EXERCISES, LIE ALGEBRAS, UNIVERSITY OF HAMBURG, SUMMER SEMESTER 2016

P. SOSNA

Sheet 3

Exercise 1. Prove the following proposition stated in the lecture.

Let A and B be algebras.

(1) If $\varphi\colon A \longrightarrow B$ is a homomorphism, then $\ker(\varphi)$ is an ideal in A.
(2) If $I \subset A$ is an ideal, then there exists a unique algebra structure on the quotient vector space A/I such that the canonical projection $\pi\colon A \longrightarrow A/I$ is an algebra homomorphism.
(3) If $\varphi\colon A \longrightarrow B$ is an algebra homomorphism and $I \subset A$ is an ideal contained in the kernel of φ, then there exists a unique algebra homomorphism $\widetilde{\varphi}\colon A/I \longrightarrow B$ such that $\widetilde{\varphi} \circ \pi = \varphi$. In particular, $A/\ker(\varphi) \simeq \operatorname{im}(\varphi)$.
(4) If $\varphi\colon A \longrightarrow B$ is an algebra homomorphism and $J \subset B$ is an ideal, then $\varphi^{-1}(J)$ is an ideal in A.
(5) If $\varphi\colon A \longrightarrow B$ is a *surjective* algebra homomorphism and $I \subset A$ is an ideal, then $\varphi(I)$ is an ideal in B.

Exercise 2. Let I be an ideal in a Lie algebra $\mathfrak{g}$. Show that $I^{(k)}$ and I^k is an ideal in $\mathfrak{g}$ for all k.

Exercise 3. Let $\mathfrak{g}$ be a three-dimensional Lie algebra with the property $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. Prove that $\mathfrak{g}$ is simple.

Exercise 4. Let K be a field and $\mathfrak{g}$ be a Lie algebra over K. Show that the following are equivalent.

(1) $\mathfrak{g}$ is solvable.
(2) There is a finite collection of subalgebras $\mathfrak{g}_k \subset \mathfrak{g}$ such that (i) $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_{n-1} \supset \mathfrak{g}_n = \{0\}$, (ii) $\mathfrak{g}_{k+1}$ is an ideal in $\mathfrak{g}_k$, and (iii) $\mathfrak{g}_k/\mathfrak{g}_{k+1}$ is abelian.

Exercise 5.

(1) Show that the Lie algebra
$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in K \right\}$$
from Exercise 2 on Sheet 2 is solvable but not nilpotent.
(2) Let K be a field of characteristic two. Show that the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,K)$ is nilpotent. In particular, it is solvable and therefore not simple.

Remark. The following is meant as a brief recollection of some facts from abstract algebra which might come useful in the course of the lecture. If you are not familiar with a statement and do not understand it completely, work out the details for yourself, ask in the next exercise session or write an email.

A *ring* is a set R together with maps $+\colon R \times R \longrightarrow R$ and $\cdot\colon R \times R \longrightarrow R$ called *addition* and *multiplication*, respectively, such that the following conditions are satisfied 1) $(R, +)$ is an abelian group, 2) $(R, \cdot)$ is a semigroup and 3) the distributive laws are satisfied (for instance $(x + y) \cdot z = x \cdot z + y \cdot z$). A ring is called *unital* if there exists an element $1 \in R$ such that $x \cdot 1 = 1 \cdot x = x$ for all $x \in R$. A ring is called *commutative* if $x \cdot y = y \cdot x$ for all $x, y \in R$. Note that any field is in particular a commutative unital ring where every nonzero element has a multiplicative inverse. In general, one defines the *units* in a ring as the set $R^* = \{x \in R \mid \exists y : xy = yx = 1\}$.

A *homomorphism* of rings is a map $f\colon R \longrightarrow S$ such that $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in R$. If R and S are unital, one requires the condition $f(1_R) = 1_S$.

Recall that we defined an associative unital K-algebra to be a K-vector space A together with a bilinear associative operation which admits a unit. In particular, A is a unital ring. Now define a map of K-vector spaces $K \longrightarrow A$ by sending $1 \longrightarrow 1_A$. This map is easily seen to be a unital ring homomorphism. Note that this map is injective because its kernel is an ideal in K and K has no non-trivial ideals. It follows that the image is contained in the *center* of A, namely the subring (this can be defined for any ring) $Z(A) = \{x \in A \mid xy = yx \ \forall y \in A\}$. A ring R is commutative if and only if $Z(R) = R$.

Note that $\mathbb{Z}$ is a commutative unital ring. Also note that there exists exactly one ring homomorphism from $\mathbb{Z}$ to an arbitrary unital ring R, namely the morphism which sends 1 to 1_R.

Now recall that a vector space over a field is an abelian group endowed with a scalar multiplication by the field. Substituting the field by a commutative unital ring leads to the concept of a *module*. So, if R is a commutative unital ring, then a module over R is an abelian group M with a scalar multiplication $R \times M \longrightarrow M$ satisfying the same properties as are required in the definition of a vector space. In particular, R itself is a module over R and so is the group R^n for any n. If R is not commutative, one can still talk about modules but now there are *right modules* and *left modules* corresponding to the cases $M \times R \longrightarrow M$ and $R \times M \longrightarrow M$. As in the case of vector spaces, one defines linear maps between (left/right) modules.

So, a module over a field is just a vector space. On the other hand, a module over $\mathbb{Z}$ is actually just an abelian group. Conversely, any abelian group is a $\mathbb{Z}$-module.

A module M over R is called *free* if it is isomorphic to a module of the form R^n for some n. In this case, there is a basis for M. Note that there are modules which are not free and hence do not have a basis. An example is the abelian group $\mathbb{Z}/2\mathbb{Z}$. It is a $\mathbb{Z}$-module, but it cannot be free, because it is finite.

As in the case of fields we can define matrices over a ring. If the ring is commutative and unital, one defines the determinant of a square matrix as in the case of matrices over a field. A square matrix turns out to be invertible if and only if its determinant is a unit. In particular, an injective map $R^n \longrightarrow R^n$ need not be surjective. For instance, $\mathbb{Z} \longrightarrow \mathbb{Z}$, $1 \longmapsto 2$ is an injective homomorphism of $\mathbb{Z}$-modules (equivalently, an injective group homomorphism), but it is not surjective which is to be expected since the determinant of this map is $2 \notin \mathbb{Z}^* = \{\pm 1\}$.

Given a (left/right) module M over a ring R, a (left/right) *submodule* is a subgroup $N \subset M$ such that the restriction of the scalar multiplication makes N into a (left/right) module over R. In particular, a (left/right) submodule of the module R is called a (left/right sided) ideal in R. The reader can check that this is exactly the definition given in the lecture.