EXERCISES, LIE ALGEBRAS, UNIVERSITY OF HAMBURG, SUMMER SEMESTER 2016

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Sheet 9

Exercise 1. Let $\mathfrak{g}$ be a semisimple Lie algebra and write $\mathfrak{g} = \oplus_{i=1}^{t} \mathfrak{g}_i$, where the $\mathfrak{g}_i$ are the simple ideals of $\mathfrak{g}$. Prove that the semisimple and nilpotent parts of $x \in \mathfrak{g}$ are the sums of the semisimple and nilpotent parts of the components of x in the various $\mathfrak{g}_i$.

Exercise 2. Recall from Exercise 1 on Sheet 5 that the map $\rho\colon \mathfrak{sl}(2, K) \longrightarrow \mathfrak{gl}(K[s, t])$ defined by

$$\rho(H) = s\tfrac{\partial}{\partial s} - t\tfrac{\partial}{\partial t}, \quad \rho(X) = s\tfrac{\partial}{\partial t}, \quad \rho(Y) = t\tfrac{\partial}{\partial s}.$$

is a representation of $\mathfrak{sl}(2, K)$ (in the exercise K was $\mathbb{C}$ but the proof works without changes for any algebraically closed field of characteristic zero). Show that, for any integer $m \geq 0$, the subspace of polynomials of degree m is an irreducible subrepresentation, denoted by $V(m)$.

Remark: In particular, we have now constructed, for any integer $m \geq 1$, an irreducible $\mathfrak{sl}(2, K)$-module of dimension m.

Exercise 3. Let V be an $(m + 1)$-dimensional vector space over K with basis $(v_0, \ldots, v_m)$. Show that for $\lambda \in K$ and with the convention $v_{-1} = v_{m+1} = 0$, the formulas $H.v_i = (\lambda - 2i)v_i$, $Y.v_i = (i + 1)v_{i+1}$ and $X.v_i = (\lambda - i + 1)v_{i-1}$ of Lemma 11.3 make V into an irreducible representation of $\mathfrak{sl}(2, K)$, denoted by $V(m)$.

Exercise 4. Suppose that $\mathrm{char}(K) = p > 0$ and consider $\mathfrak{g} = \mathfrak{sl}(2, K)$. Show that the representation $V(m)$ constructed in the previous exercises is irreducible as long as the highest weight m is strictly smaller than p, but reducible if $m = p$.

Exercise 5. Consider the Lie algebra $\mathfrak{h} = \mathfrak{sl}(3, K)$. It contains $\mathfrak{g} = \mathfrak{sl}(2, K)$ in its upper left-hand 2×2-corner and is a $\mathfrak{g}$-module via the adjoint action. Write $\mathfrak{h}$ as a direct sum of irreducible $\mathfrak{g}$-modules.