EXERCISES, LIE ALGEBRAS, UNIVERSITY OF HAMBURG, SUMMER SEMESTER 2016

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Bonus sheet

The goal of this sheet is to thoroughly investigate the symplectic algebra $\mathfrak{sp}(2n,\mathbb{C})$.

Exercise 1. Let $K=\mathbb{R}$ or $K=\mathbb{C}$. Given an $n\times n$-matrix $M\in\mathrm{Mat}(n,K)$, denote by $M^\dagger$ the transpose if $K=\mathbb{R}$ and the hermitian conjugate (that is, $M^\dagger=\overline{M}^t$) if $K=\mathbb{C}$.

Let $\mathfrak{g}\subset\mathrm{Mat}(n,K)$ be a Lie subalgebra with the property that whenever $M\in\mathfrak{g}$, then also $M^\dagger\in\mathfrak{g}$.

(1) Define $(-,-)\colon\mathfrak{g}\times\mathfrak{g}\longrightarrow K$ as $(M,N)=\mathrm{tr}(MN^\dagger)$. Show that this is a scalar product on $\mathfrak{g}$.
(2) Given a subset S in $\mathfrak{g}$, define $S'=\{x\in\mathfrak{g}\mid (x,s)=0\ \forall s\in S\}$. Show that if I is an ideal, so is I'.
(3) Show that if $I\subset\mathfrak{g}$ is an ideal, then $\mathfrak{g}=I\oplus I'$. Conclude that $\mathfrak{g}=\bigoplus_i\mathfrak{g}_i$ where each $\mathfrak{g}_i$ has no proper ideals.
Remark: $\mathfrak{g}_i$ is not necessarily simple as it may still be abelian.
(4) Show that if $\mathfrak{g}$ has trivial centre, then $\mathfrak{g}$ is semisimple.
(5) Let K be any field of characteristic 0. Show that $\mathfrak{sp}(2n,K)$ has trivial centre. In particular, $\mathfrak{sp}(2n,\mathbb{C})$ is semisimple.

Exercise 2. Consider the semisimple Lie algebra $\mathfrak{sl}(2n,\mathbb{C})$. Define, for $1\le i,j\le n$, the $(2n\times 2n)$-matrices

$$X_{i,j}=E_{i,j}-E_{n+j,n+i}\ ,\quad Y_{i,j}=E_{i,n+j}+E_{j,n+i}\ ,\quad Z_{i,j}=E_{n+i,j}+E_{n+j,i}\ .$$

Let

$$\mathfrak{h}=\mathrm{span}_{\mathbb{C}}\{\,X_{i,i}\mid i=1,\dots,n\,\}.$$

(1) Define the sets $X=\{X_{i,j}\mid i,j=1,\dots,n\}$, $Y=\{Y_{i,j}\mid 1\le i\le j\le n\}$, $Z=\{Z_{i,j}\mid 1\le i\le j\le n\}$. Show that $X\cup Y\cup Z$ is a basis of $\mathfrak{sp}(2n,\mathbb{C})$.
(2) Give the adjoint action of

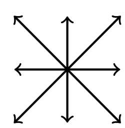
$$h=\sum_{i=1}^n h_i\,X_{i,i}\ \in\mathfrak{h}$$

($h_i\in\mathbb{C}$) on each of the $X_{i,j}$, $Y_{i,j}$, $Z_{i,j}$. Conclude that $\mathfrak{h}$ is a toral subalgebra of $\mathfrak{sp}(2n,\mathbb{C})$.
(3) Show that $\mathfrak{h}$ is a maximal toral subalgebra (also called Cartan algebra) of $\mathfrak{sp}(2n,\mathbb{C})$.

(4) Define the basis $\{\chi_i\}_{i=1,\ldots,n}$ of the dual space $\mathfrak{h}^*$ via $\chi_i(X_{j,j}) = \delta_{i,j}$. Give the roots of $\mathfrak{sp}(2n,\mathbb{C})$ in terms of the χ_i.
Hint: You should find $2n^2$ roots.

Exercise 3. Recall the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sp}(2n,\mathbb{C})$ defined in the previous exercise, as well as the matrices $X_{i,i}$.

(1) Let $h, k \in \mathfrak{h}$ be given by $h = \sum_{i=1}^n h_i X_{i,i}$ and $k = \sum_{i=1}^n k_i X_{i,i}$ (with $h_i, k_i \in \mathbb{C}$). Show that $\kappa_{\mathfrak{sp}(2n,\mathbb{C})}(h,k) = 4(n+1)\sum_{i=1}^n h_i\, k_i$.
Hint: In the previous exercise you found that the roots of $\mathfrak{sp}(2n,\mathbb{C})$ are
$$\{\pm\chi_i \pm \chi_j \,|\, 1 \le i < j \le n\} \cup \{\pm 2\chi_i \,|\, i = 1,\ldots,n\}\ .$$
(here $\pm\chi_i \pm \chi_j$ stands for four distinct terms).
(2) Show that $E = \{\sum_{i=1}^n l_i\, \chi_i \mid l_i \in \mathbb{R}\}$ and that $(\chi_i, \chi_j) = \frac{1}{4(n+1)}\delta_{i,j}$.
(3) In the case of $\mathfrak{sp}(4,\mathbb{C})$ (so $n = 2$), verify that there is an isometry $E \longrightarrow \mathbb{R}^2$ such that the image of the root system is:



Exercise 4. Using the roots of $\mathfrak{sp}(2n,\mathbb{C})$ found in Exercise 2 and recalled in Exercise 3, show that a possible choice of base Δ for Φ is $\{\alpha_1,\ldots,\alpha_n\}$ with
$$\alpha_i = \chi_i - \chi_{i+1} \text{ (for } i = 1,\ldots,n-1) \quad \text{and} \quad \alpha_n = 2\chi_n\ .$$