EXERCISES, LIE ALGEBRAS, UNIVERSITY OF HAMBURG, SUMMER SEMESTER 2016

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Sheet 5

Exercise 1. Let $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be the standard basis of $\mathfrak{sl}(2,\mathbb{C})$. Let $\mathbb{C}[s,t] = \mathrm{span}_{\mathbb{C}}\big\{ s^m t^n \,\big|\, m,n = 0,1,2,\dots \big\}$ be the complex vector space of polynomials in s and t. Define the map $\rho\colon \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathfrak{gl}(\mathbb{C}[s,t])$ as

$$\rho(H) = s\tfrac{\partial}{\partial s} - t\tfrac{\partial}{\partial t}\ ,\ \rho(X) = s\tfrac{\partial}{\partial t}\ ,\ \rho(Y) = t\tfrac{\partial}{\partial s}.$$

Show that $(\mathbb{C}[s,t],\rho)$ is a representation of $\mathfrak{sl}(2,\mathbb{C})$. State whether or not this representation is irreducible and prove your answer.

Exercise 2. A representation U of a Lie algebra $\mathfrak{g}$ is called *indecomposable* if it is not isomorphic to a direct sum of representations $V \oplus W$ such that both V and W are non-zero. Consider the one-dimensional complex Lie algebra $\mathfrak{g} = \mathbb{C}$. Give an indecomposable representation of $\mathfrak{g}$ which is not irreducible and prove your answer.

Exercise 3. Let $\mathfrak{h}$ be a Lie algebra over a field K and let V be an $\mathfrak{h}$-module. Suppose V is a direct sum of simultaneous eigenspaces for $\mathfrak{h}$, that is, there is a subset $S \subset \mathfrak{h}^*$ such that

$$V = \bigoplus_{\lambda \in S} V_\lambda$$

and where $h.v = \lambda(h)\, v$ for all $v \in V_\lambda$ and $h \in \mathfrak{h}$. Let $W \subset V$ be an $\mathfrak{h}$-submodule. Let $w \in W$ be an arbitrary non-zero element. By the direct sum decomposition above, we can write $w = \sum_{i=1}^n v_i$ with $v_i \in V_{\lambda_i}$, $v_i \neq 0$, and mutually distinct $\lambda_i \in S$. Show that $v_i \in W$ for all i.

Hint: One can proceed as follows. Assume that at least one v_i is not in W.

(1) Let $T \subset \{1,\dots,n\}$ be a minimal subset such that
(i) $v_i \notin W$ for all $i \in T$,
(ii) $\mathrm{span}_K\{v_i \mid i \in T\} \cap W \neq \{0\}$. Here, minimal means that one cannot omit any element from T such that (ii) remains true. Show that such a T exists and that it contains at least 2 elements.

(2) Let $\alpha,\beta \in T$ be two distinct elements (which exist by (1)). Write $R = T \setminus \{\alpha,\beta\}$. By (ii) there is a nonzero $w' \in W$ such that $w' = a\, v_\alpha + b\, v_\beta + \sum_{i\in R} c_i\, v_i$ for some $a, b, c_i \in K$. Show that a, b and all c_i are non-zero.

(3) Pick an $h \in \mathfrak{h}$ such that $\lambda_\alpha(h) - \lambda_\beta(h) \neq 0$. Act with $h - \lambda_\beta(h)\,\mathrm{id}_V$ on the two sides of the definition of w' in (2). Find a contradiction to the assumptions in (1) and conclude that all v_i are in W.

Exercise 4. Let $\mathfrak{g} = \mathfrak{gl}(n, K)$ for K a field with $\mathrm{char}(K) \neq 2$ and $n > 1$. Let $\mathfrak{h} \subset \mathfrak{g}$ be the diagonal matrices, that is, $\mathfrak{h} = \mathfrak{d}(n, K)$. For $1 \leq i, j \leq n$ define the element $\alpha_{i,j} \in \mathfrak{h}^*$ on the basis element $E_{m,m}$ as $\alpha_{i,j}(E_{m,m}) = \delta_{i,m} - \delta_{j,m}$.

(1) Show that $\mathrm{ad}(h)(E_{i,j}) = \alpha_{i,j}(h)\, E_{i,j}$ for all $h \in \mathfrak{h}$.
(2) The algebra $\mathfrak{g}$ is an $\mathfrak{h}$-module via the adjoint action. Show that $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \bigoplus_{\lambda \in S} \mathfrak{g}_\lambda$ into simultaneous $\mathfrak{h}$-eigenspaces $\mathfrak{g}_\lambda$ as in the previous exercise. Give the set $S \subset \mathfrak{h}^*$ and the $\mathfrak{g}_\lambda$ for all $\lambda \in S$.
(3) Let I be an ideal in $\mathfrak{g}$. Show that if $E_{i,j} \in I$ for some pair of indices (i, j) with $i \neq j$ implies $\mathfrak{sl}(n, K) \subset I$.
(4) Let I be an ideal in $\mathfrak{g}$. Show that if I is contained in the diagonal matrices $\mathfrak{d}(n, K)$, then I is contained in the multiples of the identity, $I \subset \mathfrak{s}(n, K)$.
(5) Show that the full list of ideals in $\mathfrak{gl}(n, K)$ is $\{0\}$, $\mathfrak{s}(n, K)$, $\mathfrak{sl}(n, K)$, $\mathfrak{gl}(n, K)$.
(6) Show that $\mathfrak{sl}(n, K)$ is simple if $\mathrm{char}(K) = 0$.