EXERCISES, LIE ALGEBRAS, UNIVERSITY OF HAMBURG, SUMMER SEMESTER 2016

P. SOSNA

Sheet 7

Exercise 1. Let K be any field and $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in K$ be elements such that $\alpha_i \neq \alpha_j$ for $i \neq j$. Show that there exists a polynomial $r \in K[T]$ of degree at most $n-1$ such that $r(\alpha_i) = \beta_i$ for all i.

Exercise 2. Let K be a field of characteristic $p > 0$. Consider the $p \times p$-matrices $y = \mathrm{diag}(0, \dots, p-1)$ (that is, y is a diagonal matrix with $0, \dots, p-1$ on the diagonal) and

$$x = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 1 \\ 1 & 0 & \dots & \dots & 0 \end{pmatrix}.$$

Prove that $[x, y] = x$ and conclude that $\mathfrak{g} = \mathrm{span}_K(x, y)$ is a solvable Lie algebra. Show that x and y have no common eigenvector.

Remark: This example shows the failure of Lie's theorem in finite characteristic.

Exercise 3.

(1) Let $\mathfrak{g}$ be a nilpotent algebra. Show that the Killing form of $\mathfrak{g}$ is identically zero.
(2) Show that a Lie algebra $\mathfrak{g}$ is solvable of and only if $[\mathfrak{g}, \mathfrak{g}]$ is contained in the radical of the Killing form.

Exercise 4. Let $\mathfrak{g}$ be the unique (up to isomorphism) non-abelian two dimensional Lie algebra. Show that the Killing form of $\mathfrak{g}$ is non-trivial.

Exercise 5. Let $\mathfrak{g}$ be a three dimensional vector space with basis (x, y, z) which becomes a Lie algebra with respect to the bracket $[x, y] = z$, $[x, z] = y$ and $[y, z] = 0$. Show that this Lie algebra is solvable but not nilpotent. Compute the radical of its Killing form.