EXERCISES, LIE ALGEBRAS, UNIVERSITY OF HAMBURG, SUMMER SEMESTER 2016

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Sheet 11

Exercise 1. Prove that any three-dimensional semisimple Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}(2, K)$.

Exercise 2. Compute all the root strings and Cartan integers of $\mathfrak{sl}(n, K)$ with respect to the maximal toral subalgebra $\mathfrak{h}$ of Exercise 1 on Sheet 10.

Exercise 3. Show that there do not exist semisimple Lie algebras of dimension four, five or seven.

Exercise 4. Let $\mathfrak{g}$ be a semisimple algebra, $\mathfrak{h}$ be a toral subalgebra and $h \in \mathfrak{h}$. Recall the Cartan decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$.

(1) Show that $\mathfrak{h} \subset C_{\mathfrak{g}}(h) =: C$.
(2) Set $\Phi_h = \{\alpha \in \Phi \mid \alpha(h) = 0\}$. Show that $C = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_h} \mathfrak{g}_\alpha$. Note that $\beta \in \Phi_h$ if and only if $-\beta \in \Phi_h$.
(3) Prove that $Z(C) = \{h' \in \mathfrak{h} \mid \alpha(h') = 0 \;\forall \alpha \in \Phi_h\}$.
(4) Prove that $C_{\mathfrak{g}}(h)$ is a reductive Lie algebra.
(5) Now let $\mathfrak{g} = \mathfrak{sl}(n, K)$. Recall that as a maximal toral subalgebra one can take $\mathfrak{d}(n, K) \cap \mathfrak{sl}(n, K) = \mathfrak{h}$. Give an example of an element $x \in \mathfrak{h}$ such that $\mathfrak{h} \subsetneq C_{\mathfrak{g}}(x)$.