EXERCISES, LIE ALGEBRAS, UNIVERSITY OF HAMBURG, SUMMER SEMESTER 2016

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Sheet 12

Exercise 1. Let E be a finite-dimensional Euclidean vector space and let $E' \subset E$ be a subspace. Assume that a reflection s_α leaves E' invariant. Show that either $\alpha \in E'$ or $E' \subset P_\alpha$.

Exercise 2. If Φ is a root system in E, prove that $\Phi^\vee = \{\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)} \mid \alpha \in \Phi\}$ is also a root system in E. Furthermore, show that the Weyl groups of Φ and $\Phi^\vee$ are canonically isomorphic and that $\langle \alpha^\vee, \beta^\vee \rangle = \langle \beta, \alpha \rangle$ for all $\alpha, \beta \in \Phi$.

Exercise 3. If Φ is a root system and W the associated Weyl group, prove that W is a normal subgroup of $\mathrm{Aut}(\Phi)$.

Exercise 4. Let $\alpha, \beta \in \Phi$ and consider $E' = \mathrm{span}_{\mathbb{R}}(\alpha, \beta) \subset E$. Show that $\Phi' = E' \cap \Phi$ is a root system in E'. Also prove that $\Phi'' = \Phi \cap (\mathbb{Z}\alpha + \mathbb{Z}\beta)$ is a root system in E'. Are these two root systems always equal?