EXERCISES, LIE ALGEBRAS, UNIVERSITY OF HAMBURG, SUMMER SEMESTER 2016

P. SOSNA

Sheet 6

Exercise 1. Let (V, ρ) be a finite dimensional representation of a finite dimensional Lie algebra $\mathfrak{g}$. Show that V is a direct sum of irreducible representations if and only if every subrepresentation $U \subset V$ admits a complement, that is, there exists a subrepresentation W such that $U \oplus W = V$.

Exercise 2. Let V be a finite dimensional vector space over an algebraically closed field and let $x, y \in \mathrm{End}(V)$ be two commuting endomorphisms. Prove that $(x+y)_s = x_s + y_s$ and $(x+y)_n = x_n + y_n$. Give an example of two non-commuting endomorphisms where this conclusion does not hold.

Hint: It might be helpful to use the following well-known fact from linear algebra: Two commuting diagonalisable endomorphisms can be diagonalised simultaneously.

Exercise 3. Let K be an algebraically closed field of characteristic zero and let $\mathfrak{g}$ be a finite dimensional solvable Lie algebra over K. Suppose that $V \neq 0$ is an irreducible finite dimensional representation of $\mathfrak{g}$. Prove that $\dim_K V = 1$.

Exercise 4. The Heisenberg Lie algebra over the field $\mathbb{C}$ is spanned by three elements a, $\bar{a}$, and k. All Lie brackets between these elements, except $[a, \bar{a}]$ and $[\bar{a}, a]$, are zero. For the latter, we have $[a, \bar{a}] = k$. Denote this Lie algebra by $\mathfrak{g}$.

(1) Is $\mathfrak{g}$ nilpotent? Is $\mathfrak{g}$ solvable? Prove your answers.
(2) Let (V, ρ) be be an irreducible representation of $\mathfrak{g}$ with $V \neq 0$. Prove the following statements.
  (a) If V is finite-dimensional, then it is in fact one-dimensional and k must act as zero.
  (b) Suppose k does not act as zero on V (thus, V is infinite dimensional). Suppose further that there is a nonzero $\Omega \in V$ with $a.\Omega = 0$, and that Ω is unique in the sense that $av = 0$ implies $v = \lambda\Omega$ for some $\lambda \in \mathbb{C}$ (in other words, $\ker(\rho(a)) = \mathbb{C}\Omega$). Show that k acts as a multiple of the identity on V.