EXERCISES, LIE ALGEBRAS, UNIVERSITY OF HAMBURG, SUMMER SEMESTER 2016

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Sheet 10

Exercise 1. Let $\mathfrak{g} = \mathfrak{sl}(n, K)$.

(1) Show that the set $\mathfrak{h}$ of diagonal matrices with trace zero is a maximal toral subalgebra of $\mathfrak{g}$.
(2) Compute the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Exercise 2.

(1) Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Show that for all $h, k \in \mathfrak{h}$ we have
$$\kappa_{\mathfrak{g}}(h, k) = \sum_{\alpha \in \Phi} \alpha(h)\, \alpha(k),$$
where Φ is the set of roots of the corresponding Cartan decomposition.
Hint: Use that a basis of $\mathfrak{g}$ can be obtained as the union of a basis of $\mathfrak{h}$ and the bases of the spaces $\mathfrak{g}_\alpha$ (which we know to be one-dimensional).
(2) Recall from Exercise 1 that set of all diagonal matrices with trace zero is a maximal toral subalgebra of $\mathfrak{sl}(n, K)$. Let $h = \sum_{i=1}^n h_i E_{i,i}$ and $k = \sum_{i=1}^n k_i E_{i,i}$ with $h_i, k_i \in K$ such that $\operatorname{tr}(h) = 0 = \operatorname{tr}(k)$. Show that
$$\kappa_{\mathfrak{sl}(n,K)}(h, k) = 2n(\sum_{i=1}^n h_i k_i).$$

Exercise 3. Let $\mathfrak{g} = \mathfrak{sl}(2, K)$. Show that each maximal toral subalgebra of $\mathfrak{g}$ is one-dimensional.

Exercise 4. Let $\mathfrak{g}$ be a semisimple Lie algebra and let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. Show that $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$, that is, $\mathfrak{h}$ is self-normalising.