EXERCISES, LIE ALGEBRAS, UNIVERSITY OF HAMBURG, SUMMER SEMESTER 2016

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Sheet 8

Exercise 1.

(1) Let $\mathfrak{g}$ be a Lie algebra over a field K and let β be an associative bilinear form on $\mathfrak{g}$. Show that the map $\tilde{\beta}\colon \mathfrak{g} \longrightarrow \mathfrak{g}^*$, $x \longmapsto \beta(x, -)$ is an intertwiner of $\mathfrak{g}$-modules.

(2) Let $\mathfrak{g}$ be a finite-dimensional complex simple Lie algebra and let β be an associative bilinear form on $\mathfrak{g}$. Show that there is a $\lambda \in \mathbb{C}$ such that $\beta = \lambda \cdot \kappa_{\mathfrak{g}}$.
Remark: In particular, β is automatically symmetric, and if β is non-zero, it is already non-degenerate.

Exercise 2. What is the smallest dimension of a $\mathbb{C}$-vector space on which $\mathfrak{sl}(n, \mathbb{C})$ (with $n \geq 2$) can act nontrivially? Prove your answer.

Exercise 3. Let $\mathfrak{g}$ act on $(\mathfrak{g} \otimes \mathfrak{g})^*$ via the adjoint representation. Recall that if V is any vector space, then $(V \otimes V)^*$ can be identified with the space of bilinear forms on V. Prove that β is an associative bilinear form if any only if $\mathfrak{g} \cdot \beta = 0$.

Exercise 4. Prove the following statements.

(1) If $\mathfrak{g}$ is a reductive Lie algebra, then $\mathfrak{g}$ is a completely reducible $\operatorname{ad}\mathfrak{g}$-module.
(2) If $\mathfrak{g}$ is a reductive Lie algebra, then $\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ with $[\mathfrak{g}, \mathfrak{g}]$ semisimple.
(3) If $\mathfrak{g}$ is a completely reducible $\operatorname{ad}\mathfrak{g}$-module, then $\mathfrak{g}$ is reductive.
(4) If V is a finite dimensional representation of a reductive Lie algebra $\mathfrak{g}$ on which $Z(\mathfrak{g})$ acts by semisimple endomorphisms, then V is completely reducible.