# EXERCISES, LIE ALGEBRAS, UNIVERSITY OF HAMBURG, SUMMER SEMESTER 2016

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## SHEET 1

**Exercise 1.**

(1) Let $(A, \cdot)$ be an associative algebra. Verify the claim made in the lecture that setting $[x, y] = x \cdot y - y \cdot x$ defines a Lie algebra structure on $A$.

(2) Let $E_{i,j}$ be the matrix with a 1 in $(i, j)$-position and 0 everywhere else. Prove the following equation:
$$E_{i,j} E_{k,l} = \delta_{jk} E_{i,l}.$$

(3) Prove the following claim made in the lecture: the space of all derivations on an algebra $A$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

**Exercise 2.** Let $K$ be a field of characteristic 0 (that is, $m \cdot 1 \neq 0$ for all $m \in \mathbb{Z}$). Let $n \geq 2$. Show that
$$\big[\mathfrak{sl}(n, K), \mathfrak{sl}(n, K)\big] = \mathfrak{sl}(n, K).$$
Furthermore, show that the statement is false when $n = 2$ and $\mathrm{char}(K) = 2$.

**Exercise 3.** Let $\mathfrak{s}(n, K) = \{\lambda \cdot \mathrm{Id}_n \mid \lambda \in K\}$. Show that if $\mathrm{char}(K) = 0$, then $\mathfrak{gl}(n, K) \simeq \mathfrak{sl}(n, K) \oplus \mathfrak{s}(n, K)$ as vector spaces. Furthermore, show that
$$[\mathfrak{s}(n, K), \mathfrak{gl}(n, K)] = 0.$$

**Exercise 4.** Recall the following vector spaces: $\mathfrak{t}(n, K)$, the set of all upper triangular $n \times n$-matrices; $\mathfrak{n}(n, K)$, the set of all strictly upper triangular matrices; and $\mathfrak{d}(n, K)$, the set of all diagonal matrices.

(1) Check the following claim from the lecture: all the above spaces are Lie subalgebras of $\mathfrak{gl}(n, K)$.

(2) Compute the dimensions of the above Lie algebras.

(3) Show that $\mathfrak{t}(n, K) = \mathfrak{d}(n, K) \oplus \mathfrak{n}(n, K)$ as vector spaces.

(4) Prove that $[\mathfrak{d}(n, K), \mathfrak{n}(n, K)] = \mathfrak{n}(n, K)$.

**Exercise 5.** Show that the Lie algebras $\mathfrak{so}(3, \mathbb{C})$, $\mathfrak{sp}(2, \mathbb{C})$ and $\mathfrak{sl}(2, \mathbb{C})$ are isomorphic.