EXERCISES, LIE ALGEBRAS, UNIVERSITY OF HAMBURG, SUMMER SEMESTER 2016

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Sheet 2

Exercise 1.

(1) Show that, for any $\lambda \in \mathbb{C}$, the following equation holds:
$$\exp\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = e^{\lambda} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(2) Let $A \in \mathrm{Mat}(n, \mathbb{C})$. Show that for any $U \in \mathrm{GL}(n, \mathbb{C})$ we have
$$U^{-1} \exp(A) U = \exp(U^{-1} A U).$$

(3) Compute $\exp\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}$ for $\lambda \in \mathbb{C}$.

Exercise 2. Let K be a field. Consider the following two subsets of $\mathrm{Mat}(2, K)$:
$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in K \right\}, \quad \mathfrak{g}_2 = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in K \right\}.$$

(1) Show that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie subalgebras of $\mathfrak{gl}(2, K)$.
(2) Show that neither $\mathfrak{g}_1$ nor $\mathfrak{g}_2$ is simple.
(3) Show that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are not isomorphic as Lie algebras.
(4) Let $\mathfrak{g}$ be a two-dimensional Lie algebra over K. Show that $\mathfrak{g}$ is isomorphic to either $\mathfrak{g}_1$ or to $\mathfrak{g}_2$.
Hint: Distinguish the two cases that $\mathfrak{g}$ is abelian/not abelian.

Exercise 3. Let $\mathfrak{g}$ be a Lie algebra. The *center* of $\mathfrak{g}$ is defined to be the set
$$Z(\mathfrak{g}) := \{ z \in \mathfrak{g} \mid [x, z] = 0 \ \forall x \in \mathfrak{g} \}.$$

(1) Show that $Z(\mathfrak{g})$ is an ideal in $\mathfrak{g}$.
(2) Prove that the center of $\mathfrak{gl}(n, K)$ is $\mathfrak{s}(n, K)$, the set of all multiples of the identity matrix.
(3) Show that the center of $\mathfrak{sl}(n, K)$ is trivial if $\mathrm{char}(K) = 0$.