom}_R(M, R)$. There is a similar construction for sheaves. Namely, let $\mathcal{F}$ be a quasi-coherent sheaf on a smooth projective variety X. The quasi-coherent sheaf $\underline{Hom}(\mathcal{F}, \mathcal{E})$ is given by $X \supset U \longmapsto \mathrm{Hom}(\mathcal{F}_{|U}, \mathcal{E}_{|U})$. This gives a left exact endofunctor of $\mathbf{QCoh}(X)$, which can be derived since $\mathbf{QCoh}(X)$ has enough injectives. Similarly, one can also plug in a complex of sheaves. Under our assumption we get a bifunctor $\mathrm{D^b}(X)^{op} \times \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X)$. The dual of a complex $\mathcal{F}^\bullet \in \mathrm{D^b}(X)$ is defined to be $(\mathcal{F}^\bullet)^\vee := R\underline{Hom}(\mathcal{F}^\bullet, \mathcal{O}_X) \in \mathrm{D^b}(X)$. If $\mathcal{F}^\bullet$ is, for example, a line bundle (more generally a vector bundle), then the dual is just the dual line bundle (resp. vector bundle). Similarly, for a complex of vector bundles, taking the derived dual amounts to taking the usual dual $\underline{Hom}(-, \mathcal{O}_X)$ of each term in the complex.

8.5. **The construction and Orlov's result.** Using the above, we can associate an exact functor to any object $\mathcal{P} \in \mathrm{D^b}(X \times Y)$

$$\Phi_{\mathcal{P}} \colon \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(Y).$$

Note that our notation is slightly ambiguous, since we could have equally well defined a functor in the opposite direction. Before we come to examples, let us state an important result, due to Orlov, which shows why functors of this type are important.

Theorem 8.1. *Let $F\colon \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(Y)$ be an exact equivalence between bounded derived categories of two smooth projective varieties. Then there exists an object $\mathcal{P} \in \mathrm{D^b}(X \times Y)$, unique up to isomorphism, and an isomorphism of functors $F \simeq \Phi_{\mathcal{P}}$.* □

In fact, Orlov proves a more general statement, since he only needs F to be fully faithful and to admit a left and a right adjoint functor. The theorem was later extended to other situations.

Example 8.2. Let us consider some autoequivalences which always exist and the kernels associated to them.

(1) The identity functor is naturally isomorphic to the Fourier–Mukai transform with kernel $\mathcal{O}_\Delta$, the structure sheaf of the diagonal $\Delta \subset X \times X$. More generally, if f is an automorphism of X, then f_* (which is the inverse of f^*) is an exact equivalence on $\mathbf{Coh}(X)$ and hence induces an autoequivalence of $\mathrm{D^b}(X)$. The associated kernel is the structure sheaf $\mathcal{O}_{\Gamma_f}$ of the graph Γ_f of f.
(2) If L is a line bundle on X, then tensoring with it gives an exact equivalence of $\mathbf{Coh}(X)$ (the inverse is, of course, $L^{-1} \otimes (-)$) and hence we get an autoequivalence of $\mathrm{D^b}(X)$. The kernel here turns out to be $\iota_* L$, where $\iota\colon X \longrightarrow X \times X$ is the diagonal embedding.
(3) The shift functor $T\colon \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(X)$ can be described as the Fourier–Mukai transform with kernel $\mathcal{O}_\Delta[1]$.

If X is a smooth projective variety we a priori only have the above three types of autoequivalences of $\mathrm{D^b}(X)$ at our disposal. So, we have three injective maps $\mathbb{Z} \longrightarrow \mathrm{Aut}(\mathrm{D^b}(X))$, $\mathrm{Pic}(X) \longrightarrow \mathrm{Aut}(\mathrm{D^b}(X))$ and $\mathrm{Aut}(X) \longrightarrow \mathrm{Aut}(\mathrm{D^b}(X))$. In fact, sometimes this is all there is, as shown by the following result due to Bondal and Orlov.

Theorem 8.3. *Let X be as always a smooth projective complex variety and assume that either the canonical bundle or its dual is ample (that is, a power of it defines an embedding of X into projective space). Then*

$$\mathrm{Aut}(\mathrm{D^b}(X)) \simeq \mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)).$$

□

Note that two of the three types above come from the abelian level and the shift only moves the heart of the standard t-structure.

Remark 8.4. Let us collect some nice properties of Fourier–Mukai transforms. As always, X and Y are smooth projective complex varieties. We write ω for the canonical bundle.

(1) Let $\mathcal{P} \in \mathrm{D^b}(X \times Y)$ be an object, denote the associated Fourier–Mukai transform by F and define $\mathcal{P}_L := \mathcal{P}^\vee \otimes p^*\omega_Y[\dim(Y)]$ and $\mathcal{P}_R := \mathcal{P}^\vee \otimes q^*\omega_X[\dim(X)]$. Consider the induced FM transforms $G := \Phi_{\mathcal{P}_L}$ and $H := \Phi_{\mathcal{P}_R}$ from $\mathrm{D^b}(Y)$ to $\mathrm{D^b}(X)$. Then G is left adjoint to F and H is right adjoint to F, as follows from Grothendieck–Verdier duality.
(2) Let, in addition, Z be another variety and $\mathcal{Q} \in \mathrm{D^b}(Y \times Z)$ be an object. Write, for example, π_{XY} for the projection $X \times Y \times Z \longrightarrow X \times Y$. Then the composition of $\Phi_{\mathcal{P}}$ and $\Phi_{\mathcal{Q}}$ is isomorphic to the FM transform with kernel

$$\mathcal{R} = \pi_{XZ*}(\pi_{XY}^*\mathcal{P} \otimes \pi_{YZ}^*\mathcal{Q}).$$

We cannot expect the kernel of the composition to be uniquely determined if the composition is not an equivalence.

(3) We have, in fact, a functor Φ from $\mathrm{D}^{\mathrm{b}}(X \times Y)$ to the category of exact functors from $\mathrm{D}^{\mathrm{b}}(X)$ to $\mathrm{D}^{\mathrm{b}}(Y)$, where the morphisms in the latter are natural transformations compatible with shifts. This category is not expected to have a triangulated structure, because the cone is not functorial. However, it is additive and has a shift functor. The above functor Φ is, in general, neither full nor faithful.

9. Spherical twists

In this section we will construct a class of autoequivalences of $\mathrm{D}^{\mathrm{b}}(X)$ whose existence was predicted by homological mirror symmetry and which have the interesting property that they do not come from a functor on the level of $\mathbf{Coh}(X)$. These equivalences will be built from a certain class of objects.

Definition. Let X be a smooth complex projective variety of dimension d. An object $E \in \mathrm{D}^{\mathrm{b}}(X)$ is called *spherical* if (i) $E \otimes \omega_X \simeq E$ and (ii) $\mathrm{Hom}(E, E[i]) = \mathbb{C}$ for $i = 0, d$ and 0 otherwise.

Recall that X is a *Calabi–Yau variety* if $\omega_X \simeq \mathcal{O}_X$ and if $H^i(X, \mathcal{O}_X) = 0$ for $i \neq i, d$ (for example, the vanishing set of a generic quartic polynomial in $\mathbb{P}^3$ or a generic quintic polynomial in $\mathbb{P}^4$). It follows that $\mathcal{O}_X$ and, more generally, any line bundle on X is a spherical object.

Note that condition (ii) can be reformulated by saying that the Ext-groups of E have the same pattern as the cohomology of a d-dimensional sphere. This explains the name.

Given a spherical object E, we can now consider $\mathcal{P}_E$ defined by the following triangle

$$q^*E^\vee \otimes p^*E \longrightarrow \mathcal{O}_\Delta \longrightarrow \mathcal{P}_E \longrightarrow q^*E^\vee \otimes p^*E[1].$$

Here $\mathcal{O}_\Delta$ is the structure sheaf of the diagonal, that is, $\iota_*\mathcal{O}_X$ for the diagonal embedding $\iota\colon X \longrightarrow X \times X$ and q and p are the projections. The first map in the triangle is given as the composition of the restriction map

$$q^*E^\vee \otimes p^*E \longrightarrow \iota_*\iota^*(q^*E^\vee \otimes p^*E) = \iota_*(E^\vee \otimes E)$$

and the trace map $E^\vee \otimes E \longrightarrow \mathcal{O}_X$.

Definition. The *spherical twist* associated to a spherical object E is the Fourier–Mukai functor $\Phi_{\mathcal{P}_E}$ with kernel $\mathcal{P}_E$. This will be denoted by T_E.

Note that the object $\mathcal{P}_E$ is defined only up to a non-unique isomorphism, since the cone is not functorial.

Remark 9.1. A useful description of the action of the spherical twist on objects is the following

$$T_E(F) \simeq \mathrm{cone}(\oplus_i(\mathrm{Hom}(E, F[i]) \otimes E[-i]) \longrightarrow F).$$

This implies that

$$T_E(E) \simeq E[1 - \dim(X)] \quad \text{and} \quad T_E(F) \simeq F$$

for $F \in E^\perp$.

In order to prove that a spherical twist is an equivalence we will need some general notions that we will now recall.

Definition. A collection Ω of objects in a triangulated category $\mathcal{D}$ is called a *spanning class* if for all $B \in \mathcal{D}$ the following two conditions are satisfied.

(1) If $\mathrm{Hom}(A, B[i]) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \simeq 0$.
(2) If $\mathrm{Hom}(B[i], A) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \simeq 0$.

We will need the following general result.

Lemma 9.2. *Let $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ be a functor between arbitrary categories and let G be a left adjoint. Then the functor morphism $g\colon G \circ F \longrightarrow \mathrm{id}_{\mathcal{A}}$ induces the following commutative diagram for any $A, A' \in \mathcal{A}$:*

$$\begin{tikzcd}
\mathrm{Hom}(A, A') \arrow[d, "\circ g_A"'] \arrow[dr, "F"] & \\
\mathrm{Hom}(GF(A), A') \arrow[r, "\sim"] & \mathrm{Hom}(F(A), F(A')).
\end{tikzcd}$$

A similar statement holds for a right adjoint H (see below for the appropriate triangle).

Proof. Let $f\colon A \longrightarrow A'$ be an arbitrary map and let $B \in \mathcal{B}$. Then the following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}(G(B), A) \arrow[r, "\sim"] \arrow[d, "f\circ"'] & \mathrm{Hom}(B, F(A)) \arrow[d, "F(f)\circ"] \\
\mathrm{Hom}(G(B), A') \arrow[r, "\sim"] & \mathrm{Hom}(B, F(A')).
\end{tikzcd}$$

Apllying this to $B = F(A)$ gives

$$\begin{tikzcd}
\mathrm{Hom}(G(F(A)), A) \arrow[r, "\sim"] \arrow[d, "f\circ"'] & \mathrm{Hom}(F(A), F(A)) \arrow[d, "F(f)\circ"] \\
\mathrm{Hom}(G(F(A)), A') \arrow[r, "\sim"] & \mathrm{Hom}(F(A), F(A')).
\end{tikzcd}$$

The vertical map on the right sends $\mathrm{id}_{F(A)}$ to $F(f)$. Its image running in the opposite direction is $f \circ g_A$. The proof for the right adjoint is similar. $\square$

Proposition 9.3. *Let $F\colon \mathcal{D} \longrightarrow \mathcal{D}'$ be an exact functor admitting a left adjoint G and a right adjoint H. Suppose Ω is a spanning class of $\mathcal{D}$ such that for any two objects $A, A' \in \Omega$ and all $i \in \mathbb{Z}$ the natural homomorphisms*

$$F\colon \mathrm{Hom}(A, A'[i]) \longrightarrow \mathrm{Hom}(F(A), F(A')[i])$$

are bijective. Then F is fully faithful.

Proof. The above lemma gives the following commutative diagram for arbitrary $A, B \in \mathcal{D}$ (g is as above and $h\colon \mathrm{id}_{\mathcal{B}} \to F \circ H$):

$$\begin{tikzcd} \mathrm{Hom}(A,B) \arrow[r, "h_B\circ"] \arrow[d, "\circ g_A"'] \arrow[dr, "F"] & \mathrm{Hom}(A, FH(B)) \arrow[d, "\sim"] \\ \mathrm{Hom}(GF(A),B) \arrow[r, "\sim"] & \mathrm{Hom}(F(A),F(B)). \end{tikzcd} \tag{9.1}$$

Let $A \in \Omega$. Complete the map $GF(A) \to A$ to a triangle

$$GF(A) \xrightarrow{g_A} A \longrightarrow C \longrightarrow GF(A)[1].$$

Applying $\mathrm{Hom}(-, B)$ for an arbitrary object $B \in \mathcal{D}$ gives, in combination with the lower commutative triangle, the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(C,B[i]) \arrow[r] & \mathrm{Hom}(A,B[i]) \arrow[r, "\circ g_A"] \arrow[dr, "F"] & \mathrm{Hom}(GF(A),B[i]) \arrow[d, "\sim"] \\ & & \mathrm{Hom}(F(A),F(B)[i]). \end{tikzcd}$$

If $B \in \Omega$, the map F is an isomorphism, hence the map $\mathrm{Hom}(A, B[i]) \to \mathrm{Hom}(GF(A), B[i])$ is an isomorphism. Thus $\mathrm{Hom}(C, B[i]) = 0$ for all $i \in \mathbb{Z}$, thus $C \simeq 0$ and $g_A\colon GF(A) \to A$ is an isomorphism for all $A \in \Omega$.

Therefore, if $A \in \Omega$, all the maps in (9.1) are isomorphisms. In particular,

$$h_B\circ\colon \mathrm{Hom}(A,B) \to \mathrm{Hom}(A, HF(B))$$

is an isomorphism for any $B \in \mathcal{D}$. Consider a triangle

$$B \longrightarrow HF(B) \longrightarrow C \longrightarrow B,$$

apply $\mathrm{Hom}(A, -)$ for $A \in \Omega$ to it to conclude that $C \simeq 0$ and hence $h_B\colon B \simeq HF(B)$ for any $B \in \mathcal{D}$. Thus, $h_B\circ\colon \mathrm{Hom}(A,B) \simeq \mathrm{Hom}(A, HF(B))$ for any $A \in \mathcal{D}$. Looking at the upper triangle in (9.1) we see that this implies that F gives a bijection for any $A, B \in \mathcal{D}$ as claimed. □

Proposition 9.4. *If E is a spherical object, then T_E is fully faithful.*

Proof. We will use Proposition 9.3. Note that any FM-transform has a right and a left adjoint. Set $\Omega = E \cup E^{\perp}$, where $E^{\perp} = \{F \mid \mathrm{Hom}(E, F[i]) = 0\ \forall i \in \mathbb{Z}\}$. Let us first check that this is indeed a spanning class. Suppose that F is an object such that $\mathrm{Hom}(G, F[i]) = 0$ for all $G \in \Omega$ and all $i \in \mathbb{Z}$. In particular, this holds for $G = E$, so $F \in \Omega$. Hence, $\mathrm{Hom}(F, F) = 0$ and thus $F \simeq 0$.

If F is such that $\mathrm{Hom}(F, G[i]) = 0$ for all $G \in \Omega$ and all $i \in \mathbb{Z}$, we use Serre duality which states that

$$\mathrm{Hom}(A,B) \simeq \mathrm{Hom}(B, A \otimes \omega_X[\dim(X)])^*$$

for all $A, B \in \mathrm{D^b}(X)$. So, the above condition translates to $\mathrm{Hom}(G, F \otimes \omega_X[i]) = 0$ (here we use that $\mathrm{D^b}(X)$ has finite-dimensional Hom-spaces). As before, this gives $F \otimes \omega_X \simeq 0$ and hence $F \simeq 0$.

We now have to check that

$$T_E \colon \mathrm{Hom}(G_1, G_2[i]) \simeq \mathrm{Hom}(T_E(G_1), T_E(G_2)[i])$$

for all $G_1, G_2 \in \Omega$ and all i. This is clear if $G_1 = E$ and $G_2 \in E^\perp$, since both sides are then zero. Next one considers the case $E = G_1 = G_2$. The image of $\mathrm{id} \in \mathrm{Hom}(E, E)$ is the identity map on $T_E(E) \simeq E[1 - \dim(X)]$. The argument for $i = \dim(X)$ is similar.

Lastly, we have the case where G_1 and G_2 are in $E^\perp$. Here one uses that $T_E(G_1) \simeq G_1$ and can check that the above map is indeed an isomorphism. □

We are left with checking that T_E is an equivalence. Again, we first need some general notions and results.

Definition. A *decomposition* of a triangulated category $\mathcal{D}$ is given by non-trivial triangulated subcategories $\mathcal{D}_1$ and $\mathcal{D}_2$ such that

(1) $\mathrm{Hom}(D_1, D_2) = 0 = \mathrm{Hom}(D_2, D_1)$ for all $D_i \in \mathcal{D}_i$.
(2) For all X in $\mathcal{D}$ there exists a triangle

$$D_1 \longrightarrow X \longrightarrow D_2 \longrightarrow D_1[1]$$

with $D_i \in \mathcal{D}_i$.

A triangulated category is called *indecomposable* if it does not admit a decomposition.

Remark 9.5. It can be shown that $\mathrm{D^b}(X)$ is indecomposable if and only if X is connected.

Proposition 9.6. *Let $F \colon \mathcal{D} \longrightarrow \mathcal{D}'$ be a fully faithful exact functor between triangulated categories. Suppose that $\mathcal{D} \neq 0$ and that $\mathcal{D}'$ is indecomposable. Then F is an equivalence if it has a left adjoint G and a right adjoint H such that for any $Y \in \mathcal{D}'$ one has: $H(Y) \simeq 0$ implies $G(Y) \simeq 0$.*

Proof. Define $\mathcal{D}'_1$ to be the image of F and $\mathcal{D}'_2$ to be the kernel of H. These are triangulated subcategories of $\mathcal{D}'$. The former can be equivalently described as the subcategory of objects Y such that $F(H(Y)) \simeq Y$ (any such object is, of course, in the image of F). This is seen as follows. If $Y \simeq F(X)$, then $H(Y) \simeq H(F(X)) \simeq X$, since

$$\mathrm{Hom}(X, X) \simeq \mathrm{Hom}(F(X), F(X)) \simeq \mathrm{Hom}(X, HF(X)),$$

so $Y \simeq F(X) \simeq FH(Y)$. Now, let Y be an object in $\mathcal{D}'$. Complete the natural map $FH(Y) \longrightarrow Y$ to a triangle

$$FH(Y) \longrightarrow Y \longrightarrow W \longrightarrow FH(Y)[1].$$

Since, by purely categorical arguments, $HFH(Y) \simeq H(Y)$, the object W has the property that $H(W) \simeq 0$. Hence, any object Y can be decomposed by a distinguished triangle as in the definition. Write $FH(Y) = D_1$ and $W = D_2$. For any $D_i \in \mathcal{D}'_i$ we have

$$\mathrm{Hom}(D_1, D_2) \simeq \mathrm{Hom}(FH(Y), W) \simeq \mathrm{Hom}(H(Y), H(W)) \simeq 0$$

and

$$\mathrm{Hom}(D_2, D_1) \simeq \mathrm{Hom}(W, FH(Y)) \simeq \mathrm{Hom}(G(W), H(Y)) \simeq 0,$$

since by assumption $H(D_2) \simeq 0$ implies $G(D_2) \simeq 0$. Since $\mathcal{D}'$ is indecomposable, either $\mathcal{D}'_1$ or $\mathcal{D}'_2$ is trivial. If $\mathcal{D}'_1$ is trivial, then $F(X) \simeq 0$ for all $X \in \mathcal{D}$, so $HF(X) \simeq 0$ and thus $X \simeq 0$ which is a contradiction. Hence, $\mathcal{D}'_2$ has to be trivial, therefore F is an equivalence. □

This is applied in the geometric setting as follows.

Proposition 9.7. *Let $\Phi_{\mathcal{P}} \colon \mathrm{D^b}(X) \longrightarrow \mathrm{D^b}(Y)$ be a fully faithful Fourier–Mukai transform between smooth projective varieties. If $\dim(X) = \dim(Y)$ and $\mathcal{P} \otimes q^*\omega_X \simeq \mathcal{P} \otimes p^*\omega_Y$, then $\Phi_{\mathcal{P}}$ is an equivalence.*

Proof. By the previous proposition we need to check that the adjoint functors $G = \Phi_{\mathcal{P}_L}$ and $H = \Phi_{\mathcal{P}_R}$ satisfy: $H(\mathcal{F}^\bullet) \simeq 0$, then $G(\mathcal{F}^\bullet) \simeq 0$. Recalling that $\mathcal{P}_L := \mathcal{P}^\vee \otimes p^*\omega_Y[\dim(Y)]$ and $\mathcal{P}_R := \mathcal{P}^\vee \otimes q^*\omega_X[\dim(X)]$, we dualise our assumption and get that $G \simeq H$. □

Remark 9.8. In fact, the converse of the statement in the proposition also holds.

We are now ready to prove the

Theorem 9.9. *If E is a spherical object, then T_E is an equivalence.*

Proof. We only need to check that $\mathcal{P}_E \otimes q^*\omega_X \simeq \mathcal{P}_E \otimes p^*\omega_X$. This basically follows from $E \otimes \omega_X \simeq E$, since $\mathcal{P}_E$ is a cone of $q^*(E^\vee) \otimes p^*(E) \longrightarrow \mathcal{O}_\Delta$, so

$$\begin{aligned}
\mathcal{P} \otimes q^*\omega_X &= C\big(q^*(E^\vee \otimes \omega_X) \otimes p^*(E) \longrightarrow \mathcal{O}_\Delta \otimes q^*\omega_X\big) \\
&\simeq C(q^*E^\vee \otimes p^*E \longrightarrow \mathcal{O}_\Delta \otimes p^*\omega_X) \simeq \mathcal{P}_E \otimes p^*\omega_X.
\end{aligned}$$

□

Any Fourier–Mukai transform induces a group homomorphism on the level of Grothendieck groups. For this we need the group homomorphism $f_! \colon K(X) \longrightarrow K(Y)$ induced by a map $X \longrightarrow Y$, which is defined as $f_!(\mathcal{F}) = \sum (-1)^i R^i f_*(\mathcal{F})$. Given a class $e \in K(X \times Y)$ one then defines

$$\Phi_e^K \colon K(X) \longrightarrow K(Y), \quad \alpha \longmapsto p_!(e \otimes q^*(\alpha)).$$

It can be checked that the following diagram commutes

$$\begin{tikzcd}
\mathrm{D^b}(X) \arrow[r, "\Phi_{\mathcal{P}}"] \arrow[d] & \mathrm{D^b}(Y) \arrow[d] \\
K(X) \arrow[r, "\Phi_{\mathcal{P}}^K"] & K(Y).
\end{tikzcd}$$

In fact, one can descend even further and consider the induced map on rational cohomology. The action of the spherical twist on cohomology can be described explicitly and is going to be done next.

First, let $H^*(X, \mathbb{Q})$ be the rational cohomology of the complex manifold associated to a smooth projective variety X. It has a natural ring structure. Any map $f \colon X \longrightarrow Y$ induces a map $f^*\colon H^*(Y,\mathbb{Q}) \longrightarrow H^+(X,\mathbb{Q})$ and using Poincaré duality one sees that there also exists a map $f_*\colon H^*(X,\mathbb{Q}) \longrightarrow H^{*+2\dim(Y)-2\dim(X)}(Y,\mathbb{Q})$. Using this, one can define, for any class $\alpha \in H^*(X \times Y,\mathbb{Q})$, the map $\Phi^H_\alpha\colon H^*(X,\mathbb{Q}) \longrightarrow H^*(Y,\mathbb{Q}), \beta \longmapsto p_*(q^*\beta.\alpha)$, where p and q are the projections as usual.

One option to pass from $K(X)$ to $H^*(X,\mathbb{Q})$ is given by the Chern character. Recall that the Chern character is additive and the Chern character of a line bundle L is given by

$$\mathrm{ch}(L) = \sum_i c_1(L)^i/i!,$$

where the first Chern class c_1 of a line bundle is defined by using the map $H^1(X,\mathcal{O}_X^*) = \mathrm{Pic}(X) \longrightarrow H^2(X,\mathbb{Z})$ induced by the exponential sequence. One can reduce the computation of the Chern character of a vector bundle to line bundles by using the splitting principle.

Also recall that the Todd class is a characteristic class having the following properties: $\mathrm{td}(E_1 \oplus E_2) = \mathrm{td}(E_1)\mathrm{td}(E_2)$ and $\mathrm{td}(L) = \frac{c_1(L)}{1-\exp(-c_1(L))}$ for any line bundle L. One usually writes $\mathrm{td}(X)$ for $\mathrm{td}(T_X)$.

Definition. Let $E^\bullet \in \mathrm{D^b}(X)$. Its *Mukai vector* is the cohomology class

$$v(E^\bullet) = \mathrm{ch}(E^\bullet)\sqrt{\mathrm{td}(X)}.$$

The square root in the above formula is the cohomology class whose square is $\mathrm{td}(X)$. Its existence is shown by a formal power series calculation (note that the degree 0 term of $\mathrm{td}(X)$ is 1).

The induced map $v\colon K(X) \longrightarrow H^*(X,\mathbb{Q})$ is additive.

To establish compatibilities between the FM-transform and the transform it induces on the cohomological level, one needs the Grothendieck–Riemann–Roch formula. It states that for a projective map $f\colon X \longrightarrow Y$ and any $e \in K(X)$ we have

$$\mathrm{ch}(f_!(e))\mathrm{td}(Y) = f_*(\mathrm{ch}(e)\mathrm{td}(X)).$$

This can be used to check that, given a class $e \in K(X \times Y)$, the following diagram commutes

$$\begin{tikzcd} K(X) \arrow[r, "\Phi^K_e"] \arrow[d, "v"'] & K(Y) \arrow[d, "v"] \\ H^*(X,\mathbb{Q}) \arrow[r, "\Phi^H_{v(e)}"] & H^*(Y,\mathbb{Q}). \end{tikzcd}$$

Using Hirzebruch–Riemann–Roch we have

$$\begin{aligned} \chi(E^\bullet,F^\bullet) := {} & \sum_i \dim_{\mathbb{C}} \mathrm{Hom}(E^\bullet,F^\bullet[i]) = \chi(X,(E^\bullet)^\vee \otimes F^\bullet) \\ = {} & \int_X \mathrm{ch}((E^\bullet)^\vee)\mathrm{ch}(F^\bullet)\mathrm{td}(X) = \int_X \mathrm{ch}((E^\bullet)^\vee)\sqrt{\mathrm{td}(X)}\mathrm{ch}(F^\bullet)\sqrt{\mathrm{td}(X)} \\ = {} & \int_X \mathrm{ch}((E^\bullet)^\vee)\sqrt{\mathrm{td}(X)}v(F^\bullet) \end{aligned}$$

One would like to express $\mathrm{ch}((E^\bullet)^\vee)\sqrt{\mathrm{td}(X)}$ in terms of $v(E^\bullet)$ and this is indeed possible. Namely,

$$v(E^\bullet) = v(E^\bullet)^\vee \exp(\frac{c_1(X)}{2}),$$

where for any $v = \sum_k v^k \in \oplus H^{2k}(X, \mathbb{Q})$, we set $v^\vee := \sum_k (-1)^k v_k$.

Based on the above we define the *Mukai pairing* on $H^*(X, \mathbb{Q})$ by

$$\langle v, v' \rangle = \int_X \exp(\frac{c_1(X)}{2})(v^\vee . v').$$

By construction, we have

$$\chi(E^\bullet, F^\bullet) = \langle v(E^\bullet), v(F^\bullet) \rangle.$$

With all this information one can prove the following

Proposition 9.10. *Let $\Phi_{\mathcal{P}}\colon \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(Y)$ be an equivalence. Then the cohomological Fourier–Mukai transform*

$$\Phi_{\mathcal{P}}^H\colon H^*(X, \mathbb{Q}) \longrightarrow H^*(Y, \mathbb{Q})$$

is an isometry. □

Note that if E is spherical, then $\langle v(E), v(E) \rangle$ is equal to 2 if the $\dim_{\mathbb{C}}(X)$ is even and 0 otherwise. One can prove the

Proposition 9.11. *Let E be spherical object in $\mathrm{D}^{\mathrm{b}}(X)$. Then the induced isometry T_E^H is given by $v \longmapsto v - \langle v(E), v \rangle v$.* □

In particular, if X is even-dimensional, then T_E acts on $H^*(X, \mathbb{Q})$ by reflection in the hyperplane orthogonal to $v(E)$.

Appendix: Stability conditions

Tilting theory originated in representation theory, in particular of hereditary algebras. Lately, it was used in a different context, namely in the construction of stability conditions.

A *stability condition* on a triangulated category $\mathcal{D}$ consists of a heart $\mathcal{A}$ of a bounded t-structure and a stability function with HN-property on $\mathcal{A}$. Let us explain these notions. A *stability function* on an abelian category $\mathcal{A}$ is a group homomorphism $Z\colon K(\mathcal{A}) \longrightarrow \mathbb{C}$ such that for all $0 \neq E \in \mathcal{A}$ the complex number $Z(E)$ lies in the space

$$H := \{r \exp(i\pi\phi) \mid r > 0 \text{ and } 0 < \phi \leq 1\} \subset \mathbb{C}.$$

The *phase* of an object $E \in \mathcal{A}$ is then defined to be

$$\phi(E) = \frac{1}{\pi} \arg(Z(E)) \in (0, 1].$$

Given a function Z one says that an object $E \in \mathcal{A}$ is *semistable* with respect to Z if for any subobject $0 \neq F \subset E$ one has $\phi(F) \leq \phi(E)$.

A stability function Z is said to have the *Harder–Narasimhan property* if any object possesses a HN-filtration: A *Harder–Narasimhan filtration* of a nonzero object $E \in \mathcal{A}$ is a finite chain of subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

such that the factors $F_i = E_i/E_{i-1}$ are semistable objects in $\mathcal{A}$, called the HN-factors of E, of $\mathcal{A}$ and

$$\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n).$$

This concept was introduced by Bridgeland inspired by work of Douglas on Π-stability in string theory. The set of all stability conditions satisfying a further technical assumption called local finiteness is a (possibly infinite-dimensional) complex manifold. The interest in this space comes, as alluded to above, from string theory. To be more precise, we consider the bounded derived category of coherent sheaves $\mathrm{D^b}(X)$ on a smooth complex projective variety X and study its stability manifold. In fact, in this context one studies numerical stability conditions. Define a bilinear form on $\mathrm{D^b}(X)$, the *Euler form*, by

$$\chi(E, F) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}(E, F[i]).$$

The *numerical Grothendieck group* $N(X)$ is the quotient of $K(\mathrm{D^b}(X)) \simeq K(\mathbf{Coh}(X))$ by the nullspace of the Euler form. It is a fact that $N(X)$ is a free abelian group of finite rank. The manifold of numerical stability conditions is then finite-dimensional.

If X is a Calabi–Yau threefold, then a suitable quotient of the space of numerical stability conditions is expected to be the stringy Kähler moduli space of X.

Thus, in order to construct examples of stability conditions one needs a bounded t-structure plus a function on it. If X is one-dimensional we can indeed take the standard t-structure with heart $\mathbf{Coh}(X)$ and construct a stability condition with this heart. But if $\dim(X) \geq 2$, then $\mathbf{Coh}(X)$ does not admit a function with HN-property. The method of tilting gives us the possibility to construct new hearts.

Let X be a surface. One takes $\mathbb{R}$-divisors β and ω so that ω is in the ample cone

$$\mathrm{Amp}(X) = \left\{\omega \in \mathrm{NS}(X) \otimes \mathbb{R} \mid \omega^2 > 0 \text{ and } \omega \cdot C > 0 \text{ for any curve } C \subset X\right\}.$$

Recall that the slope $\mu_\omega(E)$ of a torsion-free sheaf E on X with respect to ω is defined by

$$\mu_\omega(E) = \frac{c_1(E) \cdot \omega}{\mathrm{rk}(E)}.$$

This gives us the possibility to define semistability with respect to the slope. It turns out that any torsion-free sheaf has a HN-filtration with respect to μ_ω. One can then show that there exists a torsion pair $(\mathcal{T}, \mathcal{F})$ on $\mathrm{Coh}(X)$ defined as follows: The category $\mathcal{T}$ consists of those sheaves whose torsion-free parts have μ_ω-semistable HN-factors of slope $\mu_\omega > \beta \cdot \omega$ and $\mathcal{F}$ consists of torsion-free sheaves on S all of whose μ_ω-semistable HN-factors have slope $\mu_\omega \leq \beta \cdot \omega$.

It follows from this construction that the torsion pair does not depend on β, but only on ω and the product $\beta \cdot \omega$. Tilting with respect to this torsion pair gives a heart $\mathcal{B} \subset \mathrm{D^b}(X)$ on which a stability condition can be constructed.

Unfortunately, there is so far no example of a stability condition on a Calabi–Yau threefold. It is known that tilting the standard heart $\mathbf{Coh}(X)$ twice gives a bounded t-structure and there is a candidate for a stability condition on it. However, the HN-property is not yet known to hold.

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