SOME TOPICS IN THE REPRESENTATION THEORY OF FINITE-DIMENSIONAL ALGEBRAS, SUMMER SEMESTER 2013

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Contents

These are the lecture notes for a course about the representation theory of finite-dimensional algebras held at at the University of Hamburg. The rough idea is to first give an introduction to some basic concepts, such as radicals, semisimple modules and path algebras, before showing that any basic connected algebra is the quotient of a path algebra by an admissible ideal. A very useful observation is that representations of a quiver are the same as modules over the associated path algebra. After establishing these facts, we will look at Gabriel's theorem which classifies the representation finite hereditary algebras as those associated to Dynkin diagrams.

In Sections 1-4 we will, for the most part, closely follow [1]. When dealing with Gabriel's theorem our reference will be [5].

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1. Basic concepts

Let K be an algebraically closed field. Recall that a *K-algebra* is a ring A with an identity element such that A has a vector space structure compatible with the ring multiplication. The algebra is called *commutative* if it is commutative as a ring. We say that A is a *finite-dimensional K-algebra* if its dimension as a K-vector space is finite. A morphism of K-algebras is a ring homomorphism which is linear over K.

Unless otherwise stated, all algebras will be assumed to be finite-dimensional.

A *right ideal* of a K-algebra A is a K-vector subspace I such that $xa \in I$ for all $x \in I$ and $a \in A$. A *left ideal* is defined dually and a *two-sided ideal*, or simply *ideal*, is a K-vector subspace which is both a left and right ideal. A (right or left) ideal I is *maximal* if it not equal to A and if $I \subset I'$ for an ideal I', then $I = I'$. It is straightforward to see that the K-vector space A/I is a K-algebra if I is an ideal and the quotient map is a morphism of K-algebras. Given an ideal I and $1 \leq m \in \mathbb{N}$, the ideal I^m consists of finite sums of elements of the form $x_1 \cdots x_m$ with $x_i \in I$ and I is called *nilpotent* if for some m we have $I^m = 0$. This also makes sense for right (or left) ideals.

Example 1.1.
- The set $M_n(K)$ of all $(n \times n)$-matrices with K-coefficients is a K-algebra with respect to the usual matrix addition and multiplication. Its dimension is n^2. The subset of $M_n(K)$ consisting of all lower triangular matrices is a K-subalgebra of dimension $\frac{n(n+1)}{2}$. To be even more specific, consider the $n = 2$ case. The subspace $\begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix}$ is easily seen to be a right but not a left ideal. On the other hand $\begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ K & K \end{pmatrix}$ are both maximal ideals. The case of upper triangular matrices is left to the reader.
- For an infinite-dimensional example consider the rings $K[t]$ or $K[t_1, \dots, t_n]$. These algebras are commutative.
- Given two K-algebras A_1 and A_2, their product is the space $A = A_1 \times A_2$ with componentwise addition and multiplication.

Definition. The *radical* $\operatorname{rad} A$ of a K-algebra A is the intersection of all maximal right ideals of A.

We can describe elements in the radical as follows.

Lemma 1.2. *For an element $a \in A$ the following conditions are equivalent.*

(1) $a \in \operatorname{rad} A$,
(2) *a is in the intersection of all maximal left ideals of A,*
(3) *for any $b \in A$, the element $1 - ab$ has a two-sided inverse,*
(4) *for any $b \in A$, the element $1 - ab$ has a right inverse,*
(5) *for any $b \in A$, the element $1 - ba$ has a two-sided inverse,*
(6) *for any $b \in A$, the element $1 - ba$ has a left inverse.*

Proof. It is clear that (3) implies (4) and that (5) implies (6). To prove that (1) implies (4) assume that $x = 1 - ab$ has no right inverse. Then there exists a maximal right ideal I such that $x \in I$. But $ab \in I$ and hence $x + ab = 1 \in I$, a contradiction.

To see that (4) implies (1), assume that $a \notin \operatorname{rad} A$, so there exists a maximal right ideal I with $a \notin I$. It follows that $A = I + aA$, hence $1 = x + ab$ for some $x \in I$ and $b \in A$. Thus $x = 1 - ab \in I$ has no right inverse. The equivalence between (2) and (6) is proved similarly and the equivalence of (3) and (5) follows from the following easily checked statements:

$$(1 - cd)x = 1 \Longrightarrow (1 - dc)(1 + dxc) = 1,$$

$$y(1 - cd) = 1 \Longrightarrow (1 + dyc)(1 - dc) = 1.$$

Finally, let us see that (4) implies (3). For any $b \in A$ there exists $c \in A$ such that $(1-ab)c = 1$, hence $c = 1 - a(-bc) := 1 - ab'$. Applying (4) to c, gives an element d such that $1 = cd = d + abcd = d + ab$. Hence, $d = 1 - ab$ and c is its left inverse. The proof that (6) implies (5) is similar. □

Example 1.3. Consider again the lower triangular (2×2)-matrices. It is easily checked that the intersection $\begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix}$ of the ideals $\begin{pmatrix} K & 0 \\ K & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ K & K \end{pmatrix}$ is the radical of this algebra.

Corollary 1.4. *If* $\operatorname{rad} A$ *is the radical of an algebra* A*, then the following hold:*

1. $\operatorname{rad} A$ *is the intersection of all maximal left ideals of* A*.*
2. $\operatorname{rad} A$ *is a two-sided ideal and* $\operatorname{rad}(A/\operatorname{rad} A) = 0$*.*
3. *If* I *is a two-sided nilpotent ideal of* A*, then* $I \subseteq \operatorname{rad} A$*. If, in addition,* $A/I \simeq K \times \ldots \times K$*, then* $I = \operatorname{rad} A$*.*

Proof. (1) is clear. To see that (2) holds, assume $a' \in \operatorname{rad}(A/\operatorname{rad} A)$. Using the lemma, we see that for a representative a of a' and any $b \in A$ there exists $c \in A$ such that $(1-ab)c = 1 - x$ for some $x \in \operatorname{rad} A$. Applying the lemma to $1-x$, we get an element $d \in A$ such that $(1-x)d = 1$, hence $a \in \operatorname{rad} A$ and so $a' = 0 \in A/\operatorname{rad} A$.

To see that (3) holds, let $m > 0$ be such that $I^m = 0$. If $x \in I$ and $a \in A$, then $ax \in I$, hence there exists an $r > 0$ such that $(ax)^r = 0$. Now

$$(1 + ax + (ax)^2 + \ldots + (ax)^{r-1})(1 - ax) = 1,$$

so $x \in \operatorname{rad} A$ and, therefore, $I \subseteq \operatorname{rad} A$. Note that the proof also works if I is only a right (or left) ideal.

Assume now that $A/I \simeq K \times \ldots \times K$. In particular, $\operatorname{rad}(A/I) = 0$. Now note that any surjective algebra homomorphism $f\colon B \twoheadrightarrow B'$ induces a map $\operatorname{rad} B \longrightarrow \operatorname{rad} B'$. Indeed, if $b \in \operatorname{rad} B$, then $1 - bc$ is invertible for all $c \in B$ and hence $f(1-bc)$ is invertible in B'. Applying this to the canonical map $A \twoheadrightarrow A/I$ gives $\operatorname{rad} A \subseteq I$. □

Example 1.5. Let $A = K[t_1, \ldots, t_n]/(t_1^{s_1}, \ldots, t_n^{s_n})$ for some positive integers s_i. The ideal $I = (\bar{t}_1, \ldots, \bar{t}_n)$ of A generated by the cosets of the indeterminates t_i is clearly nilpotent, hence $I \subseteq \operatorname{rad} A$. On the other hand, $A/I \simeq K$, hence $I = \operatorname{rad} A$.

Recall that a *right module* over an algebra A is a K-vector space admitting a scalar multiplication by A from the right satisfying the usual properties. A *left module* is defined dually. Note that A can be considered as a right or a left module over itself. Write A_A for the former and ${}_AA$ for the latter. We will usually consider right modules from here on. A module is *finite-dimensional* if its dimension as a K-vector space is finite. All well-known notions such as submodules, module homomorphisms, finite generation, etc., are the same as for modules over commutative rings. In particular, the category $\operatorname{Mod}(A)$ of all right modules is an abelian category. Given an algebra A, the *opposite algebra* A^{op} is defined by reversing the order of the multiplication. It follows that $\operatorname{Mod}(A^{\mathrm{op}})$

is equivalent to the category of left modules over A and vice versa. The subcategory $\mathrm{mod}(A)$ of $\mathrm{Mod}(A)$ has as objects the finite-dimensional modules.

Note that a module M over A is finitely generated if and only if it is finite-dimensional.

Lemma 1.6 (Nakayama). *Let A be a K-algebra, M a finitely generated A-module and $I \subseteq \operatorname{rad} A$ a two-sided ideal. If $MI = M$, then $M = 0$.*

Proof. Let M be generated by $m_1, \dots, m_s$. We use induction on s. If $s = 1$, then $M = m_1A = m_1I$ implies $m_1 = m_1x_1$ for some $x_1 \in I$. Thus $m_1(1 - x_1) = 0$, hence $m_1 = 0$, because $1 - x_1$ is invertible. Assume now that $s \geq 2$. Since $MI = M$, we have $m_1 = \sum_{i=1}^{s} m_ix_i$ for some $x_i \in I$. It follows that m_1 can be generated by $m_2, \dots, m_s$, since $(1 - x_1)$ is invertible, and, therefore, M can be generated by $m_2, \dots, m_s$. By induction, $M = 0$. □

Corollary 1.7. *The radical of any (finite-dimensional, as usual) algebra A is nilpotent.*

Proof. The chain

$$A \supset \operatorname{rad} A \supset (\operatorname{rad} A)^2 \supset (\operatorname{rad} A)^3 \supset \dots$$

has to become stationary, since A is finite-dimensional. Therefore, there exists n such that $(\operatorname{rad} A)^n = (\operatorname{rad} A)^{n+1} = (\operatorname{rad} A)^n \operatorname{rad} A$. By Nakayama, $(\operatorname{rad} A)^n = 0$. □

Example 1.8. If $A = A_1 \times A_2$ is the product of two K-algebras, we have $1_A = (1, 1) = (1, 0) + (0, 1) =: e_1 + e_2$. Furthermore, $e_1e_2 = e_2e_1 = 0$. Given any A-module M, it is easily checked that Me_i is an A_i-module for $i = 1, 2$. This eventually leads to an equivalence $\mathrm{Mod}(A) \simeq \mathrm{Mod}(A_1) \times \mathrm{Mod}(A_2)$.

If A is an algebra and $M \in \mathrm{mod}(A)$, consider the dual space $M^* = \mathrm{Hom}(M, K)$. This becomes a left A-module by setting $(a\varphi)(m) := \varphi(ma)$ for $a \in A$, $m \in M$ and $\varphi \in M^*$. Given a module homomorphism $M \longrightarrow N$, the map on dual spaces is again a homomorphism of (now left) modules. This leads to the *duality* functor

$$D\colon \mathrm{mod}(A) \longrightarrow \mathrm{mod}(A^{\mathrm{op}}).$$

This functor is an equivalence with quasi-inverse defined in the same way, that is, for a left module Y, we consider the dual vector space Y^* and endow it with a right module structure given by $\varphi a(y) := \varphi(ay)$.

Definition. Let A and B be two K-algebras. An *A-B-bimodule* is a triple ${}_AM_B = (M, *, \cdot)$ such that ${}_AM = (M, *)$ is a left A-module, $M_B = (M, \cdot)$ is a right B-module, and $(a * m) \cdot b = a * (m \cdot b)$ for all $m \in M$, $a \in A$ and $b \in B$. One usually suppresses $*$ and $\cdot$ in the notation.

Example 1.9. Any right module M can be considered as an $\mathrm{End}(M)$-A-bimodule by noting that the left $\mathrm{End}(M)$-module structure is defined by $\varphi m := \varphi(m)$.

Note that if ${}_AM_B$ is an A-B-bimodule and N_B is a right B-module, the vector space $\mathrm{Hom}_B({}_AM_B, N_B)$ is a right A-module by setting $fa(m) := f(am)$ for all $a \in A$, $m \in M$

and $f \in \mathrm{Hom}_B({}_AM_B, N_B)$. Using this observation, we have a covariant functor

$$\mathrm{Hom}_B({}_AM_B, -)\colon \mathrm{Mod}(B) \longrightarrow \mathrm{Mod}(A).$$

Similarly, we have a contravariant functor

$$\mathrm{Hom}_B(-, {}_AM_B)\colon \mathrm{Mod}(B) \longrightarrow \mathrm{Mod}(A^{\mathrm{op}}).$$

Furthermore, given ${}_AM_B$ as above there are the *tensor product functors*

$$(-) \otimes_A M_B\colon \mathrm{Mod}(A) \longrightarrow \mathrm{Mod}(B), \quad {}_AM \otimes_B (-)\colon \mathrm{Mod}(B^{\mathrm{op}}) \longrightarrow \mathrm{Mod}(A^{\mathrm{op}})$$

and an *adjunction isomorphism*

$$\mathrm{Hom}_B(X \otimes_A M_B, Z_B) \simeq \mathrm{Hom}_A(X_A, \mathrm{Hom}_B({}_AM_B, Z_B))$$

defined for a φ in the left hand space by sending it to the map ψ given by $\psi(x)(m) = \varphi(x \otimes m)$. The inverse map sends ψ in the right hand space to the map $\varphi\colon x \otimes m \longmapsto \psi(x)(m)$. To quote Atiyah-Macdonald, "in the language of abstract nonsense" the functor $(-) \otimes_A M_B$ is left adjoint to $\mathrm{Hom}_B(-, {}_AM_B)$ and $\mathrm{Hom}_B(-, {}_AM_B)$ is right adjoint to $(-) \otimes_A M_B$.

Definition. An A-module S is called *simple* if it is nonzero and the only submodules of S are 0 and S. A module M is *semisimple* if it is a direct sum of simple modules. A module is called *indecomposable* if in a decomposition $M = M_1 \oplus M_2$ either $M_1 = 0$ or $M_2 = 0$.

Clearly, any simple module is indecomposable. The next result describes some restrictions on maps between simple modules.

Lemma 1.10 (Schur)**.** *Let S and S' be A-modules and $f\colon S \longrightarrow S'$ be a non-trivial homomorphism. If S is simple, then f is a monomorphism and if S' is simple, then f is an epimorphism. If both are simple, then f is an isomorphism.*

Proof. Just note that $\mathrm{im}(f) \subseteq S'$ and $\ker(f) \subseteq S$ are submodules of S' and S, respectively. □

Corollary 1.11. *If S is a simple A-module, then* $\mathrm{End}(S) \simeq K$.

Proof. By Schur's lemma, $\mathrm{End}(S)$ is a skew field. Since A is simple, any map $A \longrightarrow S$ is an epimorphism, hence $\dim_K S < \infty$. Therefore, also $\dim_K \mathrm{End}(S) < \infty$. Hence, for any $0 \neq \varphi \in \mathrm{End}(S)$ there exists an irreducible polynomial $f \in K[t]$ such that $f(\varphi) = 0$. Since K is algebraically closed, f is of degree 1, hence φ corresponds to a scalar $\lambda_\varphi \in K^*$, which gives the desired isomorphism. □

Lemma 1.12. *A finite-dimensional module M is semisimple iff for any submodule N of M there exists a submodule L of M such that $L \oplus N \simeq M$. In particular, a submodule of a semisimple module is semisimple.*

Proof. Assume that $M = S_1 \oplus \ldots \oplus S_m$ where the S_i are simple modules. Let $0 \neq N \subseteq M$ be a submodule and consider the maximal family $\{S_{j_1}, \ldots, S_{j_k}\}$ of the S_i such that $N \cap L = 0$, where $L = S_{j_1} \oplus \ldots \oplus S_{j_k}$. Then $N \cap (L + S_t) \neq 0$, for any $t \notin \{j_1, \ldots, j_k\}$.

From this it follows that $(L+N) \cap S_t \neq 0$ for all $t \notin \{j_1, \dots, j_k\}$, hence $S_t \subseteq L+N$ for all $t \notin \{j_1, \dots, j_k\}$. Therefore, $M = L+N$ and hence $M = L \oplus N$. The reverse implication follows by induction on $\dim_K M$. □

Definition. Let M be a right A-module. The (Jacobson) *radical* $\operatorname{rad} M$ of M is the intersection of all maximal submodules of M.

Let us study some basic properties of the radical:

Proposition 1.13. *Let L, M and N be finite-dimensional A-modules.*

(1) $m \in \operatorname{rad} M$ *iff* $f(m) = 0$ *for all* $f \in \operatorname{Hom}_A(M, S)$ *and all simple modules* S.
(2) $\operatorname{rad}(M \oplus N) = \operatorname{rad} M \oplus \operatorname{rad} N$.
(3) *If* $f \in \operatorname{Hom}_A(M, N)$, *then* $f(\operatorname{rad} M) \subseteq \operatorname{rad} N$.
(4) $M \operatorname{rad} A = \operatorname{rad} M$.
(5) *If L and M are submodules of a finite-dimensional module N with* $L \subseteq \operatorname{rad} N$ *and $L + M = N$, then $M = N$.*

Proof. To see (1), note that $L \subseteq M$ is a maximal submodule iff M/L is simple. (2) follows immediately from (1). To prove (3), consider any map $g \in \operatorname{Hom}_A(N, S)$ and use that $gf(m) = 0$.

Now, take any $m \in M$ and define a homomorphism $f_m \colon A \longrightarrow M$ of A-modules by $f_m(a) = ma$ for $a \in A$. Part (3) gives that $f_m(\operatorname{rad} A) \subseteq \operatorname{rad} M$ for all m, hence $M \operatorname{rad} A \subseteq \operatorname{rad} M$. To see the reverse inclusion, note that $M/M \operatorname{rad} A$ is a module over $A/\operatorname{rad} A$. The Wedderburn–Artin theorem tells us, in particular, that an algebra B has trivial radical if and only if any right module is semisimple if and only if B_B is semisimple. We apply this to $A/\operatorname{rad} A$ and conclude that $M/M \operatorname{rad} A$ is a direct sum of simple modules. Clearly, the radical of a simple module is trivial, hence $\operatorname{rad}(M/M \operatorname{rad} A) = 0$. Now (3) gives us that $\operatorname{rad} M \subseteq M \operatorname{rad} A$.

Finally, assume L and M are as in (5), but $M \neq N$. Since $\dim_K N < \infty$, M is contained in a maximal submodule $P \subseteq N$. Therefore, $L \subseteq \operatorname{rad} N \subseteq P$, so $N = L + M \subseteq P + M = P$, a contradiction. □

Note that any homomorphism $f \colon M \longrightarrow N$ induces a map $M/\operatorname{rad} M \longrightarrow N/\operatorname{rad} N$, by part (3) of Proposition 1.13.

Corollary 1.14. *For any finite-dimensional module M, the module $M/\operatorname{rad} M$, called the* top *of M and sometimes denoted by* $\operatorname{top}(M)$, *is semisimple and a module over* $A/\operatorname{rad} A$. *Furthermore, if $L \subseteq M$ such that M/L is semisimple, then* $\operatorname{rad} M \subseteq L$.

Proof. The statement about semisimplicity follows from the Wedderburn–Artin theorem mentioned above. Considering the homomorphism $M \longrightarrow M/L$ and using part (3) of the lemma, gives the second statement. □

Corollary 1.15. *A homomorphism $f \colon M \longrightarrow N$ is surjective if and only if the morphism* $\operatorname{top}(M) \longrightarrow \operatorname{top}(N)$ *is surjective. If S is a simple module, then* $S \operatorname{rad} A = 0$ *and S is a simple* $A/\operatorname{rad} A$*-module. Finally, an A-module M is semisimple if and only if* $\operatorname{rad} M = 0$.

Proof. The second statement is clear, by Nakayama's lemma and since $S \operatorname{rad} A$ is a submodule of the simple module S. This also immediately implies the "only if" direction in the third statement. The "if" direction follows from the previous corollary. Finally, assume that $M/\operatorname{rad} M \twoheadrightarrow N/\operatorname{rad} N$ is surjective. Then $\operatorname{im} f + \operatorname{rad} N = N$, hence, by (5) of Proposition 1.13 applied to $L = \operatorname{rad} N$ and $M = \operatorname{im} f$, $\operatorname{im} f = N$. The other direction is trivial. $\square$

A *composition series* of a finite-dimensional module M is a chain $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$ such that M_{j+1}/M_j is simple for $j = 0, \ldots, n-1$. The modules M_{j+1}/M_j are called the *composition factors* of M.

The *Jordan–Hölder theorem* tells us that any two composition series $(M_i)_{i=1}^n$ and $(N_j)_{j=1}^l$ have the property that $n = l$ and that there exists a permutation σ of $\{1, \ldots, n\}$ such that $M_{j+1}/M_j \simeq N_{\sigma(j+1)}/N_{\sigma(j)}$ for $j \in \{0, \ldots, n-1\}$. In particular, the number n of modules in a composition series is well-defined and called the *length* of M, denoted by $l(M)$. It is easily checked that for a submodule N of M, we have $l(N) + l(M/N) = l(M)$ and for any two submodules L and N of M, we have $l(L+N) + l(L \cap N) = l(L) + l(N)$.

Definition. An element $e \in A$ is called an *idempotent* if $e^2 = e$. The idempotent e is called *central* if $ea = ae$ for all $a \in A$. Two idempotents $e_1, e_2 \in A$ are called *orthogonal* if $e_1e_2 = e_2e_1 = 0$. Finally, an idempotent e is called *primitive* if it cannot be written as a sum of nonzero orthogonal idempotents.

Note that any algebra has two trivial idempotents, namely 0 and 1. If e is a non-trivial idempotent, then so is $1-e$, and the idempotents e and $1-e$ are orthogonal. Furthermore, there exists a decomposition $A_A = eA \oplus (1-e)A$ of right A-modules. Conversely, if $A_A = M_1 \oplus M_2$, then $m_i \in M_i$ with $1 = m_1 + m_2$ are orthogonal idempotents and $M_i = e_iA$ is an indecomposable module if and only if e_i is primitive.

Given a central idempotent e, the modules eA and $(1-e)A$ are in fact algebras (with identity e resp. $(1-e)$) and the above decomposition of A as a module exhibits A as a product of the algebras eA and $(1-e)A$. An algebra A is called *connected* (or indecomposable) if 0 and 1 are the only central idempotents of A or, equivalently, the algebra A is a not a product of two algebras.

Since A is finite-dimensional, the right module A_A admits a decomposition $A_A = P_1 \oplus \ldots \oplus P_n$, where the P_i are indecomposable right ideals. It is clear that $P_i = e_iA$ for primitive pairwise orthogonal idempotents e_i such that $1 = \sum_{i=1}^n e_i$. Conversely, every set of idempotents with these properties determines a decomposition of A_A as above. Such a decomposition is called an *indecomposable decomposition* of A and the e_i are called a *complete set of primitive orthogonal idempotents* of A.

Consider a right A-module M and an idempotent $e \in A$. Note that the K-vector subspace eAe of A is a K-algebra with identity e. Of course, it is a subalgebra of A iff $e = 1$. We can define an eAe-module structure on the subspace Me of M by setting $me(eae) := meae$ for all $m \in M$ and $a \in A$. In particular, Ae is a right eAe-module and eA is a left eAe-module. This implies that $\operatorname{Hom}_A(eA, M)$ is a right eAe-module with respect to the action $(\varphi \cdot eae)(x) = \varphi(eaex)$ for $x \in eA$, $a \in A$ and $\varphi \in \operatorname{Hom}_A(eA, M)$.

The following easy lemma will turn out to be very useful.

Lemma 1.16. *Let A be a K-algebra, M a right A-module and $e \in A$ an idempotent. Define a K-linear map $\theta_M\colon \operatorname{Hom}_A(eA, M) \longrightarrow Me$ by $\varphi \longmapsto \varphi(e) = \varphi(e)e$. Then θ_M is an isomorphism of right eAe-modules and it is functorial in M. The isomorphism $\theta_{eA}\colon \operatorname{End}(eA) \simeq eAe$ of right eAe-modules induces an isomorphism of K-algebras.*

Proof. First note that the second statement follows easily from the first. On the other hand, it is straightforward to see that θ_M is functorial in M and a homomorphism of right eAe-modules. To see that it is an isomorphism, we will define an inverse map ψ_M as follows. Given $me \in Me$ and $ea \in eA$, set $\psi_M(me)(ea) := mea$. The details are left to the reader. □

The next result states, roughly speaking, that under the canonical quotient map $A \longrightarrow A/\operatorname{rad} A$ idempotents can be lifted.

Lemma 1.17. *Given an idempotent $\eta = g + \operatorname{rad} A$ ($g \in A$) of $B = A/\operatorname{rad} A$, there exists an idempotent e of A such that $g - e \in \operatorname{rad} A$.*

Proof. By Corollary 1.7, $\operatorname{rad} A$ is nilpotent. Since $\eta^2 - \eta = 0$ in B, we have $g - g^2 \in \operatorname{rad} A$, so $(g - g^2)^m = 0$ for some $m > 0$. By Newton's binomial formula, $0 = (g - g^2)^m = g^m - g^{m+1}t$ with $t = \sum_{j=1}^m (-1)^{j-1}\binom{m}{j} g^{j-1}$. Hence, $g^m = g^{m+1}t$ and $gt = tg$. The first equation immediately implies that $e = (gt)^m$ is an idempotent. Next note that $g - g^m \in \operatorname{rad} A$, since

$$g - g^m = g(1 - g^{m-1}) = g(1-g)(1 + g + \ldots + g^{m-2}) = (g - g^2)(1 + g + \ldots + g^{m-2}).$$

Furthermore, $g - gt \in \operatorname{rad} A$, because modulo $\operatorname{rad} A$ we have the equalities $g = g^m = g^{m+1}t = gg^m t = g^2 t = gt$. This then implies that

$$e + \operatorname{rad} A = (gt)^m + \operatorname{rad} A = (gt + \operatorname{rad} A)^m = (g + \operatorname{rad} A)^m = g^m + \operatorname{rad} A = g + \operatorname{rad} A.$$

Hence, e is as desired. □

Proposition 1.18. *Consider the algebra $B = A/\operatorname{rad} A$.*

1. *Every right ideal I of B is a direct sum of simple right ideals of the form eB, where e is a primitive idempotent of B. In particular, the right B-module B_B is semisimple.*
2. *Any finite-dimensional B-module is isomorphic to a direct sum of simple right ideals as in (1).*
3. *If $e \in A$ is a primitive idempotent, then the B-module $eA/\operatorname{rad} eA$ is simple and $\operatorname{rad} eA = e \operatorname{rad} A \subseteq eA$ is the unique proper submodule of eA.*

Proof. (1) Let S be a nonzero right ideal of B contained in I which is of minimal dimension. The minimality implies that S is a simple B-module and, furthermore, $S^2 \neq 0$, since if $S^2 = 0$, then, by Lemma 1.4, $0 \neq S \subseteq \operatorname{rad} B = 0$, so we get a contradiction. Therefore, $S^2 = S$ and there exists $x \in S$ such that $xS \neq 0$, $S = xS$ and $x = xe$ for some $0 \neq e \in S$. Now, Schur's lemma implies that the homomorphism $\varphi\colon S \longrightarrow S$ given by $y \mapsto xy$ is an isomorphism. Since $\varphi(e^2 - e) = x(e^2 - e) = 0$, $e^2 - e = 0$, e is an idempotent and $S = eB$, hence $B = eB \oplus (1-e)B$ and similarly $I = eI \oplus (1-e)I$. By induction on $\dim_K I$, (1) follows.

(2) Note that any $M \in \operatorname{mod} B$ is a quotient of the free module B^k for some k. By (1), B^k is a direct sum of simple right ideals, hence semisimple. By Lemma 1.12, $B^k \simeq \ker p \oplus L$, where $p\colon B^k \twoheadrightarrow M$ is the quotient map. Therefore, M can be considered as a submodule of B^k, and (2) follows from Lemma 1.12.

(3) Given e, the element $\overline{e} = e + \operatorname{rad} A$ is an idempotent of B and $eA/\operatorname{rad} eA = \overline{e}B$. If $\overline{e}B$ is not simple, there exist, by (1), primitive orthogonal idempotents $\overline{e}_1$ and $\overline{e}_2$ of B such that $\overline{e}B \simeq \overline{e}_1 B \oplus \overline{e}_2 B$. Now compute that $\overline{e}_1 = \overline{e}_1^2 = \overline{e}\overline{e}_1$, so $\overline{e}_1 = g_1 + \operatorname{rad} A$ for some $g_1 \in eA$. Since idempotents lift, $e_1 = (g_1 t)^m$ and $\overline{e}_1 = e_1 + \operatorname{rad} A$, for some $t \in A$ and $m \geq 0$. Since $g_1 \in eA$, also $e_1 \in eA$, so $e_1 A \subseteq eA$. The decomposition $A_A = e_1 A \oplus (1-e_1)A$, gives a decomposition of $eA = e_1 A \oplus ((1-e_1 A) \cap eA)$. Since e is primitive, $eA = e_1 A$. Thus, $\overline{e}B = \overline{e}_1 B$. Consequently, $eA/\operatorname{rad} eA$ is simple, so $\operatorname{rad} eA = e \operatorname{rad} A$ is a maximal proper submodule of eA. Any other proper submodule N is contained in $\operatorname{rad} eA$, by (5) of Proposition 1.13 applied to $L = \operatorname{rad} eA$. $\square$

Definition. An algebra is called *local* is it has a unique maximal right ideal.

We will now give several equivalent characterisations of a local algebra.

Proposition 1.19. *Let A be a K-algebra. The following are equivalent.*

(1) *A is local.*
(2) *A has a unique maximal left ideal.*
(3) *The set of all noninvertible elements of A is a two-sided ideal.*
(4) *For any $a \in A$, either a or $1-a$ is invertible.*
(5) *A has only two idempotents, namely 0 and 1.*
(6) *The algebra $A/\operatorname{rad} A$ is isomorphic to K.*

Proof. If A is local, then $\operatorname{rad} A$ is the unique maximal right ideal of A. Hence, $x \in \operatorname{rad} A$ iff x has no right inverse. Now, if x is right invertible, so $xy = 1$ for some y, then $(1 - yx)y = 0$. The element y has to have a right inverse, because otherwise $y \in \operatorname{rad} A$, so $1 - yx$ is invertible by Lemma 1.2, hence $y = 0$, a contradiction. But if y has a right inverse, $1 - yx = 0$, so x is invertible. Summarising, $x \in \operatorname{rad} A$ iff x has no right inverse iff x is not invertible. Therefore, (1) implies (2). Similar arguments show that (2) implies (3). It is obvious that (3) implies (4). Next, if e is an idempotent, so is $1 - e$ and $e(1-e) = 0$, so if (4) holds, then so does (5). If (5) holds, then the algebra $B = A/\operatorname{rad} A$ has only two idempotents. By Proposition 1.18, the module B_B is simple and, by Corollary 1.11, $\operatorname{End}(B_B) = K$. Therefore, $B \simeq \operatorname{End}(B_B) \simeq K$, hence (5) implies (6). Finally, if (6) holds, then clearly so does (1) (or (2)). $\square$

Remark 1.20. Note that the algebra $K[t]$ has only two idempotents but is not local. Hence, the proposition does not hold for infinite-dimensional algebras.

Corollary 1.21. *An idempotent $e \in A$ is primitive iff the algebra $eAe \simeq \operatorname{End}(eA)$ has only 0 and e as idempotents.* $\square$

Corollary 1.22. *Let A be an algebra and M a right A-module. If* $\mathrm{End}(M)$ *is local, then M is indecomposable. If M is finite-dimensional and indecomposable, then* $\mathrm{End}(M)$ *is local and any element in* $\mathrm{End}(M)$ *is either nilpotent or an isomorphism.*

Proof. If M decomposes as $M = M_1 \oplus M_2$, then $e_1 + e_2 = p_1 i_1 + p_2 i_2 = \mathrm{id}_M$, where the p_j are the canonical projections, the i_j are the canonical injections and the e_j are nonzero idempotents. Therefore, $\mathrm{End}(M)$ is not local.

Now assume that $M \in \operatorname{mod} A$ is indecomposable. If $\mathrm{End}(M)$ is not local, there exists a non-trivial idempotent e, hence $M \simeq \operatorname{im} e_1 \oplus \operatorname{im} e_2$, where $e_1 = e$ and $e_2 = 1 - e$. This is a contradiction, hence $\mathrm{End}(M)$ is local. The last statement is clear, since any non-invertible element in a local finite-dimensional algebra, which we know $\mathrm{End}(M)$ to be, belongs to the radical and is therefore nilpotent by Corollary 1.7. □

The following result reduces the study of finite-dimensional modules to the study of indecomposable ones.

Theorem 1.23. *Every finite-dimensional module M over A has a decomposition $M \simeq M_1 \oplus \ldots \oplus M_n$, where the M_i are indecomposable modules, and hence have local endomorphism algebras. Furthermore, if $M \simeq M_1 \oplus \ldots \oplus M_n$ and $M \simeq N_1 \oplus \ldots \oplus N_k$ with M_i and N_j indecomposable, then $m = n$ and there exists a permutation σ of* $\{1, \ldots, n\}$ *such that $M_i \simeq N_{\sigma(i)}$ for all i.*

Proof. The first statement is clear, because $\dim_K M$ is finite. To see the second, we proceed by induction. If $n = 1$, then there is nothing to show. So assume that $n > 1$ and consider $M' := \oplus_{i>1} M_i$. We have the decomposition $M = M_1 \oplus M'$ with the corresponding projections and injections p, p' and ι, ι', respectively. Denote the projections and injections corresponding to $M = \oplus N_j$ by p_j and ι_j. We know that $1_{M_1} = p\iota = p(\sum_j \iota_j p_j)\iota = \sum_j p\iota_j p_j \iota$. Since $\mathrm{End}(M_1)$ is local, there exists an index j, which without loss of generality can be assumed to be 1, such that $v := p\iota_1 p_1 \iota$ is invertible. Now set $w := v^{-1} p\iota_1 \colon N_1 \longrightarrow M_1$ and note that $w p_1 \iota = 1_{M_1}$. Hence, $p_1 \iota w$ is an idempotent in $\mathrm{End}(N_1)$. The latter is a local algebra, so $p_1 \iota w$ is 0 or 1. It cannot be equal to 0, because then $p_1 \iota = 0$, since w is an epimorphism, but $v := p\iota_1 p_1 \iota$ is invertible. Therefore, $p_1 \iota w = 1_{N_1}$ and hence $p_1 \iota$ gives $M_1 \simeq N_1$. Writing $M \simeq M_1 \oplus M' = N_1 \oplus N'$, where $N' = \oplus_{j>1} N_j$, we are done by induction if we can show that $M' \simeq N'$. But this is clear, since N' is the kernel of $p_1 \colon M \longrightarrow N_1$ and M' is the kernel of $p \colon M \longrightarrow M_1$ and it is obvious that they coincide via the above isomorphism $p_1 \iota \colon M_1 \simeq N_1$. □

Corollary 1.24. *If $A_A = P_1 \oplus \ldots \oplus P_n$ is an indecomposable decomposition, it is unique.* □

Definition. A finite-dimensional algebra A is *representation finite* or an *algebra of finite representation type* if the number of the isomorphism classes of indecomposable finite-dimensional right modules is finite.

One of the goals of the course is to classify the representation finite hereditary algebras.

Recall that a module P is projective if and only if it is a direct summand of a free module. So a consequence of Theorem 1.23 is

Corollary 1.25. *Assume $A_A = e_1A \oplus \dots e_nA$ is a decomposition with respect to a complete set of primitive orthogonal idempotents. Then the indecomposable projective modules are precisely the modules $P(i) = e_iA$.* $\square$

Note that the proof of part (3) of Proposition 1.18 shows that $\{\pi(e_1), \dots, \pi(e_n)\}$ is a complete set of primitive orthogonal idempotents of $B = A/\operatorname{rad} A$, where $\pi\colon A \twoheadrightarrow B$ is the canonical quotient map. Consider the corresponding decomposition $B_B = \oplus_i \pi(e_i)B$ and note that the modules $\pi(e_i)B \simeq \operatorname{top} e_iA$ are simple by Proposition 1.18, part (3). Furthermore, the epimorphism $\pi_i\colon e_iA \twoheadrightarrow \operatorname{top} e_iA$ induced by π is a so-called *projective cover* of $\operatorname{top} e_iA$, which, by definition, means that e_iA is a projective module and π_i has the property that for any submodule N of e_iA the equality $\ker \pi_i + N = e_iA$ implies that $N = e_iA$. This property is satisfied, since $\operatorname{rad} e_iA = \ker \pi_i$ is the unique maximal submodule of e_iA.

Since $\operatorname{top} M$ is a semisimple B-module for any A-module M by Corollary 1.14, it is a direct sum of (copies of) the modules $\pi(e_i)B$, say

$$\operatorname{top} M \simeq \bigoplus_{i=1}^{n} ((\pi(e_i))B)^{\oplus s_i}$$

for some $s_i \geq 0$. Set

$$P(M) := \bigoplus_{i=1}^{n} (e_iA)^{\oplus s_i}.$$

The module $P(M)$ is, of course, projective. Note that $\operatorname{top} P(M) \simeq \operatorname{top} M$ and by the projectivity of $P(M)$ we get a map $P(M) \twoheadrightarrow M$ such that the diagram

$$\begin{tikzcd} P(M) \arrow[r, two heads] \arrow[d, two heads, "t"'] & M \arrow[d, two heads, "t'"] \\ \operatorname{top} P(M) \arrow[r, two heads, "\simeq"] & \operatorname{top} M \end{tikzcd}$$

is commutative. Since the lower map is an isomorphism, the upper one is an epimorphism by Corollary 1.15. Furthermore, its kernel is contained in $\operatorname{rad} P(M) = \ker t$, hence the map is in fact a projective cover. Summarising, for any module M in $\operatorname{mod} A$ there exists a projective cover $P(M)$ and $P(M)/\operatorname{rad} P(M) \simeq M/\operatorname{rad} M$.

The next step is to show that the projective cover is unique, i.e., if $P' \xrightarrow{p'} M \longrightarrow 0$ is a projective cover, then $P' \simeq P(M)$. The projectivity of P' gives us a morphism $g\colon P' \twoheadrightarrow P(M)$ such that $pg = p'$. Since p' is surjective, $\operatorname{im} g + \ker p = P(M)$. Since $\ker p = \operatorname{rad} M$, this implies the surjectivity of g. Therefore, $l(P') \geq l(P(M))$. Reversing the situation, we get $l(P(M)) \geq l(P')$, hence an equality. Thus, $P' \simeq P(M)$. Summarising

Proposition 1.26. *Any module M in $\operatorname{mod} A$ has a unique projective cover $P(M)$ satisfying $P(M)/\operatorname{rad} P(M) \simeq M/\operatorname{rad} M$.* □

Corollary 1.27. *If P is a projective module in $\operatorname{mod} A$, then $P \longrightarrow \operatorname{top} P$ is a projective cover. In particular, $e_iA \longrightarrow \operatorname{top} e_iA$ is a projective cover for any primitive idempotent e_i of A. By the uniqueness of projective covers, $e_iA \simeq e_jA$ if and only if $\operatorname{top} e_iA \simeq \operatorname{top} e_jA$.* □

Corollary 1.28. *The simple modules in $\operatorname{mod} A$ are precisely the modules $S(i) = \operatorname{top} e_iA = \operatorname{top}(P(i))$.*

Proof. Take a simple module S. It has a projective cover $P(S)$ which is a direct sum of copies of the $P(i)$. Since $P(S)/\operatorname{rad} P(S) \simeq S$, the left hand side is a direct sum of the $S(i)$. But S is simple, so the assertion follows. □

Definition. Let A be an algebra with a complete set of primitive idempotents $\{e_1, \dots, e_n\}$. The algebra is called *basic* if $e_iA \not\simeq e_jA$ for all $i \neq j$.

Clearly, a local algebra is basic. Basicness of an algebra A can be detected by the algebra $A/\operatorname{rad} A$:

Proposition 1.29. *A finite-dimensional K-algebra A is basic iff $B = A/\operatorname{rad} A \simeq K \times \dots \times K$.*

Proof. Let $A_A = \oplus_{i=1}^n e_iA$ for a complete set of primitive orthogonal idempotents and $B_B = \oplus_{i=1}^n \pi(e_i)B$ the corresponding decomposition. Since $e_iA \simeq e_jA$ if and only if $\pi(e_i)B \simeq \operatorname{top} e_iA \simeq \operatorname{top} e_jA \simeq \pi(e_j)B$, we conclude that B is basic if A is. Schur's lemma gives that $\operatorname{Hom}(\pi(e_i)B, \pi(e_j)B) = 0$ for $i \neq j$ and, since these modules are simple, $\operatorname{End}(\pi(e_i)B) \simeq K$ for all i. Using this, we get

$$B \simeq \operatorname{End}_B(B_B) \simeq \oplus_{i=1}^n \operatorname{End}(\pi(e_i)B) \simeq K \times \dots \times K.$$

For the converse, assume that B is isomorphic to a product of n copies of K. Then B is a commutative algebra and admits n central primitive pairwise orthogonal idempotents $\overline{e}_i$. Hence, $e_iB \not\simeq e_jB$ for $i \neq j$ and therefore $P(e_iB) \simeq e_iA \not\simeq P(e_jB) \simeq e_jA$ for $i \neq j$. □

Corollary 1.30. *Any simple module S over a basic algebra is one-dimensional.*

Proof. First note that a simple module S' over any algebra A satisfies $S' \operatorname{rad} A = 0$ and, consequently, S' is a simple $A/\operatorname{rad} A$-module. Indeed, Nakayama's lemma gives that $S' \neq S' \operatorname{rad} A$, hence the latter has to be zero since S' is simple.

Using this and the proposition we see that S is a simple module over the algebra $A/\operatorname{rad} A \simeq K \times \dots \times K$ and the corollary follows at once. □

Definition. Let A be an algebra with a complete set of primitive idempotents $\{e_1, \dots, e_n\}$. A *basic algebra* associated to A is the algebra $A^b = e_AAe_A$, where $e_A = e_{j_1} + \dots + e_{j_a}$ and e_{j_k} are chosen such that $e_{j_i}A \not\simeq e_{j_t}A$ for $i \neq t$ and each module e_sA is isomorphic to one of the modules $e_{j_1}A, \dots, e_{j_k}A$.

In other words, we consider all modules e_kA and if $e_kA \simeq e_lA$, only e_k or e_l will be part of e_A. Hence, a priori, A^b is not unique, since it depends on which idempotents we keep.

Lemma 1.31. *Let A^b be a basic algebra associated to A. The element $e_A \in A^b$ is the identity of A^b and $A^b \simeq \mathrm{End}(e_{j_1}A \oplus \ldots \oplus e_{j_a}A)$. Furthermore, the algebra A^b does not depend on the choice of the sets $(e_i)_i$ and $e_{j_1}, \ldots, e_{j_a}$.*

Proof. The first statement is clear. To see the second, apply Lemma 1.16 to e_AA and use that $e_AA \simeq e_{j_1}A \oplus \ldots \oplus e_{j_a}A$. Theorem 1.23 tells us that e_AA does not depend on the choices, so the third statement follows from the second. □

For an idempotent $e \in A$, consider the algebra $B := eAe \simeq \mathrm{End}(eA)$ with identity e. Given an A-module M, note that Me is a B-module. If $f\colon M \longrightarrow M'$ is a homomorphism of A-modules, we get a homomorphism between the B-modules Me and $M'e$ by setting $me \longmapsto f(m)e$. This defines a *restriction functor*

$$\mathrm{res}_e\colon \ \mathrm{mod}\, A \longrightarrow \mathrm{mod}\, B.$$

We now define two functors from $\mathrm{mod}\, B$ to $\mathrm{mod}\, A$ as follows. We have seen before that eA is a left $B = eAe$-module. It is, of course, also a right A-module. Therefore, we have the functor $T_e(-) := - \otimes_B eA$. On the other hand, Ae is a left A-module and a right eAe-module, hence we have the functor $L_e(-) = \mathrm{Hom}_B(Ae, -)$.

The next result collects some properties of these functors.

Proposition 1.32. *Let A be an algebra, let e be an idempotent of A and $B = eAe$. Then the following holds.*

1. *T_e and L_e are fully faithful K-linear functors such that $\mathrm{res}_e T_e \simeq \mathrm{id}_{\mathrm{mod}\, B} \simeq \mathrm{res}_e L_e$, the functor L_e is right adjoint to res_e and T_e is left adjoint to res_e.*
2. *T_e is right exact, L_e is left exact and res_e is exact.*
3. *T_e and L_e preserve indecomposability, T_e respects projectives and L_e respects injectives.*
4. *A right A-module M is in the image of T_e iff there exists an exact sequence $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$, where P_1 and P_0 are direct sums of summands of eA.*

Proof. (1) Recall from Lemma 1.16 that we have a functorial B-module isomorphism

$$\mathrm{Hom}_A(eA, M) \simeq Me$$

for any right A-module M. Using the adjointness properties of the tensor and Hom functors we have, for a B-module N,

$$\begin{aligned}\mathrm{Hom}_A(T_e(N), M) &\simeq \mathrm{Hom}_A(N \otimes_B eA, M) \simeq \mathrm{Hom}_B(N, \mathrm{Hom}_A(eA, M))\\ &\simeq \mathrm{Hom}_B(N, Me) \simeq \mathrm{Hom}_B(N, \mathrm{res}_e(M))\end{aligned}$$

Hence, T_e is left adjoint to res_e. Also note that

$$\mathrm{res}_e T_e(N) = (N \otimes_B eA)e \simeq N \otimes_B B \simeq N,$$

consequently, $\mathrm{Hom}_B(N, N') \simeq \mathrm{Hom}_A(T_e(N), T_e(N'))$. Hence, T_e is fully faithful. The proof that L_e satisfies the stated properties is completely analogous.

(2) is obvious.

(3) Since T_e and L_e are fully faithful, $\mathrm{End}(N) \simeq \mathrm{End}(T_e(N)) \simeq \mathrm{End}(L_e(N))$. So if N is indecomposable, then its endomorphism algebra is local, hence the same holds for $T_e(N)$ and $L_e(N)$ and these modules are indecomposable by Corollary 1.22.

Now consider a projective B-module P and an epimorphism $h\colon M \twoheadrightarrow M'$ in $\mathrm{mod}\, A$. We have the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_A(T_e(P), M) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Hom}_A(T_e(P), M') \arrow[d, "\simeq"] \\ \mathrm{Hom}_B(P, \mathrm{res}_e(M)) \arrow[r] & \mathrm{Hom}_B(P, \mathrm{res}_e(M')). \end{tikzcd}$$

Since P is projective, the lower map is an epimorphism, hence so is the upper map. Therefore, $T_e(P)$ is a projective A-module if P is a projective B-module. Dually, we can show the statement for L_e.

(4) Assume that $e = e_{j_s} + \ldots + e_{j_s}$ and the e_{j_k} are primitive idempotents. This implies that $B = e_{j_1}B \oplus \ldots \oplus e_{j_s}B$ and the modules $e_{j_k}B$ are indecomposable, because the e_{j_k} are primitive.

Consider the map

$$m_{j_i}\colon e_{j_i}B \otimes_B eA \longrightarrow e_{j_i}A, \quad e_{j_i}x \otimes ea \longmapsto e_{j_i}xea.$$

Note that this map is the restriction of the A-module isomorphism $B \otimes_B eA \longrightarrow eA$ to the direct summand $e_{j_i}B \otimes_B eA$, hence it is a well defined homomorphism of A-modules and injective and $e_{j_i}A$ is clearly the image of the restriction. Therefore, m_{j_i} is an isomorphism.

Now assume that $Q_1 \longrightarrow Q_0 \longrightarrow N \longrightarrow 0$ is an exact sequence in $\mathrm{mod}\, B$ and the Q_i are projective. Applying the right exact functor T_e to this sequence, we note that the modules $T_e(Q_i)$ are projective. Recalling that a module is projective if and only if it is a direct summand of a free module, that B decomposes into the modules $e_{j_k}B$ and using the maps m_{j_i}, shows that $T_e(Q_i)$ satisfy the properties required in (4).

Conversely, assume a sequence as in (4) is given. Note that $P_ie = \mathrm{res}_e(P_i)$ are projective B-modules, since $\mathrm{res}_e e$ is exact. Applying T_e gives back the P_i. Denote by N the cokernel of $P_1e \longrightarrow P_0e$. We derive the existence of a commutative diagram

$$\begin{tikzcd} P_1 \arrow[r] \arrow[d, "\simeq"] & P_0 \arrow[r] \arrow[d, "\simeq"] & M \arrow[r] & 0 \\ T_e(P_1) \arrow[r] & T_e(P_0) \arrow[r] & T_e(N) \arrow[r] & 0. \end{tikzcd}$$

Therefore, $M \simeq T_e(N)$. □

We will use the above to prove the following

Theorem 1.33. *Let $A^b = e_A A e_A$ be a basic algebra associated with A. The algebra A^b is basic and the functor T_{e_A} gives an equivalence* $\operatorname{mod} A^b \simeq \operatorname{mod} A$*, with quasi-inverse* res_e.

Proof. We know that $A^b = e_A A^b = e_{j_1} A^b \oplus \ldots \oplus e_{j_a} A^b$ and, clearly, $e_{j_t} A^b e_{j_t} = e_{j_t} A e_{j_t}$ for all t. Since $e_{j_t} A$ is indecomposable in $\operatorname{mod} A$, the algebra $\operatorname{End}(e_{j_t} A^b) \simeq e_{j_t} A^b e_{j_t}$ is local. Therefore, e_{j_t} is a primitive idempotent of A^b. Now assume that $e_{j_t} A^b \simeq e_{j_r} A^b$. Using the isomorphisms m_{j_i} from the proof of the previous proposition, we see that

$$e_{j_t} A \simeq e_{j_t} A^b \otimes_{A^b} e_A A \simeq e_{j_r} A^b \otimes_{A^b} e_A A \simeq e_{j_r} A,$$

so $t = r$ by the choice of $e_{j_1}, \ldots, e_{j_a}$.

We already know that T_e is fully faithful. Now any module $M \in \operatorname{mod} A$ has a resolution $P' \longrightarrow P \longrightarrow M \longrightarrow 0$, with P', P projective. It remains to note that P and P' are direct sums of summands of $e_A A$. By part (4) of Proposition 1.32, T_e is essentially surjective, and hence an equivalence. ☐

If we are only interested in finite-dimensional modules, the theorem tells us that we can restrict our attention to basic algebras.

Example 1.34. Let $B = M_n(A)$ be the algebra of $(n \times n)$-matrices over an algebra A. Clearly, the matrices M_{ij} having 1 on the positions (i, j) and 0 everywhere else are a complete set of idempotents. Furthermore, $M_{ij} B$ does not depend on (i, j) and hence the associated basic algebra is $M_{11} B M_{11} \simeq A$. Thus, $\operatorname{mod} A \simeq \operatorname{mod} M_n(A)$.

2. Quivers and path algebras

Definition. A *quiver* $Q = (Q_0, Q_1, s, t)$ is given by a set of *vertices* Q_0, a set of *arrows* Q_1 and two maps $s, t \colon Q_1 \longrightarrow Q_0$ associating to any arrow α its *source* $s(\alpha)$ and its *target* $t(\alpha)$. One frequently just writes Q. A quiver is called *finite* if Q_0 and Q_1 are finite sets.

A *subquiver* is a quadruple $Q' = (Q'_0, Q'_1, s', t')$ such that $Q'_0 \subseteq Q_0$, $Q'_1 \subseteq Q_1$ and s', t' are the restrictions of s, t to Q'_1. A subquiver is called *full* if any arrow with source and target in Q'_0 belongs to Q'_1.

If a and b are elements in Q_0, a *path* from a to b of length l is a sequence of arrows $\alpha_1, \ldots, \alpha_l$ such that $s(\alpha_1) = a$, $t(\alpha_k) = s(\alpha_{k+1})$ for all $1 \leq k < l$ and $t(\alpha_l) = b$. We will write this as $\alpha_1 \ldots \alpha_l$. Note that with this convention the composition is not like that of functions. Of course, one could define it the other way around which is the same as considering the opposite algebra.

A *cycle* is a path such that source and target coincide. A cycle is a *loop* if it is of length 1. A quiver is called *acyclic* if it contains no cycles.

For any vertex a we have the *trivial path* ϵ_a of length 0.

Definition. Let Q be a quiver. The *path algebra* KQ of Q is the K-algebra whose underlying K-vector space has as basis all paths in Q and where the composition of two paths $\alpha_1 \ldots \alpha_k$ and $\beta_1 \ldots \beta_l$ is defined by

$$(\alpha_1 \ldots \alpha_k)(\beta_1 \ldots \beta_l) = \delta_{bc} \alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_l,$$

where $b = t(\alpha_k)$ and $c = s(\beta_1)$.

Note that KQ is an associative *graded* algebra since the composition of a path of length k and one of length l is a path of length $k+l$ (or 0). In symbols, $KQ = KQ_0 \oplus KQ_1 \oplus KQ_2 \oplus \ldots$, where KQ_i is the subspace generated by paths of length i.

Example 2.1. (1) The path algebra of the quiver with one vertex and one loop is isomorphic to $K[t]$, with t corresponding to the loop.
(2) If Q has one vertex and two loops, then KQ is the free associative algebra in two noncommuting indeterminates.
(3) Consider the quiver Q given by $1 \xrightarrow{\rho} 2$ which is 3-dimensional as a k-vector space and the multiplication rules are, for example, $e_1\rho = \rho$, $\rho e_2 = \rho$ etc. It is easily checked that KQ corresponds to the algebra of lower triangular (2×2)-matrices.

Lemma 2.2. *Let Q be a quiver and KQ its path algebra. The algebra KQ has an identity element if and only if Q_0 is finite. KQ is finite-dimensional if and only if Q is finite and acyclic.*

Proof. If Q is finite, say $Q_0 = \{1, \ldots, n\}$, then it is easily checked that $\sum_{i=1}^n \epsilon_i$ is the identity of KQ. To see the converse of the first statement, assume that Q_0 is not finite and let $1 = \sum_i \lambda_i \omega_i$, where $\lambda_i \in K$ and ω_i are paths, be the identity element. The paths ω_i have only finitely many sources, so take a vertex a not in this set. Then $\epsilon_a 1 = 0$, a contradiction.

If Q is finite and acyclic, there are only finitely many paths, hence KQ is finite-dimensional. Conversely, if Q_0 is infinite, then so is KQ. If Q is not acyclic, then take a cycle ω. Considering all its powers gives that KQ is infinite-dimensional. □

Corollary 2.3. *Let Q be a finite quiver. The set of all stationary paths $\epsilon_a, a \in Q_0$, is a complete set of primitive orthogonal idempotents of KQ.*

Proof. It is clear that the ϵ_a are orthogonal idempotents. To check that they are primitive, it is enough to show that the algebra $B = \epsilon_a KQ \epsilon_a$ is local, see Corollary 1.21. Note that this algebra is clearly K if Q has no cycles. In any case, an idempotent ϵ of B can be written as $\epsilon = \lambda\epsilon_a + \omega$, where $\lambda \in K$ and ω is a linear combination of cycles through a of length at least 1. Then

$$0 = \epsilon^2 - \epsilon = (\lambda^2 - \lambda)\epsilon_a + (2\lambda - 1)\omega + \omega^2$$

shows that $\omega = 0$ and $\lambda^2 = \lambda$, hence $\lambda = 0$ or $\lambda = 1$. Hence, $\epsilon = \epsilon_a$ or $\epsilon = 0$. □

Remark 2.4. The set of primitive idempotents exhibited above is, in general, not unique, consider, for instance, the lower triangular matrices.

Lemma 2.5. *Let A be an algebra and assume that $\{e_1 \ldots, e_n\}$ is a complete set of primitive orthogonal idempotents. Then A is connected if and only if there does not exist a nontrivial partition $I \amalg J$ of the set $\{1, \ldots, n\}$ such that for any $i \in I$ and $j \in J$ $e_i A e_j = 0 = e_j A e_i$.*

Proof. Assume that such a partition does exist and let $c = \sum_{j\in J} e_j$. By assumption, c is non-trivial. Furthermore, it is an idempotent, $ce_i = e_i c = 0$ for each $i \in I$ and $ce_j = e_j c = e_j$ for each $j \in J$. By our hypothesis, $e_i a e_j = 0 = e_j a e_i$ for any $a \in A$. Therefore,

$$\begin{aligned} ca &= \sum_{j\in J} e_j a = (\sum_j e_j a)\cdot 1 = (\sum_j e_j a)(\sum_{i\in I} e_i + \sum_{k\in J} e_k) \\ &= \sum_{j,k} e_j a e_k = (\sum_j e_j + \sum_i e_i) a (\sum_{k\in J} e_k) = ac. \end{aligned}$$

Hence, c is a non-trivial central idempotent and so A is not connected.

Conversely, if A is not connected, there exists a central non-trivial idempotent c. Since c is central, we have $c = \sum_{i=1}^n e_i c e_i$. Let $c_i = e_i c e_i$. Then $c_i^2 = c_i$, so $c_i \in e_i A e_i$ is an idempotent. Since e_i is primitive, $c_i = 0$ or $c_i = e_i$. Set $I = \{i \mid c_i = 0\}$ and $J = \{j \mid c_j = e_j\}$. This clearly is a partition of $\{1, \dots, n\}$ and, since $ce_j = e_j = e_j c$ and $ce_i = 0 = e_i c$, we have $e_i A e_j = 0 = e_j A e_i$. □

Using this we can now prove the

Lemma 2.6. *Let Q be a finite quiver. The path algebra KQ is connected if any only if Q is a connected quiver, which, by definition, means that the graph obtained by forgetting the orientation of the arrows is connected.*

Proof. If Q is not connected, let Q' be a connected component and let Q'' be the full subquiver of Q having as vertices $Q_0 \setminus Q_0'$. Take a in Q_0' and $b \in Q_0''$. Any path ω in Q is either contained in Q' or (in a connected component) of Q''. Therefore, either $\omega\epsilon_b = 0$ or $\epsilon_a \omega = 0$. In any case, $\epsilon_a \omega \epsilon_b = 0$. By Lemma 2.5, KQ is not connected.

Conversely, let Q be connected but not KQ. Thus we have a partition $Q_0 = Q_0' \amalg Q_0''$ as in the lemma. Since Q is connected, there exist $a \in Q_0'$ and $b \in Q_0''$ with an arrow α from a to b. Then $\alpha = \epsilon_a \alpha \epsilon_b = 0$, a contradiction. □

We record the following obvious

Proposition 2.7. *Let Q be a finite connected quiver and A an associative algebra with identity. For any pair of maps $\varphi_0 \colon Q_0 \longrightarrow A$ and $\varphi_1 \colon Q_1 \longrightarrow A$ satisfying (1) $\sum_{a\in Q_0} \varphi_0(a) = 1$, (2) $\varphi_0(a)^2 = \varphi_0(a)$, (3) $\varphi_0(a) \neq \varphi_0(b)$ for $a \neq b$ and (4) if $\alpha \colon a \longrightarrow b$, then $\varphi_1(a) = \varphi_0(a)\varphi_1(\alpha)\varphi_0(b)$, there exists a unique K-algebra homomorhpism $\varphi \colon KQ \longrightarrow A$ such that $\varphi(\epsilon_a) = \varphi_0(a)$ for any $a \in Q_0$ and $\varphi(\alpha) = \varphi_1(\alpha)$ for any $\alpha \in Q_1$.* □

Definition. Let Q be a finite and connected quiver. The two-sided ideal of KQ generated by the arrows of Q is called the *arrow ideal* and denote by R_Q or simply R.

Clearly, $R_Q = KQ_1 \oplus KQ_2 \oplus \dots$ as a K-vector space. This implies that $R_Q^l = \oplus_{m\geq l} KQ_m$.

Proposition 2.8. *Let Q be a finite connected quiver, R the arrow ideal of KQ and ϵ_a the trivial paths associated to the vertices of Q. Consider the canonical algebra homomorphism $\pi \colon KQ \longrightarrow KQ/R$ and the set of the images $e_a := \pi(\epsilon_a)$. Then this is a complete* *set of primitive orthogonal idempotents for KQ/R and the latter algebra is isomorphic to $K \times \ldots \times K$. If Q is acyclic, then* $\operatorname{rad} KQ = R$ *and KQ is a finite-dimensional basic algebra.*

Proof. As a K-vector space we have

$$KQ/R = \oplus_{a,b \in Q_0} e_a(KQ/R)e_b = \oplus_{a \in Q_0} e_a(KQ/R)e_a,$$

where the second equality stems from the fact that R contains all paths of length at least 1. Hence, KQ/R is a Q_0-dimensional vector space. The elements e_a give a compete set of primitive orthogonal idempotents of KQ/R and every piece $e_a(KQ/R)e_a$ is isomorphic to K. Therefore, the first statement holds.

If Q is acyclic, then KQ is finite-dimensional and the length of paths in Q is bounded by some integer l. Hence, $R^{l+1} = 0$, so $R \subseteq \operatorname{rad} KQ$, by Corollary 1.4. Since $KQ/R \simeq K \times \ldots \times K$, Corollary 1.4 gives that $R = \operatorname{rad} KQ$ and it follows from Proposition 1.29 that KQ is basic. □

Remark 2.9. If Q is not acyclic, then $\operatorname{rad} KQ$ need not be equal to R_Q. As an example consider the quiver with one vertex and one loop. Then the radical is trivial, but R_Q is not.

Definition. Let Q be a finite quiver and R be the arrow ideal of the path algebra KQ. A two-sided ideal I of KQ is called *admissible* if there exists an $m \geq 2$ such that $R^m \subseteq I \subseteq R^2$.

If I is an admissible ideal of KQ, we call the pair (KQ, I) a *bound quiver*. The quotient algebra KQ/I is said to be a *bound quiver algebra*.

It is clear that an ideal $I \subseteq R^2$ is admissible if and only if it contains all paths whose length is large enough. In fact, this is the case if and only if for each cycle σ there exists an $s \geq 1$ such that $\sigma^s \in I$. In particular, if Q is acyclic, any ideal $I \subseteq R^2$ is admissible.

Example 2.10. (1) The ideal R^m is admissible for any $m \geq 2$.
(2) The zero ideal is admissible if and only if Q is acyclic.
(3) Let Q be the quiver

$$\begin{tikzcd}
 & 2 \arrow[dl, "\beta"'] & \\
1 & & 4 \arrow[ul, "\alpha"'] \arrow[ll, "\lambda"'] \arrow[dl, "\gamma"] \\
 & 3 \arrow[ul, "\delta"] &
\end{tikzcd}$$

The ideal $I = \langle \alpha\beta - \gamma\delta \rangle$ is admissible but $I' = \langle \alpha\beta - \lambda \rangle$ is not, since $\alpha\beta - \lambda \notin R^2$.

Definition. Let Q be a quiver. A *relation* ρ in Q with coefficients in K is a K-linear combination of paths ω_i of length at least two having the same source and target. In symbols, $\rho = \sum_{i=1}^{n} \lambda_i \omega_i$. If $(\rho_j)_{j \in J}$ is a set of relations such that the ideal they generate is admissible, then we say that the quiver Q is *bound by the relations* $\rho_j = 0$ for all $j \in J$.

Lemma 2.11. *Let Q be a finite quiver and I be an admissible ideal of KQ. The set $e_a = \pi(\epsilon_a)$, where $\pi\colon KQ \twoheadrightarrow KQ/I$, is a complete set of primitive orthogonal idempotents of KQ/I.*

Proof. It is clear that the given set is a complete set of orthogonal idempotents. It therefore remains to check that each e_a is primitive or, equivalently, that the algebra $B_a = e_a(KQ/I)e_a$ has only the trivial idempotents 0 and 1 for any $a \in Q_0$. Note that any idempotent e in B_a can be written in the form $e = \lambda\epsilon_a + \omega + I$, where ω is a linear combination of cycles through a of length ≥ 1 and $\lambda \in K$. Since $e^2 = e$, we get

$$(\lambda^2 - \lambda)\epsilon_a + (2\lambda - 1)\omega + \omega^2 \in I.$$

Since $I \subseteq R^2$, $\lambda^2 - \lambda = 0$, hence $\lambda = 1$ or $\lambda = 0$. If $\lambda = 0$, then $e = \omega + I$, so ω is idempotent modulo I. Since $R^m \subseteq I$ for some $m \geq 2$, $\omega^m \in I$, so $\omega \in I$ and hence $e = 0$. If $\lambda = 1$, then $e_a - e = -\omega + I$ is an idempotent in B_a, so ω is idempotent modulo I, thus nilpotent as before, so is an element in I. Thus $e_a = e$. □

Lemma 2.12. *Let Q be a finite quiver and I be an admissible ideal of KQ. The bound quiver algebra KQ/I is connected if and only if Q is a connected quiver.*

Proof. If Q is not connected, neither is KQ, so there exists a central non-trivial idempotent γ which is a sum of paths of length 0. Then its image is a central non-trivial idempotent in KQ/I, since if $\pi(\gamma) = 1$, then $1-\gamma \in I$, which is impossible, since $I \subseteq R^2$. The reverse implication is proved as in Lemma 2.6. □

Proposition 2.13. *Let Q be a finite quiver and I an admissible ideal. Then KQ/I is a finite-dimensional algebra.*

Proof. We have a surjective homomorphism $KQ/R^m \twoheadrightarrow KQ/I$. The former algebra is finite-dimensional, since the finitely many paths of length at most m form a basis of KQ/R^m as a K-vector space. □

Example 2.14. Consider the quiver Q having one vertex and two loops α and β, and the ideal $I = \langle \beta\alpha, \beta^2 \rangle$. Then I is not admissible, since $\alpha^m \notin I$ for any $m \geq 1$. Consider $A = KQ/I$ and the subspace J of A generated by elements of the form $\pi(\alpha^n)\pi(\beta)$, $n \geq 1$, where as usual $\pi\colon KQ \twoheadrightarrow KQ/I$. Clearly, J is a right ideal of A, since $J\pi(\alpha) \subseteq J$ and similarly for $\pi(\beta)$. Hence J is a submodule of A_A, but it is not finitely generated. Indeed, assume J has a finite set of generators and take m to be the largest exponent of $\pi(\alpha)$ among this set of generators. Then $\pi(\alpha)^{m+1}\beta$ cannot be a K-linear combination of the generators. Hence, A is not only not finitely generated, but not even right Noetherian.

Lemma 2.15. *Let Q be a finite quiver. Every admissible ideal I of KQ is finitely generated.*

Proof. Consider the short exact sequence of KQ-modules

$$0 \longrightarrow R^m \longrightarrow I \longrightarrow I/R^m \longrightarrow 0.$$

Clearly, R^m is finitely generated and so is I/R^m, being an ideal in KQ/R^m. Hence, I is a finitely generated KQ-module. □

Corollary 2.16. *If I is an admissible ideal of a finite quiver Q, then it is generated by a finite set of relations.*

Proof. We know that I is generated by $\{\sigma_1, \ldots, \sigma_n\}$, but the σ_i need not have the same source and target. However, the set $\{\epsilon_a \sigma_i \epsilon_b \mid 1 \leq i \leq n,\ a, b \in Q_0\}$ is as desired. $\square$

Lemma 2.17. *Let Q be a finite quiver and I an admissible ideal of KQ. Then $R/I = \operatorname{rad}(KQ/I)$. Furthermore, the algebra KQ/I is basic.*

Proof. We know that $R^m \subseteq I$ for some $m \geq 2$. Hence, $(R/I)^m = 0$ and $R/I \subseteq \operatorname{rad}(KQ/I)$. Since $(KQ/I)/(R/I) \simeq KQ/R \simeq K \times \ldots \times K$, the assertions follow by Corollary 1.4 and Proposition 1.29. $\square$

Remark 2.18. For each $l \geq 1$, $\operatorname{rad}^l(KQ/I) = (R/I)^l$. Therefore,

$$\operatorname{rad}(KQ/I)/\operatorname{rad}^2(KQ/I) \simeq R/R^2.$$

Example 2.19. It can be checked that if Q is the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$, its path algebra is isomorphic to the lower triangular (3×3)-matrices. The ideal $I = \langle \alpha\beta \rangle$ is easily seen to be equal to R^2, which is the set of matrices generated by the matrix M_{31}.

Our next goal is to show that any basic and connected finite-dimensional algebra can be described as the bound quiver algebra of a finite connected quiver. We begin with the

Definition. Let A be a basic and connected finite-dimensional algebra and $\{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents. The *(ordinary) quiver* of A, denoted by Q_A, is defined as follows:

(1) The vertices of Q_A are the numbers $\{1, \ldots, n\}$.
(2) Given two points $a, b \in (Q_A)_0$ the arrows $\alpha \colon a \longrightarrow b$ are in bijective correspondence with the vectors in a basis of $e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$.

Note that Q_A is finite, since A is finite-dimensional and, therefore, the vector spaces $e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$ are also finite-dimensional.

Lemma 2.20. *Let A be as in the definition. Then*

(1) *The quiver Q_A does not depend on the choice of a complete set of primitive orthogonal idempotents of A.*
(2) *For any pair e_a, e_b of primitive orthogonal idempotents of A the K-linear map*

$$\psi \colon e_a(\operatorname{rad} A)e_b/e_a(\operatorname{rad}^2 A)e_b \longrightarrow e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$$

defined by

$$e_a x e_b + e_a \operatorname{rad}^2 A e_b \longmapsto e_a(x + \operatorname{rad}^2 A)e_b$$

is an isomorphism $\forall x \in \operatorname{rad} A$.

Proof. By Theorem 1.23, the number of points of Q_A is uniquely determined, since it equals the number of indecomposable direct summands of A_A. The same theorem also gives that for distinct complete sets of primitive orthogonal idempotents, say e_a and e'_a, there is a bijection $e_aA \simeq e'_aA$ for all a. Define an A-module homomorphism $\varphi\colon e_a(\operatorname{rad} A) \longrightarrow e_a(\operatorname{rad} A/\operatorname{rad}^2 A)$ by $e_ax \longmapsto e_a(x + \operatorname{rad}^2 A)$. It is easy to see that its kernel is $e_a(\operatorname{rad}^2 A)$. Hence, using that $(\operatorname{rad}(eA)/\operatorname{rad}^2(eA))e_b \simeq \operatorname{Hom}_A(e_bA, \operatorname{rad}(eA)/\operatorname{rad}^2(eA))$, we conclude that

$$e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b \simeq e'_a(\operatorname{rad} A/\operatorname{rad}^2 A)e'_b.$$

This proves part (1) and (2) is trivial. □

Lemma 2.21. *For each arrow $\alpha\colon i \longrightarrow j$ in $(Q_A)_1$, let $x_\alpha \in e_i(\operatorname{rad} A)e_j$ be such that the set $\{x_\alpha + \operatorname{rad}^2 A \mid \alpha\colon i \longrightarrow j\}$ is a basis of $e_i(\operatorname{rad} A/\operatorname{rad}^2 A)e_j$. Then*

(1) *for any two points $a, b \in (Q_A)_0$, every element $x \in e_a(\operatorname{rad} A)e_b$ can be written in the form $x = \sum x_{\alpha_1} \dots x_{\alpha_l}\lambda_{\alpha_1\dots\alpha_l}$, where $\lambda_{\alpha_1\dots\alpha_l} \in K$ and the sum is taken over all paths $\alpha_1 \dots \alpha_l$ in Q_A from a to b.*
(2) *for each arrow $\alpha\colon i \longrightarrow j$, the element x_α uniquely determines a nonzero nonisomorphism $\tilde{x}_\alpha \in \operatorname{Hom}_A(e_jA, e_iA)$ such that $\tilde{x}_\alpha(e_j) = x_\alpha$, $\operatorname{im} \tilde{x}_\alpha \subseteq e_i(\operatorname{rad} A)$ and $\operatorname{im} \tilde{x}_\alpha \nsubseteq e_i(\operatorname{rad}^2 A)$.*

Proof. Recall that $\operatorname{rad} A$ is nilpotent and, as a K-vector space, $\operatorname{rad} A \simeq (\operatorname{rad} A/\operatorname{rad}^2 A) \oplus \operatorname{rad}^2 A$. Since the x_α are a basis of the first vector space, we get

$$x - \sum_{\alpha\colon a \longrightarrow b} x_\alpha\lambda_\alpha =: x' \in e_a(\operatorname{rad}^2 A)e_b,$$

for $\lambda_\alpha \in K$. Using that $e_a(\operatorname{rad}^2 A)e_b = \sum_{c\in(Q_A)_0}(e_a(\operatorname{rad} A)e_c)(e_c(\operatorname{rad} A)e_b)$, we get $x' = \sum_{c\in(Q_A)_0} x'_c y'_c$, where $x'_c \in e_a(\operatorname{rad} A)e_c$ and $y'_c \in e_c(\operatorname{rad} A)e_b$. We now apply the previous consideration to x'_c and y'_c and get

$$x = \sum_{\alpha\colon a \longrightarrow b} x_\alpha\lambda_\alpha + \sum_{\beta\colon a \longrightarrow c}\ \sum_{\gamma\colon c \longrightarrow b} x_\beta x_\gamma \lambda_\beta\lambda_\gamma \quad \text{modulo} \quad e_a(\operatorname{rad}^3 A)e_b.$$

Induction and the nilpotency of $\operatorname{rad} A$ give (1). To prove (2), use the isomorphism $e_i(\operatorname{rad} A)e_j \simeq \operatorname{Hom}_A(e_jA, e_i(\operatorname{rad} A))$. □

Corollary 2.22. *If A is a basic connected algebra, then Q_A is connected.*

Proof. Assume the converse and write $(Q_A)_0$ as a disjoint set $Q' \amalg Q''$. We will show that for $i \in Q'$ and $j \in Q''$ we have $e_iAe_j = 0 = e_jAe_i$, which means that A is not connected, a contradiction. We have already seen that $M \operatorname{rad} A = \operatorname{rad} M$ for any right module M, so $\operatorname{rad}(e_iA) = e_i \operatorname{rad} A$. Furthermore, $e_iAe_j \simeq \operatorname{Hom}(e_jA, e_iA)$ and $\operatorname{Hom}(e_jA, \operatorname{rad} e_iA) \simeq e_i(\operatorname{rad} A)e_j$. The latter space is zero by our assumption and the lemma. Hence, we are done, if we can show that

$$\operatorname{Hom}(e_jA, e_iA) \simeq \operatorname{Hom}(e_jA, \operatorname{rad} e_iA).$$

Recall that, given an idempotent $e \in A$, $\operatorname{rad}(eA)$ is the unique maximal submodule of eA (Proposition 1.18). This implies that $eA/\operatorname{rad}(eA) \simeq eA/e\operatorname{rad} A$ is simple.

Now take any map $\varphi\colon e_jA \longrightarrow e_iA$. If it is not surjective, we are done, since the image has to be in $\operatorname{rad} e_iA$. If φ is surjective, then $e_jA/\ker(\varphi) \simeq e_iA$. Since $\ker(\varphi) \subset \operatorname{rad}(e_jA)$, this gives a map $e_jA \longrightarrow S(i) := e_iA/\operatorname{rad}(e_iA)$ which is surjective. Factoring out its kernel, we get a non-trivial map $S(j) \longrightarrow S(i)$, a contradiction by Schur's lemma, since $S(j)$ cannot be isomorphic to $S(i)$ by the assumption that A is basic and Corollary 1.27. □

Example 2.23. If $A = K[t]/(t^m)$ for $m \geq 1$, then Q_A has only one point since the only nonzero idempotent of A is the identity. The radical of A is the image of the ideal generated by (t), by Corollary 1.4. Therefore, a basis of $\operatorname{rad} A/\operatorname{rad}^2 A$ is given by one element and Q_A is the quiver with one vertex and one loop.

Lemma 2.24. *Let Q be a finite connected quiver, I an admissible ideal and $A = KQ/I$. Then $Q_A = Q$.*

Proof. By Lemma 2.11, $\{e_a = \epsilon_a + I \mid a \in Q_0\}$ is a complete set of primitive orthogonal idempotents of $A = KQ/I$, so the sets of vertices of Q and Q_A are the same. On the other hand, Remark 2.18 gives that the arrows from a to b in Q are in bijective correspondence with the vectors in a basis of $e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$, that is, with the arrows from a to b in Q_A. □

Theorem 2.25. *Let A be a basic and connected finite-dimensional K-algebra. There exists an admissible ideal I of KQ_A such that $A \simeq KQ_A/I$.*

Proof. Let $\alpha\colon i \longrightarrow j$ in $(Q_A)_1$ and choose $x_\alpha \in \operatorname{rad} A$ such that $\left\{x_\alpha + \operatorname{rad}^2 A \mid \alpha\colon i \longrightarrow j\right\}$ forms a basis in $e_i(\operatorname{rad} A/\operatorname{rad}^2 A)e_j$. Consider

$$\varphi_0\colon (Q_A)_0 \longrightarrow A, \quad a \longmapsto e_a$$

and

$$\varphi_1\colon (Q_A)_1 \longrightarrow A, \quad \alpha \longmapsto x_\alpha.$$

It is clear that the conditions of Proposition 2.7 are satisfied and hence we get an algebra homomorphism $\varphi\colon KQ_A \longrightarrow A$. It remains to check that φ is surjective and that its kernel is an admissible ideal of KQ_A.

The Wedderburn–Malcev theorem tells us that $A \simeq A/\operatorname{rad} A \oplus \operatorname{rad} A$. The former space is generated by the e_a, while any element of $\operatorname{rad} A$ is in the image by Lemma 2.21. Hence, φ is surjective. By definition, $\varphi(R) \subseteq \operatorname{rad} A$, hence $\varphi(R^l) \subseteq \operatorname{rad}^l A$ for any $l \geq 1$. Since $\operatorname{rad} A$ is nilpotent, there exists an $m \geq 1$ such that $R^m \subseteq \ker(\varphi) =: I$. It remains to check that $I \subseteq R^2$. Any $x \in I$ can be written as

$$x = \sum_{a \in (Q_A)_0} \epsilon_a \lambda_a + \sum_{\alpha \in (Q_A)_1} \alpha \mu_\alpha + y,$$

where $\lambda_a, \mu_\alpha \in K$ and $y \in R^2$. If $\varphi(x) = 0$, then

$$\sum_{a \in (Q_A)_0} e_a \lambda_a = - \sum_{\alpha \in (Q_A)_1} x_\alpha \mu_\alpha - \varphi(y) \in \operatorname{rad} A.$$

Since $\operatorname{rad} A$ is nilpotent, this implies that $\lambda_a = 0$ for all a. A similar reasoning shows that $\sum_{\alpha \in (Q_A)_1} x_\alpha \mu_\alpha = -\varphi(y) \in \operatorname{rad}^2 A$, so

$$\sum_{\alpha \in (Q_A)_1} (x_\alpha + \operatorname{rad}^2 A) \mu_\alpha = 0 \quad \text{in} \quad \operatorname{rad} A / \operatorname{rad}^2 A.$$

By assumption on the x_α, all the μ_α have to be zero, hence $x = y \in R^2$.

□

Remark 2.26. We say that two algebras A and A' are *Morita equivalent* if $\operatorname{mod} A \simeq \operatorname{mod} A'$. Since any algebra A is Morita equivalent to a basic algebra by Theorem 1.33, Theorem 2.25 implies, in particular, that any connected algebra is Morita equivalent to a bound quiver algebra. Furthermore, we could deal with non-connected algebras as well by considering their connected factors.

3. Representations of quivers

Definition. Let Q be a finite quiver. A *K-linear representation* M of Q consists of the following data. For each point $a \in Q_0$ a vector-space M_a and for every arrow $\alpha\colon a \longrightarrow b$ a K-linear map $\varphi_\alpha\colon M_a \longrightarrow M_b$. A representation is called *finite-dimensional* if every M_a is a finite-dimensional vector space.

A morphism between representations M and M' consists of linear maps $f_a\colon M_a \longrightarrow M'_a$ for every $a \in Q_0$ such that

$$\begin{tikzcd} M_a \arrow[r, "\varphi_\alpha"] \arrow[d, "f_a"'] & M_b \arrow[d, "f_b"] \\ M'_a \arrow[r, "\varphi'_\alpha"] & M'_b \end{tikzcd}$$

commutes for all a, b and α.

It is clear that maps of representations can be composed and that there exist identity maps, so there is a category $\operatorname{Rep}(Q)$ of representations of Q. We can define direct sums, kernels and images componentwise and it is easily checked that this makes $\operatorname{Rep}(Q)$ into an abelian category. The full abelian subcategory of finite-dimensional representations will be denoted by $\operatorname{rep}(Q)$.

Example 3.1. Let Q be the quiver $1 \longrightarrow 2 \longrightarrow 3$. A representation of Q is, for example $M = [\, K \xrightarrow{\mathrm{id}} K \longrightarrow 0 \,]$. Another representation is $N = [\, 0 \longrightarrow K \longrightarrow 0 \,]$. It is easily checked that $\operatorname{Hom}(M, N) = 0$, while $\operatorname{Hom}(N, M) \simeq K$.

Definition. If $\omega = \alpha_1 \dots \alpha_l$ is a non-trivial path from a to b in a finite quiver Q, the *evaluation* of ω is the K-linear map

$$\varphi_\omega = \varphi_{\alpha_l} \dots \varphi_{\alpha_1} \colon M_a \longrightarrow M_b.$$

This extends to K-linear combinations of paths with the same source and target. If I is an admissible ideal of KQ, a representation M of Q is said to *satisfy the relations in* I or *to be bound by* I if $\varphi_\rho = 0$ for all relations ρ in I.

The full subcategory of $\operatorname{Rep}(Q)$ consisting of representations satisfying the relations in I will be denoted by $\operatorname{Rep}(Q, I)$, and similarly for $\operatorname{rep}(Q)$.

Example 3.2. Consider the quiver Q

$$\begin{tikzcd}
& 2 \arrow[dl, "\beta"'] & \\
1 & & 4 \arrow[ul, "\gamma"'] \arrow[dl, "\alpha"] \\
& 3 \arrow[ul, "\delta"] &
\end{tikzcd}$$

with the relation $\gamma\beta = \alpha\delta$ and the representations M and N of Q given by

$$\begin{tikzcd}
& K \arrow[dl, "i_1"'] & \\
K^2 & & 0 \arrow[ul] \arrow[dl] \\
& K \arrow[ul, "i_2"] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& K \arrow[dl, "\mathrm{id}"'] & \\
K & & K \arrow[ul, "\mathrm{id}"'] \arrow[dl, "\mathrm{id}"] \\
& K \arrow[ul, "\mathrm{id}"] &
\end{tikzcd}$$

Both are bound by I. On the other hand, changing one of the maps in the second representation to 0 gives a representation not bound by I.

Theorem 3.3. *Let Q be a finite connected quiver, I an admissible ideal of KQ and $A = KQ/I$. There exists a K-linear equivalence*

$$F \colon \operatorname{Mod} A \simeq \operatorname{Rep}(Q, I)$$

that restricts to an equivalence $F \colon \operatorname{mod} A \simeq \operatorname{rep}(Q, I)$.

Proof. We start with the construction of F on objects. Let $M \in \operatorname{Mod} A$ and $a \in Q_0$. Set M_a to be Me_a, where e_a is the image of the stationary path ϵ_a under the canonical projection $KQ \twoheadrightarrow KQ/I$. Next, if $\alpha\colon a \to b$ is an arrow and $x \in M_a = Me_a$, let $\varphi_\alpha(x) := x\overline{\alpha}$, where $\overline{\alpha}$ is the class of α modulo I. If $\rho = \sum_i \lambda_i \omega_i$ is a relation in I, then

$$\varphi_\rho(x) = \sum_i \lambda_i \varphi_{\omega_i}(x) = x\overline{\rho} = 0.$$

Hence, $F(M)$ is indeed a representation bound by I.

Let $f\colon M \to M'$ be a homomorphism of A-modules. For any $a \in Q_0$ and $x = xe_a \in M_a$ we have

$$f(xe_a) = f(xe_a^2) = f(xe_a)e_a \in M'e_a = M'_a.$$

Thus, we get a K-linear map $f_a\colon M_a \to M'_a$ for any $a \in Q_0$ which is just the restriction of f. Given an arrow $\alpha\colon a \to b$ and $x \in M_a$, we now compute

$$f_b\varphi_\alpha(x) = f(x\overline{\alpha}) = f(x)\overline{\alpha} = \varphi'_\alpha f_a(x).$$

It is obvious that F is a K-linear functor. Furthermore, it restricts to a functor $\operatorname{mod} A \to \operatorname{rep}(Q, I)$.

We will now define a functor $G\colon \operatorname{Rep}(Q, I) \to \operatorname{Mod} A$. So, let M be a representation bound by I. We set $G(M) = \oplus_{a \in Q_0} M_a$. We will define an A-module structure on $G(M)$ in two steps, first by specifying a KQ-module structure and then checking that it is annihilated by I. To define a KQ-module structure on $G(M)$, we have to say what an arbitrary path ω does. Let $x \in G(M)$. If $\omega = \epsilon_a$, then set $x\omega := x_a$. If ω is a non-trivial path from a to b, we define $x\omega$ to be the component of $\varphi_\omega(x)$ in M_b. This endows $G(M)$ with a KQ-module structure. If $\rho \in I$, by definition $x\rho = 0$, hence $G(M)$ is an A-module.

Next, given a morphism $(f_a)_{a \in Q_0}$ from $M = (M_a, f_a)$ to $M' = (M'_a, f'_a)$, we clearly have a K-linear map

$$f\colon G(M) = \oplus_a M_a \to G(M') = \oplus_a M'_a.$$

It remains to check that this map is A-linear. Without loss of generality we will do this for $x_a \in M_a \subset G(M)$ and $\overline{\omega} \in KQ/I$, where ω is a path from a to b in Q. Then

$$f(x_a\overline{\omega}) = f_b\varphi_\omega(x_a) = \varphi'_\omega f_a(x_a) = f(x)\overline{\omega}.$$

The functor G is obviously K-linear and restricts to a functor $\operatorname{rep}(Q, I) \to \operatorname{mod} A$. It is left to the reader to check that F and G are quasi-inverse to each other. Finally, note that a representation M of a finite quiver is finite-dimensional if and only if M_a is finite-dimensional for all $a \in Q_0$, which proves that F and G restrict to equivalences of the smaller categories. □

Recall that Corollaries 1.25 and 1.28 classify the indecomposable projective and simple modules in $\operatorname{mod} A$, where A is any finite-dimensional algebra.

We now consider the following situation. Let Q be a finite connected quiver with n vertices, I an admissible ideal of KQ and let KQ/I be the associated path algebra, which we know to be basic and connected, to have R/I as radical and $\pi(\epsilon_a) = e_a$, for

$a \in Q_0$ as a complete set of primitive orthogonal idempotents. We want to understand the indecomposable projective/injective and the simple modules in $\operatorname{mod} A \simeq \operatorname{rep}(Q, I)$. We will not distinguish between these two categories in what follows.

Let $a \in Q_0$ and consider the representation $S(a)$ defined by $S(a)_b = \delta_{ab} K$, where δ_{ab} is the Kronecker delta and $b \in Q_0$. In other words, $S(a)$ only has the vector space K over the vertex a. Hence, all the linear maps in $S(a)$ are zero.

Lemma 3.4. *Let $A = KQ/I$ be the bound quiver algebra of (Q, I). The A-module $S(a)$ is isomorphic to $\operatorname{top} e_a A$. In particular, the set $\{S(a) \mid a \in Q_0\}$ contains precisely the simple A-modules.*

Proof. The vector space $S(a)$ is one-dimensional for all a, hence defines a simple A-module. We also have $\operatorname{Hom}_A(e_a A, S(a)) \simeq S(a) e_a \simeq S(a)_a \neq 0$, so there exists a nonzero map $e_a A \longrightarrow S(a)$. The map is surjective by Schur's lemma and its kernel is a maximal submodule of $e_a A$, hence isomorphic to $\operatorname{rad} e_a A$. This proves the first statement. Since obviously $\operatorname{Hom}(S(a), S(b)) = 0$ for $a \neq b$, the $S(a)$ are pairwise non-isomorphic which proves the second statement. $\square$

Remark 3.5. A path algebra of a finite quiver with a cycle can have infinitely many pairwise non-isomorphic simple finite-dimensional modules. For example, take Q to be

$$1 \rightleftarrows 2$$

We have the simple modules $S(1) = K \rightleftarrows 0$ and $S(2) = 0 \rightleftarrows K$. But also $S_\lambda = K \underset{\lambda}{\overset{\mathrm{id}}{\rightleftarrows}} K$ for $\lambda \in K^*$ are simple pairwise non-isomorphic modules.

Before stating the next result, define the *socle* of a module M, denoted by $\operatorname{soc} M$, to be the submodule of M generated by all simple submodules of M. Furthermore, we say that a vertex of a quiver is a *sink* resp. a *source* if no arrow starts resp. ends in this vertex.

Lemma 3.6. *Let $M = (M_a, \varphi_\alpha)$ be a bound representation of (Q, I). Then*

(1) *M is semisimple if and only if $\varphi_\alpha = 0$ for all $\alpha \in Q_1$.*
(2) *$\operatorname{soc} M = N$ where $N = (N_a, \psi_\alpha)$ is the representation where $N_a = M_a$ when a is a sink, whereas*

$$N_a = \bigcap_{\alpha\colon a \longrightarrow b} \ker(\varphi_\alpha\colon M_a \longrightarrow M_b)$$

if a is not a sink, and $\psi_\alpha = 0$ for every arrow α.
(3) *$\operatorname{rad} M = J$, where $J = (J_a, \gamma_\alpha)$ with*

$$J_a = \sum_{\alpha'\colon b \longrightarrow a} \operatorname{im}(\varphi_{\alpha'}\colon M_b \longrightarrow M_a)$$

and $\gamma_\alpha = (\varphi_\alpha)_{|J_a}$ for every arrow α of source a.

(4) $\operatorname{top} M = L$, *where* $L = (L_a, \psi_\alpha)$ *with* $L_a = M_a$ *if* a *is a source, while*

$$L_a = \sum_{\alpha' \colon b \longrightarrow a} \operatorname{coker}(\varphi_{\alpha'} \colon M_b \longrightarrow M_a)$$

if a *is not a source, and* $\psi_\alpha = 0$ *for any arrow* α.

Proof. (1) M is semisimple if any only if it a direct sum of copies of the $S(a)$, whence (1) holds.

(2) Clearly, N is a semisimple submodule of M. Let S be a simple submodule of M, which has to be isomorphic to some $S(a)$. So given any arrow $\alpha\colon a \longrightarrow b$, we have a commutative diagram

$$\begin{tikzcd} K \simeq S(a)_a \arrow[r] \arrow[d] & S(a)_b = 0 \arrow[d] \\ M_a \arrow[r, "\varphi_\alpha"] & M_b. \end{tikzcd}$$

It follows that $S(a)_a \subseteq \ker(\varphi_\alpha)$ for all arrows $\alpha\colon a \longrightarrow b$, hence $S(a)_a \subseteq N_a$. Therefore, $S(a) \subseteq N$, hence $N = \operatorname{soc} M$.

(3) Start with the equation $J = \operatorname{rad} M = M \operatorname{rad} A = M(R/I) = \sum_{\alpha \in Q_1} M\overline{\alpha}$, where $\overline{\alpha} = \alpha + I$ and R is, as usual, the arrow ideal of KQ. This implies that $J_a = \sum_{\alpha\colon b \longrightarrow a} M\overline{\alpha}$. Given an arrow with target a,

$$M\overline{\alpha} = Me_b\overline{\alpha} = M_b\overline{\alpha} = \varphi_\alpha(M_b) = \operatorname{im} \varphi_\alpha.$$

Hence, J_a for all a is as claimed and the assertion follows.

(4) follows from (3), since $\operatorname{top} M = M/\operatorname{rad} M$. □

Example 3.7. Let Q be the *Kronecker quiver* $1 \underset{\alpha}{\overset{\beta}{\leftleftarrows}} 2$. Note that 1 is a sink and 2 is a source.

We know that the simple modules are $S(1)$ and $S(2)$ where the former has K over vertex 1 and 0 over 2 and vice versa for $S(2)$. We consider the representation M given by $K^{m-1} \underset{\pi_\alpha}{\overset{\pi_\beta}{\leftleftarrows}} K^m$, where $m \geq 2$ and the maps are given by the following $((m-1) \times m)$-matrices:

$$\pi_\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

$$\pi_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Clearly, both maps are surjective, $\ker(\pi_\alpha)$ has as basis e_2 and $\ker(\pi_\beta)$ has as basis e_1. Hence, $\operatorname{soc} M = \operatorname{rad} M = S(1)^{m-1}$ and $\operatorname{top} M = S(2)^m$.

Lemma 3.8. *Let (Q, I) be a bound quiver, $A = KQ/I$ and $P(a) = e_a A$, where $e_a = \epsilon_a + I$ and $a \in Q_0$. We have the decomposition $A_A = \oplus_{a \in Q_0} e_a A$ corresponding to the complete set of primitive orthogonal idempotents $\{e_a \mid a \in Q_0\}$.*

1. *If $P(a) = (P(a)_b, \varphi_\beta)$, then $P(a)_b$ is the vector space with basis the set of all $\overline{\omega} = \omega + I$ with ω a path from a to b, and for an arrow $\beta\colon b \longrightarrow c$ the map $\varphi_\beta\colon P(a)_b \longrightarrow P(a)_c$ is given by the right multiplication with $\overline{\beta} = \beta + I$.*
2. *Let $\operatorname{rad} P(a) = (P'(a)_b, \varphi'_\beta)$. Then $P'(a)_b = P(a)_b$ for $b \neq a$, $P'(a)_a$ is the vector space with basis set of all $\overline{\omega} = \omega + I$ with ω a non-trivial path from a to a, $\varphi'_\beta = \varphi_\beta$ for any arrow of source $b \neq a$ and φ'_α is the restriction of φ_α to $P'(a)_a$ for any arrow α with source a.*

Proof. It suffices to prove (1), since (2) follows from it and part (3) of the previous lemma. We have

$$P(a)_b = P(a)e_b = e_a A e_B = e_a (KQ/I) e_b = (\epsilon_a KQ \epsilon_b)/(\epsilon_a I \epsilon_b).$$

This proves the first statement. It follows immediately from the construction of the functor F that for an arrow $\beta\colon b \longrightarrow c$, the K-linear map φ_β is given by the right multiplication with $\overline{\beta}$, proving the second statement. $\square$

Remark 3.9. If $I = 0$ and Q is acyclic, the space $P(a)_b$ has as basis the set of all paths from a to b.

Example 3.10. Let Q be the quiver

$$\begin{tikzcd} & 1 & \\ 2 \arrow[ur] & & 3 \arrow[ul] \end{tikzcd}$$

The representation $P(1)$ is then

$$\begin{tikzcd} & K & \\ 0 \arrow[ur] & & 0 \arrow[ul] \end{tikzcd}$$

while $P(2)$ is

$$\begin{tikzcd} & K & \\ K \arrow[ur, "\mathrm{id}"] & & 0 \arrow[ul] \end{tikzcd}$$

and $P(3)$ is

$$\begin{tikzcd} & K & \\ 0 \arrow[ur] & & K. \arrow[ul, "\mathrm{id}"'] \end{tikzcd}$$

The path algebra of a finite quiver Q has the following useful property.

Proposition 3.11. *Let Q be a finite quiver, and $A = KQ$. For any right A-module M there exists a projective resolution of the form*

$$0 \longrightarrow \bigoplus_{\alpha \in Q_1} \epsilon_{t(\alpha)} A \otimes_K M \epsilon_{s(\alpha)} \xrightarrow{f} \bigoplus_{i \in Q_0} \epsilon_i A \otimes_K M \epsilon_i \xrightarrow{g} M \longrightarrow 0,$$

where $f(a \otimes m) = \alpha a \otimes m - a \otimes m\alpha$ and $g(a' \otimes m') = m'a'$.

Proof. First note that obviously $g \circ f = 0$. To see that f is injective, consider a non-zero element $a \otimes m = \sum_\alpha a_\alpha \otimes m_\alpha$ and look at the component where a_α has the maximal length with $m_\alpha \neq 0$. Then by definition of f, the image will not be zero, since we are composing a_α with α.

One way to see the surjectivity of g and the exactness in the middle, is the following. Note that $KQ \otimes_{KQ} M$ can be written as a quotient of $KQ \otimes_K M$ by the linear span of elements of the form $a\epsilon_i \otimes m - e \otimes m\epsilon_i$ and $a\alpha \otimes m - a \otimes m\alpha$, for $i \in Q_0$ and $\alpha \in Q_1$, since elements in Q_0 and Q_1 generate KQ as a vector space. Furthermore, since $A_A = \oplus_i e_i A$, we get $KQ \otimes_K M = \oplus_{i,j \in Q_0} e_i A \otimes_k M e_j$. Clearly, the linear span of $a\alpha \otimes m - a \otimes m\alpha$ is just the image of f and the linear span of $a\epsilon_i \otimes m - e \otimes m\epsilon_i$ can be seen to be $\oplus_{i \neq j} e_i A \otimes_K M e_j$. This concludes the proof. □

Theorem 3.12. *The path algebra KQ of a finite quiver Q is hereditary, that is,* $\mathrm{Ext}^i(M, N) = 0$ *for all $M, N \in \operatorname{mod} A$ and all $i \geq 2$. In particular, the global dimension of KQ is at most 1.*

Proof. The Ext-groups are computed using projective resolutions. Since any module admits a projective resolution of length one by the previous proposition, the claim follows. □

Using the duality functor D it is in fact also possible to classify the indecomposable injective modules. Since this is rather straightforward, we just note that a module in $\operatorname{mod} A$, where A is any algebra, is injective/simple if and only if the dual module in $\operatorname{mod} A^{\mathrm{op}}$ is projective/simple. Dual to the notion of a projective cover is the *injective envelope* which is a monomorphism with the property that every submodule in the target space has nonzero intersection with the image. Any module has a unique injective envelope, since projective covers correspond to injective envelopes via D. For future reference we record the

Proposition 3.13. *Every indecomposable injective module in* $\operatorname{mod} A$ *is isomorphic to $I(j) = D(Ae_j)$ for some j. Dually to the case of projective modules, the module $I(j)$ is the injective envelope of the simple module $S(j)$ for all j.* □

We now go back to our standard quiver situation. Note that, since $\mathrm{Hom}(eA, M) \simeq Me$ for any idempotent e in an algebra A, we have $\mathrm{Hom}(Ae_a, A) = D(Ae_a) = I(a)$, because the Ae_a are the projective modules in A^{op}. Hence,

Proposition 3.14. *If $A = KQ/I$ is a bound quiver algebra, the indecomposable injective modules are precisely $I(a) = D(Ae_a)$ for $a \in Q_0$.* □

We can easily prove the

Lemma 3.15. (1) *Given $a \in Q_0$, the simple module $S(a)$ is isomorphic to the simple socle of $I(a)$.*
(2) *If $I(a) = (I(a)_b, \varphi_\beta)$, then $I(a)_b$ is the dual of the K-vector space with basis the set of all $\overline{\omega} = \omega + I$ with ω a path from b to a, and for an arrow $\beta\colon b \longrightarrow c$ the map $\varphi_\beta\colon I(a)_b \longrightarrow I(a)_c$ is given by the dual of the left multiplication by $\overline{\beta}$.*
(3) *Let $I(a)/S(a) = (L_b, \psi_\beta)$. Then L_b is the quotient space of $I(a)_b$ spanned by the residual classes of paths from b to a of length at least one, and ψ_β is the induced map.*

Proof. (1) Since $S(a) = \mathrm{top}\, e_a A$, it is the socle of $I(a)$ by duality. Alternatively, apply Lemma 3.6, (2).

(2) We have $I(a)_b = I(a)e_b = D(Ae_a)e_b = D(e_bAe_a) = D(\epsilon_b(KQ)\epsilon_a)/(\epsilon_b I \epsilon_a)$. Now apply Lemma 3.8 to see the first statement and the second follows similarly.

(3) is a consequence of (2). □

Example 3.16. Let Q be the quiver from Example 3.10. Then $I(2) = S(2), I(3) = S(3)$ and the injective representation $I(1)$ is

$$\begin{tikzcd} & K & \\ K \arrow[ur, "\mathrm{id}"] & & K. \arrow[ul, "\mathrm{id}"'] \end{tikzcd}$$

Note that $I(2)/S(2) = 0 = I(3)/S(3)$, while $I(1)/S(1) = S(2) \oplus S(3)$.

Definition. Let A be an algebra. The *Nakayama functor* of $\mathrm{mod}\, A$ is defined to be the endofunctor $\nu = D\mathrm{Hom}_A(-, A)$.

Lemma 3.17. *The Nakayama functor is right exact and isomorphic to the functor $- \otimes_A DA$.*

Proof. First note that ν is the composition of two contravariant left exact functors, hence right exact and covariant. Define a functorial morphism $\phi\colon\ - \otimes_A DA \longrightarrow \nu$ by

$$\phi_M\colon M \otimes_A DA \longrightarrow D\mathrm{Hom}_A(M, A), \quad x \otimes f \longmapsto (\psi \longmapsto f(\psi(x))).$$

If $M = A_A$, then ϕ_M is an isomorphism. Hence, it is an isomorphism for any free module. It is also an isomorphism if M is projective, because both functors are linear and a projective module is a direct summand of a free module. Now recall that any module has a projective cover. This implies that it has a *projective presentation*, that is, there exists a sequence

$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0,$$

such that $P_0 \twoheadrightarrow M$ and $P_1 \twoheadrightarrow \ker(p_0)$ are projective covers. Now apply both functors to a projective presentation of M to get

$$\begin{tikzcd} P_1 \otimes_A DA \arrow[r] \arrow[d, "\phi_{P_1}"'] & P_0 \otimes_A DA \arrow[r] \arrow[d, "\phi_{P_0}"'] & M \otimes_A DA \arrow[r] \arrow[d, "\phi_M"'] & 0 \\ \nu P_1 \arrow[r] & \nu P_0 \arrow[r] & \nu M \arrow[r] & 0. \end{tikzcd}$$

The two left vertical arrows are isomorphisms, hence the third is also one. □

Proposition 3.18. *The Nakayama functor establishes an equivalence between the full subcategory of projective modules and the full subcategory of injective modules. The quasi-inverse is given by* $\mathrm{Hom}_A(D({}_AA), -)$.

Proof. If $a \in Q_0$, then $\nu P(a) = D\mathrm{Hom}(e_aA, A) = D(Ae_a) = I(a)$. On the other hand,

$$\mathrm{Hom}_A(D({}_AA), I(a)) \simeq \mathrm{Hom}_A(D({}_AA), D(Ae_a)) \simeq \mathrm{Hom}_{A^{\mathrm{op}}}(Ae_a, A) \simeq e_aA = P(a).$$

□

Lemma 3.19. *Let $A = KQ/I$, M an A-module and $a \in Q_0$. There are functorial isomorphisms of K-vector spaces*

$$\mathrm{Hom}_A(P(a), M) \simeq Me_a \simeq D\mathrm{Hom}_A(M, I(a)).$$

Proof. Since $P(a) = e_aA$, the first isomorphism is clear. As for the second one,

$$\begin{aligned} D\mathrm{Hom}_A(M, I(a)) &\simeq D\mathrm{Hom}_A(M, D(Ae_a)) \simeq D\mathrm{Hom}_{A^{\mathrm{op}}}(Ae_a, DM) \\ &\simeq D(e_aDM) \simeq D(DM)e_a \simeq Me_a. \end{aligned}$$

□

Proposition 3.20. *Let $A = KQ/I$ and $a, b \in Q_0$. There exists an isomorphism of K-vector spaces*

$$\mathrm{Ext}^1_A(S(a), S(b)) \simeq e_a(\mathrm{rad}\, A/\mathrm{rad}^2\, A)e_b.$$

Since the number of arrows in A from a to b is equal to the dimension of the right-hand side, it is equal to $\dim_K \mathrm{Ext}^1_A(S(a), S(b))$.

Proof. Let S be a simple module. It admits a projective resolution $P_\bullet \twoheadrightarrow S$ and, in fact, a minimal one, meaning that $P_j \twoheadrightarrow \mathrm{im}(p_j)$ is a projective cover for all $j \geq 1$. By definition, to compute $\mathrm{Ext}^i(S, S')$, we have to apply the functor $\mathrm{Hom}(-, S')$ to the complex $P^\bullet$ and compute the cohomology of the resulting complex

$$0 \longrightarrow \mathrm{Hom}(P_0, S') \longrightarrow \mathrm{Hom}(P_1, S') \longrightarrow \mathrm{Hom}(P_2, S') \longrightarrow \dots$$

Let $f \in \operatorname{Hom}(P_i, S')$ be a nonzero homomorphism. Then f is surjective, since S' is simple. Since P_i is projective, we can consider its decomposition into indecomposable projective modules and conclude that there exists a direct summand P' such that f is the composition $P_i \twoheadrightarrow P' \twoheadrightarrow P'/\operatorname{rad} P' \simeq S'$. Since we assumed the resolution to be minimal, we have $P_i/\operatorname{rad}(P_i) \simeq \operatorname{im} p_i/\operatorname{rad}(\operatorname{im} p_i)$, so there exists a surjection from $\operatorname{im} p_i = P_i/\ker p_i$ to $P_i/\operatorname{rad} P_i$, hence $\operatorname{im} p_{i+1} = \ker p_i \subseteq \operatorname{rad} P_i$. Since the map $\operatorname{Hom}(p_{i+1}, S')\colon \operatorname{Hom}(P_i, S') \twoheadrightarrow \operatorname{Hom}(P_{i+1}, S')$ is given by precomposing with p_{i+1}, we have, for any $i \geq 0$ and any $x \in P_i$,

$$\operatorname{Hom}(p_{i+1}, S')(f)(x) = fp_{i+1}(x) \in f(\operatorname{im} p_{i+1}) \subseteq f(\operatorname{rad} P_i) \subseteq \operatorname{rad}(S') = 0.$$

Therefore, all the maps in the above complex are zero and correspondigly, $\operatorname{Ext}^1(S, S') \simeq \operatorname{Hom}(P_1, S')$.

Assume that $S = S(a)$. The semisimple module $\operatorname{rad} P(a)/\operatorname{rad}^2 P(a)$ is a direct sum of simple modules, say

$$\operatorname{rad} P(a)/\operatorname{rad}^2 P(a) \simeq \oplus_{c \in Q_0} S(c)^{\oplus n_c},$$

for some integers n_c. Let us recall how a minimal projective resolution of $S(a)$ is contructed. First, we take the projective cover of $S(a) = \operatorname{top} P(a)$ which is just $P(a)$ and the map $P(a) \twoheadrightarrow S(a)$ is the natural projection. Next, consider the kernel of this map, namely $\operatorname{rad} P(a) =: M_1$ and take its projective cover. The approach was to consider the semisimple $A/\operatorname{rad} A = B$-module $M_1/\operatorname{rad} M_1$, take its decomposition and then "lift" to A. Hence, in our case this gives that the next term in the resolution is precisely $\oplus_{c \in Q_0} P(c)^{\oplus n_c}$. Therefore, $\operatorname{Ext}^1(S(a), S(b)) = \operatorname{Hom}(\oplus_{c \in Q_0} P(c)^{\oplus n_c}, S(b))$. Now note that for a simple module S and an arbitrary module M we have $\operatorname{Hom}(M, S) \simeq \operatorname{Hom}(M/\operatorname{rad} M, S)$, since any non-trivial map from M to S sends $\operatorname{rad} M$ to 0. Applying this to $M = \oplus_{c \in Q_0} P(c)^{\oplus n_c}$, we get $\operatorname{Ext}^1(S(a), S(b)) = \operatorname{Hom}(\operatorname{rad} P(a)/\operatorname{rad}^2 P(a), S(b))$. Since $\operatorname{rad} P(a)/\operatorname{rad}^2 P(a)$ is semisimple, it is equal to its socle. On the other hand, $S(b)$ is the socle of $I(b)$. Since any map between modules maps the socle into the socle, we conclude that $\operatorname{Hom}(\operatorname{rad} P(a)/\operatorname{rad}^2 P(a), S(b)) \simeq \operatorname{Hom}(\operatorname{rad} P(a)/\operatorname{rad}^2 P(a), I(b))$. So,

$$\begin{aligned}\operatorname{Ext}^1(S(a), S(b)) &\simeq \operatorname{Hom}(\operatorname{rad} P(a)/\operatorname{rad}^2 P(a), I(b)) \simeq D\operatorname{Hom}_A(P(b), \operatorname{rad} P(a)/\operatorname{rad}^2 P(a)) \\ &\simeq D\operatorname{Hom}_A(e_b A, e_a(\operatorname{rad} A/\operatorname{rad}^2 A)) \simeq D(e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b) \\ &\simeq e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b,\end{aligned}$$

where the second isomorphism is Lemma 3.19, the third applies the equality $M \operatorname{rad} A = \operatorname{rad} M$ to $M = e_a A$ and the forth is Lemma 1.16. □

Remark 3.21. The proposition allows us to give an alternative definition of the ordinary quiver of a basic and connected K-algebra. Namely, the vertices are in bijective correspondence to the simple modules in $\operatorname{mod} A$ and the number arrows between two vertices is equal to the dimension of the Ext^1 between the corresponding simple modules.

4. Dimension vectors and the Euler form

Let A be a basic and connected finite-dimensional K-algebra which will we write as $A = KQ/I$ for a finite and connected quiver Q and an admissible ideal I in KQ. In this section we will assume that the vertices of Q are given by the set $\{1, \dots, n\}$. Recall that for any $j \in Q_0$, e_j is the corresponding primitive idempotent, $P(j) = e_j A$ are precisely the indecomposable projective modules, $I(j) = D(Ae_j)$ the indecomposable injective modules and $S(j) = \operatorname{top} P(j)$ the simple modules. Furthermore, recall that for any representation M we have

$$M_j = Me_j \simeq \operatorname{Hom}_A(P(j), M) \simeq D\operatorname{Hom}_A(M, I(j)).$$

Definition. Let M be a finite-dimensional KQ/I-module. The *dimension vector* of M is

$$\mathbf{dim} M = (\dim_K Me_1, \dots, \dim_K Me_n)^t \in \mathbb{Z}^n.$$

Remark 4.1. As seen before, the dimension vector of $S(i)$ is precisely the i-th basis vector in $\mathbb{Z}^n$. Note that the definition of the dimension vector does not depend on the choice of a complete set of primitive orthogonal idempotents, up to permutation of the coordinates. Furthermore, since $Me_j \simeq \operatorname{Hom}_A(P(j), M)$ by Lemma 3.19, we can express the dimension vector in terms of projectives (or injectives).

Example 4.2. Consider the quiver from Example 3.10. Then $\mathbf{dim} P(1) = (1, 0, 0)^t$, $\mathbf{dim} P(2) = (1, 1, 0)^t$ and $\mathbf{dim} P(3) = (1, 0, 1)^t$.

Lemma 4.3. *If* $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ *is an exact sequence of A-modules, then* $\mathbf{dim} M = \mathbf{dim} L + \mathbf{dim} N$.

Proof. Apply the exact functor $\operatorname{Hom}(P(j), -)$ to the sequence to get $\dim_K Me_j = \dim_K Le_j + \dim_K Ne_j$ for all $j = 1, \dots, n$ and the claim follows. □

Recall that the *Grothendieck group* of a small abelian category $\mathcal{A}$ is defined to be the free abelian group $K_0(\mathcal{A})$ generated by the isomorphism classes of objects where we factor out the subgroup generated by relations: $[F] = [F'] + [F'']$, whenever there is an exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0\ .$$

This group has the following universal property. Any map α from the set of isomorphism classes of $\mathcal{A}$ to an abelian group which is *additive*, that is, $\alpha(F) = \alpha(F') + \alpha(F'')$ for an exact sequence as above, factorizes over $K_0(\mathcal{A})$.

In particular, this applies to $\mathcal{A} = \operatorname{mod} A$. Denote the image of a module M in $K_0(\operatorname{mod} A) = K_0(\mathcal{A}) =: K_0(A)$ by $[M]$.

Proposition 4.4. *The Grothendieck group $K_0(A)$ is isomorphic to $\mathbb{Z}^n$.*

Proof. Take any module M. It admits a composition series, hence its class can be written as a sum of simple modules. This shows that this set generates $K_0(A)$. The previous lemma shows that $\mathbf{dim}$ is an additive function, hence we get a group homomorphism

$K_0(A) \longrightarrow \mathbb{Z}^n$. Since the simple modules have the basis vectors as dimension vectors, the claim follows. □

Corollary 4.5. *For any module $M \in \operatorname{mod} A$ the number $c_j(M)$ of simple composition factors that are isomorphic to $S(j)$ is precisely $\dim_K Me_j$. Furthermore, $l(M) = \dim_K M$.*

Proof. Writing out the composition series for M explicitly gives $[M] = \sum_{i=1}^n c_i(M)[S(i)]$, hence $\mathbf{dim} M = \sum_{i=1}^n c_i(M) \mathbf{dim} S(i)$. Since the $S(i)$ are basis vectors, the first claim follows. The second follows from the first, since $l(M) = \sum_i c_i(M) = \sum_i \dim_K Me_i = \dim_K M$. □

Definition. The *Cartan matrix* of A is the $(n \times n)$-matrix $C_A = (c_{ij})_{1 \le i,j \le n}$ where $c_{ij} = \dim_k e_i A e_j$.

Note that if one were to choose a different complete set of primitive orthogonal idempotents, the corresponding matrix C'_A one gets is conjugate (over $\mathbb{Z}$) to C_A.

Proposition 4.6. *The i-th column of C_A is $\mathbf{dim} P(i) = C_A \mathbf{dim} S(i)$, while the i-th row of C_A is $\mathbf{dim} I(i) = C_A^t \mathbf{dim} S(i)$.*

Proof. Use $e_i A e_j = P(i)e_j = \operatorname{Hom}(P(j), P(i)) = \operatorname{Hom}(I(j), I(i))$. □

Example 4.7. The Cartan matrix of the quiver in Example 3.10 is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proposition 4.8. *If A is an algebra of finite global dimension, then the determinant of the Cartan matrix is equal to $+1$ or -1. In particular, C_A is invertible over $\mathbb{Z}$.*

Proof. Our assumption gives a finite projective resolution $P_\bullet \longrightarrow S(i)$ for any $S(i)$. Hence, $\mathbf{dim} S(i) = \sum_{j=1}^m (-1)^j \mathbf{dim} P_j$. Now the projective modules P_j can be decomposed into the $P(k)$. Hence, the i-th basis vector $\mathbf{dim} S(i)$ can be written as a $\mathbb{Z}$-linear combination of the dimension vectors $\mathbf{dim} P(k)$. Using the fact that these vectors are the columns of C_A, the claim follows. □

Definition. Let A be a basic connected algebra of finite global dimension. The *Euler form* of A is the $\mathbb{Z}$-bilinear form $\langle -, - \rangle_A \colon \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}$ defined by $\langle x, y \rangle = x^t (C_A^{-1})^t y$.

The *Euler quadratic form* of A is the quadratic form $q_A \colon \mathbb{Z}^n \longrightarrow \mathbb{Z}$ defined by $q_A(x) = \langle x, x \rangle$.

Example 4.9. Let Q be the quiver from 3.10. Then $(C_A^{-1})^t$ is the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Hence, the Euler form of A is

$$\langle x, y\rangle_A = x_1y_1 + x_2y_2 + x_3y_3 - x_2y_1 - x_3y_1$$

and the quadratic form is

$$q_A(x) = x_1^2 + x_2^2 + x_3^2 - x_2x_1 - x_3x_1.$$

Example 4.10. Let Q be the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ with relations $\alpha\beta = 0 = \beta\alpha$ and consider the algebra $A = KQ/I$. Then $P(1) = (\ K \underset{0}{\overset{\mathrm{id}}{\rightleftarrows}} K\ ) = I(2)$ and $P(2) = (\ K \underset{\mathrm{id}}{\overset{0}{\rightleftarrows}} K\ ) = I(1)$. The Cartan matrix is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, hence the algebra cannot be of finite global dimension. Another way to see this is to check that the minimal projective resolution of $S(1)$ has the form

$$\ldots \longrightarrow P(1) \longrightarrow P(2) \longrightarrow P(1) \longrightarrow P(2) \longrightarrow P(1) \longrightarrow S(1) \longrightarrow 0.$$

Note that the module A_A is injective, hence A is a so-called *self-injective algebra.*

Proposition 4.11. *Let A be of finite global dimension and $\langle -, -\rangle_A$ be its Euler form. For any two modules $M, N \in \operatorname{mod} A$ we have*

$$\langle \mathrm{dim} M, \mathrm{dim} N\rangle_A = \sum_{j=0}^{\infty} (-1)^j \dim_K \mathrm{Ext}^j(M, N)$$

and

$$q_A(\mathrm{dim} M) = \sum_{j=0}^{\infty} (-1)^j \dim_K \mathrm{Ext}_A^j(M, M).$$

Proof. Of course, it is enough to prove the first statement and we will do it by induction on the projective dimension d of M. Without loss of generality we may assume that M is indecomposable, since both sides are additive. Also note if $A = KQ$, then j in fact only runs from 0 to 1.

Assume $d = 0$. Hence M is an indecomposable projective, so $M = e_iA$ for some i. We now compute

$$\begin{aligned} \langle \mathrm{dim} M, \mathrm{dim} N\rangle_A &= \langle \mathrm{dim} P(i), \mathrm{dim} N\rangle_A = (\mathrm{dim} P(i))^t (C_A^{-1})^t (\mathrm{dim} N) \\ &= (C_A^{-1} \mathrm{dim} P(i))^t (\mathrm{dim} N) = (\mathrm{dim} S(i))^t \mathrm{dim} N \\ &= \dim_K Ne_i = \dim_K \mathrm{Hom}_A(P(i), N), \end{aligned}$$

thus showing the statement for $d = 0$. So assume that $d \geq 1$ and the result holds for all modules with projective dimension at most $d - 1$. Consider a short exact sequence

$$0 \longrightarrow L \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P projective. Then the projective dimension of L is $d-1$ and applying $\mathrm{Hom}(-, N)$ to this sequence, we get a long exact cohomology sequence from which the claim follows by induction and using $\mathrm{dim}M = \mathrm{dim}P - \mathrm{dim}L$. $\square$

Definition. Let C_A be the Cartan matrix of an algebra A of finite global dimension. The *Coxeter matrix* of A is the matrix

$$\Phi_A = -C_A^t C_A^{-1}.$$

The *Coxeter transformation* is the group homomorphism $\Phi_A \colon \mathbb{Z}^n \longrightarrow \mathbb{Z}^n$ defined by $\Phi_A(x) = \Phi_A \cdot x$ for $x = (x_1, \dots, x_n)^t \in \mathbb{Z}^n$.

Proposition 4.12. $\Phi_A \cdot \mathrm{dim}P(i) = -\mathrm{dim}I(i)$ *for all* $i \in \{1, \dots, n\}$ *and*

$$\langle x, y\rangle_A = -\langle y, \Phi_A x\rangle_A = \langle \Phi_A x, \Phi_A y\rangle_A.$$

Proof. We know that $\mathrm{dim}S(i) = C_A^{-1}\mathrm{dim}P(i)$, hence $\mathrm{dim}I(i) = C_A^t\mathrm{dim}S(i) = -\Phi_A\mathrm{dim}P(i)$.

Furthermore,

$$\begin{aligned}\langle x, y\rangle &= x^t(C_A^{-1})^t y = ((y^t C_A^{-1})x)^t = y^t C_A^{-1} x \\ &= y^t(C_A^{-1})^t C_A^t C_A^{-1} x = y^t(C_A^{-1})^t(-\Phi_A)x = -\langle y, \Phi_A x\rangle_A\end{aligned}$$

and the last equation follows by applying what we just proved. $\square$

5. Gabriel's theorem

The purpose of this section is to prove Gabriel's theorem which classifies the quiver having finitely many indecomposable representations. Given a quiver Q and the associated algebra $A = KQ$ we will secretly change our convention from before and go to the opposite algebra. This has minor effects: for instance, the projective modules will be Ae_j now and not e_jA as before.

Definition. Let Q be a finite quiver with vertex set $\{1, \dots, n\}$. If i is a vertex, the quiver $\sigma_i Q$ is obtained from Q by reversing all arrows which start or end at i.

Example 5.1. Let Q be
$$\begin{tikzcd} & 2 & \\ 1 \arrow[ur] & & 3 \arrow[ul] \end{tikzcd}$$
. Then $\sigma_1 Q$ is
$$\begin{tikzcd} & 2 \arrow[dl] & \\ 1 & & 3 \arrow[ul] \end{tikzcd}$$
.

Definition. An ordering $i_1 \dots, i_n$ of the vertices of Q is called *admissible* if for each p the vertex i_p is a sink for the quiver $\sigma_{i_{p-1}} \dots \sigma_{i_1} Q$.

It is easy to check that if $i_1, \dots, i_n$ is an admissible ordering, then $\sigma_{i_n} \dots \sigma_{i_1} Q = Q$.

Lemma 5.2. *An admissible ordering of the vertices of Q exists if and only if Q has no oriented cycles.*

Proof. We only sketch one implication by induction on n. Suppose Q has no oriented cycles and let i_n be the starting vertex of a path of maximal length. This implies that i_n is a source. The quiver obtained by deleting this vertex has an admissible ordering by the induction hypothesis, hence so does Q. $\square$

Example 5.3. Let Q be the quiver from the previous example. Then $2, 1, 3$ or $2, 3, 1$ are admissible orderings.

Let Q be a finite quiver without loops with n vertices. The *Euler form* is the bilinear form

$$\langle -, - \rangle \colon \mathbb{Z}^n \times \mathbb{Z}^n \longrightarrow \mathbb{Z}, \quad \langle x, y \rangle = \sum_{i \in Q_0} x_i y_i - \sum_{\alpha \in Q_1} x_{s(\alpha)} y_{t(\alpha)}.$$

It is an exercise to check that this coincides with the definition from the previous section when $A = KQ$: It is enough to check the equality of the two forms on the basis vectors, that is, on the simple representations. This can be done using Proposition 4.11 and Remark 3.21.

We get a symmetric bilinear form by setting $(x, y) = \langle x, y \rangle + \langle y, x \rangle$.

The *reflection* with respect to a vertex i is the map

$$\sigma_i \colon \mathbb{Z}^n \longrightarrow \mathbb{Z}^n, \quad \sigma_i(x) = x - \frac{2(x, e_i)}{(e_i, e_i)} e_i.$$

Here e_i as usual denotes the i-th coordinate vector. It is an exercise to check that σ_i is an isometry with respect to $(-, -)$ and is of order 2.

We are now close to the definition of reflection functors which will be indespensable in proving Gabriel's theorem. First note that, given two quivers Q and Q', it of course makes sense to talk about functors between the respective categories of representations.

Definition. Let i be a sink of a quiver Q and consider the quiver $Q' = \sigma_i Q$. We define the functor S_i^+ as follows.

Given a representation (M_k, φ_α) of Q, set $S_i^+(M_k, \varphi_\alpha) = (N_k, \eta_\alpha)$ to be the representation with $N_j = M_j$ for $j \neq i$ and with N_i the kernel of the map ξ in the following sequence

$$N_i \xrightarrow{\overline{\xi}} \bigoplus_{\alpha \in Q_1, t(\alpha) = i} M_{s(\alpha)} \xrightarrow{\xi} M_i \ .$$

If α is an arrow and $t(\alpha) \neq i$, we set $\eta_\alpha = \varphi_\alpha$. If $t(\alpha) = i$, we set η_α to be the map $\overline{\xi}$ followed by the projection onto $M_{s(\alpha)}$.

If $f \colon M \longrightarrow M'$ is a morphism between representations, then $S_i^+(f) = g$ is defined as follows. If $j \neq i$, then $g_j = f_j$. If $j = i$, then define $g_i \colon N_i \longrightarrow N_i'$ to be the restriction of the map

$$(f_{s(\alpha)})_\alpha \colon \bigoplus_{\alpha \in Q_1, t(\alpha) = i} M_s(\alpha) \longrightarrow \bigoplus_{\alpha \in Q_1, t(\alpha) = i} M_s'(\alpha).$$

If i is a source of Q, we will dually construct S_i^- as follows. For $j \neq i$ we again set $N_j = M_j$. For $j = i$, N_i is the defined as the cokernel of the map ξ^t in the following sequence

$$M_i \xrightarrow{\xi^t} \bigoplus_{\alpha \in Q_1, s(\alpha) = i} M_{t(\alpha)} \xrightarrow{\widehat{\xi}} N_i \ .$$

For an arrow α, $\eta_\alpha = \varphi_\alpha$ if $s(\alpha) \neq i$, while $\eta_\alpha \colon N_{t(\alpha)} = M_{t(\alpha)} \longrightarrow N_i$ is the restriction of $\widehat{\xi}$ if $s(\alpha) = i$. If f is a morphism of representations as above, then $g_j = f_j$ for $j \neq i$ and $g_i \colon N_i \longrightarrow N_i'$ is the map induced by

$$(f_{t(\alpha)})_\alpha \colon \bigoplus_{\alpha \in Q_1, s(\alpha)=i} M_t(\alpha) \longrightarrow \bigoplus_{\alpha \in Q_1, s(\alpha)=i} M_t'(\alpha).$$

Note that if i is a sink and M is any representation, then $S_i^- S_i^+ M$ exists. In fact, there exists a natural monomorphism

$$\iota_i M \colon S_i^- S_i^+ M \longrightarrow M$$

defined as the identity on the vector spaces over vertices not equal to i and where $(\iota_i M)_i$ is the canonical map

$$(S_i^- S_i^+ M)_i = \mathrm{coker} \overline{\xi} = \mathrm{im}\, \xi \longrightarrow M_i.$$

On the other hand, if i is a source, we have a natural epimorphism

$$\pi_i M \colon M \longrightarrow S_i^+ S_i^- M$$

defined as the identity on the vector spaces over vertices not equal to i and where $(\pi_i)M_i$ is the canonical map

$$M_i \longrightarrow \mathrm{im}\, \xi^t = \ker \widehat{\xi} = (S_i^+ S_i^- M)_i.$$

The following result collects some properties. It is tacitly assumed that the expressions make sense, that is, the vertices considered are sinks/sources at the correct moment.

Lemma 5.4. *The functors $S_i^\pm$ are additive. If M is any representation, then $M = (S_i^- S_i^+ M) \oplus \mathrm{coker}(\iota_i M)$ and $M = (S_i^+ S_i^- M) \oplus \ker \pi_i M$. If $\mathrm{coker} \iota_i M = 0$, then $\mathrm{dim} S_i^+ M = \sigma_i(\mathrm{dim} M)$. If $\ker \pi_i M = 0$, then $\mathrm{dim} S_i^- M = \sigma(\mathrm{dim} M)$.*

Proof. The first statement is obvious. To see the second, note that the representations $(S_i^- S_i^+ M)$ and M are the same over all vertices $j \neq i$. Now, at the i-th vertex we have a monomorphism $(\iota_i M)_i \colon (S_i^- S_i^+ M)_i \longrightarrow M_i$, hence M_i decomposes as the direct sum of the cokernel of this map and $(S_i^- S_i^+ M)_i$. Since the representation $\mathrm{coker} \iota_i M$ is concentrated at the i-th vertex, this proves the second statement and the proof of the statement concerning π_i is analogous.

Next, if $\mathrm{coker}(\iota_i M) = 0$, then $\dim(S_i^+ M)_j = \dim M_j$ for all $j \neq i$, while

$$\dim(S_i^+ M)_i = \sum_{\alpha \in Q_1, t(\alpha)=i} \dim M_{s(\alpha)} - \dim M_i,$$

which follows immediately from the exact sequence we used to define $(S_i^+ M)_i$. Since $(\mathrm{dim} M, e_i) = \dim M_i - \sum_{\alpha \in Q_1, t(\alpha)=i} \dim M_{s(\alpha)}$, it follows easily that $\dim(S_i^+ M) = \sigma_i(\mathrm{dim} M)$ and the last statement is proved similarly. $\square$

Remark 5.5. As noted in the proof, the representations $\mathrm{coker} \iota_i M$ and $\ker \pi_i M$ are concentrated at the i-th vertex. Therefore, they are direct sums of copies of $S(i)$.

Before we formulate the next result, we recall that there is a partial order on $\mathbb{Z}^n$ defined by

$$x \leq y \iff x_k \leq y_k \ \forall\, k.$$

Lemma 5.6. *Let i be a sink and M an indecomposable representation of Q. Then the following conditions are equivalent:*

(1) $M \not\cong S(i)$.
(2) $S_i^+ M$ *is indecomposable.*
(3) $S_i^+ M \neq 0$.
(4) $S_i^- S_i^+ M \simeq M$.
(5) *The map* $\xi\colon \bigoplus_{\alpha \in Q_1, t(\alpha)=i} M_{s(\alpha)} \longrightarrow M_i$ *is an epimorphism.*
(6) $\sigma_i(\mathrm{dim} M) > 0$.
(7) $\mathrm{dim} S_i^+ M = \sigma_i(\mathrm{dim} M)$.

If i is a source, corresponding statements hold for S_i^-.

Proof. This follows rather easily from Lemma 5.4. For example, assume that $S_i^+ M$ is decomposable, so $S_i^+ M = N \oplus N'$. Then $S_i^- S_i^+ M = S_i^- N \oplus S_i^- N'$ which is a direct summand of M, giving a contradiction. Hence (1) implies (2). □

We record the above discussion in the

Theorem 5.7. *The functors S_i^+ and S_i^- induce mutually inverse bijections between the isomorphism classes of indecomposable representations of Q and those of $\sigma_i Q$ except for the simple representation $S(i)$ which is annihilated by these functors. Moreover,* $\mathrm{dim} S_i^{\pm} M = \sigma_i(\mathrm{dim} M)$ *for every indecomposable representation $M \not\cong S(i)$.* □

If Q is a finite quiver and we forget the orientations of the arrows, then the resulting object is a finite graph Γ. For a graph Γ with n vertices, we get a symmetric bilinear form on $\mathbb{Z}^n$ by setting $(e_i, e_i) = 2 - 2d_{ii}$ and $(e_i, e_j) = -d_{ij}$, where d_{ij} is the number of edges joining the vertices i and j. Note that this is the same definition as we had before (for us, $d_{ii} = 0$ since we usually do no have cycles). We of course also have a quadratic form q defined by $q(x) = \frac{1}{2}(x, x)$ and

$$q(x) = \sum_{i=1}^{n} x_i^2 - \sum_{i \leq j} d_{ij} x_i x_j.$$

On the other hand, q also determines $(-,-)$, since $(x, y) = q(x + y) - q(x) - q(y)$.

Definition. The *radical* of the form q is the set

$$\mathrm{rad} q = \{x \in \mathbb{Z}^n \mid (x, -) = 0\}.$$

Definition. Let $q\colon \mathbb{Z}^n \longrightarrow \mathbb{Z}$ be an arbitrary quadratic form. It is called *positive definite* if $q(x) > 0$ for $x \neq 0$ and *positive semi-definite* if $q(x) \geq 0$ for all x.

We will use the following terminology in the next lemma and afterwards: A vector v will be called *sincere* if $v_i \neq 0$ for all i.

Lemma 5.8. *Let Γ be a connected graph, q the associated quadratic form defined above and $y \in \mathbb{Z}^n$ a positive vector contained in the radical. Then y is sincere and q is positive semi-definite. For a vector $x \in \mathbb{Z}^n$ we have*

$$q(x) = 0 \Longleftrightarrow x \in \mathbb{Q}y \Longleftrightarrow x \in \mathrm{rad}q.$$

Proof. Since y is contained in the radical and $(-,-)$ is symmetric, we have, for any $1 \leq i \leq n$,

$$0 = (e_i, y) = (2 - 2d_{ii})y_i - \sum_{j \neq i} d_{ij} y_j. \tag{5.1}$$

If $y_i = 0$, then $\sum_{j \neq i} d_{ij} y_j = 0$ and since every term is non-negative, $y_j = 0$ whenever i and j are joined by an edge. Therefore, $y = 0$, since Γ is connected, contradicting our assumption. Hence, y is sincere.

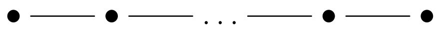
To show the next statement, we first compute, for $x \in \mathbb{Z}^n$:

$$\begin{aligned}
q(x) &= \sum_{i=1}^{n} x_i^2 - \sum_{i \leq j} d_{ij} x_i x_j \\
&= \sum_i x_i^2 - \sum_{i<j} d_{ij} x_i x_j - \sum_i d_{ii} x_i^2 \\
&= \sum_i (1 - d_{ii}) x_i^2 - \sum_{i<j} d_{ij} x_i x_j \\
&= \sum_i (2 - 2d_{ii}) y_i \frac{1}{2y_i} x_i^2 - \sum_{i<j} d_{ij} x_i x_j \\
&= \sum_{i \neq j} d_{ij} \frac{y_j}{2y_i} x_i^2 - \sum_{i<j} d_{ij} x_i x_j \\
&= \sum_{i<j} d_{ij} \frac{y_j}{2y_i} x_i^2 + \sum_{i<j} d_{ij} \frac{y_i}{2y_j} x_j^2 - \sum_{i<j} d_{ij} x_i x_j \\
&= \sum_{i<j} d_{ij} \frac{y_i y_j}{2} (\frac{x_i}{y_i} - \frac{x_j}{y_j})^2 \geq 0,
\end{aligned}$$

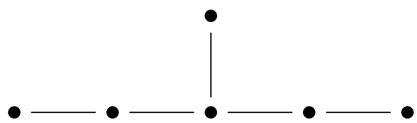
where the fifth equality uses (5.1) and the last inequality is clear. Therefore, if $q(x) = 0$, then $\frac{x_i}{y_i} = \frac{x_j}{y_j}$ whenever there is an edge joining i and j. Since Γ is connected, $x \in \mathbb{Q}y$. If $x \in \mathbb{Q}y$, then $x \in \mathrm{rad}q$, since $y \in \mathrm{rad}q$. Lastly, $x \in \mathrm{rad}q$ readily implies that $q(x) = 0$. $\square$

The following quivers will play a prominent role in Gabriel's theorem. First, the *simply laced Dynkin diagrams* with n vertices are:

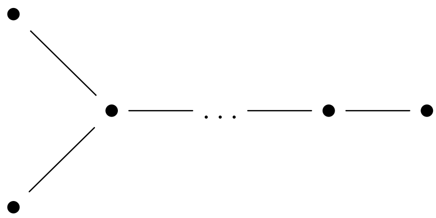
A_n



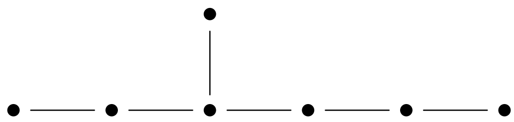
E_6



D_n

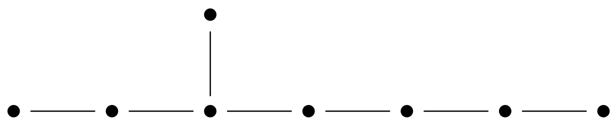


E_7



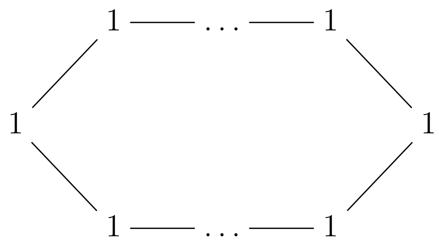
and

E_8

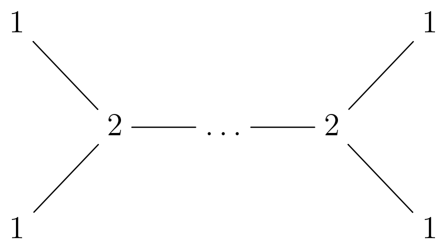


The *Euclidean diagrams* with $n = m + 1$ vertices are as follows. We mark each vertex with the value δ_i of a vector $\delta \in \mathbb{Z}^n$.

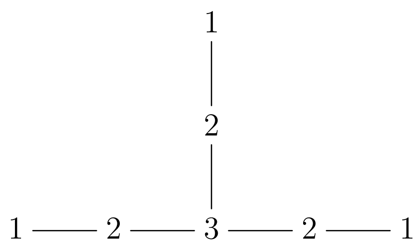
$\widetilde{A}_m$ $(m \geq 0)$



$\widetilde{D_m}$ $(m \geq 4)$



$\widetilde{E}_6$



$\widetilde{E}_7$

$$\begin{tikzcd}[arrows=dash] & & & 2 \arrow[d] & & & \\ 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] & 3 \arrow[r] & 2 \arrow[r] & 1 \end{tikzcd}$$

$\widetilde{E}_8$

$$\begin{tikzcd}[arrows=dash] & & 3 \arrow[d] & & & & & \\ 2 \arrow[r] & 4 \arrow[r] & 6 \arrow[r] & 5 \arrow[r] & 4 \arrow[r] & 3 \arrow[r] & 2 \arrow[r] & 1 \end{tikzcd}$$

Theorem 5.9. *Let Γ be a connected graph and q the corresponding quadratic form. Then Γ is a Dynkin diagram if and only if q is positive definite. It is a Euclidean diagram if and only if q is positive semi-definite and not positive definite. In this case there is a unique positive vector $\delta \in \mathbb{Z}^n$ with $\mathrm{rad} q = \mathbb{Z}\delta$.*

Proof. First we show that if Γ is Euclidean, then q is positive semi-definite and $\mathrm{rad} q = \mathbb{Z}\delta$. For this, we want to use Lemma 5.8. Hence, we have to check that δ is a radical vector, since it is positive by definition. This boils down to proving that

$$(e_i, \delta) = 2\delta_i - \sum_{j=1, d_{ij} \neq 0}^{n} \delta_j$$

is zero for $1 \leq i \leq n$. This is done by explicit computation which proves the first statement. To see the second, note that $\delta_i = 1$ for some i and therefore $\mathrm{rad} q = \mathbb{Q}\delta \cap \mathbb{Z}^n = \mathbb{Z}\delta$, since if $\lambda \in \mathbb{Q} \setminus \mathbb{Z}$, then $\lambda\delta \notin \mathbb{Z}^n$, because $\lambda\delta_i \notin \mathbb{Z}$.

Next, note that if Γ is a Dynkin diagram then q is positive definite. Indeed, there exists a Euclidean diagram $\widetilde{\Gamma}$ such that Γ is obtained by deleting some vertex e. The diagram $\widetilde{\Gamma}$ satisfies $q(x) > 0$ for every $x \neq 0$ with $x_e = 0$, proving the claim.

Finally, if Γ is neither Dynkin nor Euclidean, then there exists a vector x such that $q(x) < 0$. One proves this statement by first checking that there exists a Euclidean subgraph Γ' (this is a purely combinatorial statement) with radical vector δ. If the vertices of Γ and Γ' coincide, then $\delta = x$ will satisfy $q(x) < 0$. Otherwise we take a vertex i of $\Gamma \setminus \Gamma'$ which is connected with Γ' by an edge and set $x = 2\delta + e_i$. □

Let Γ be a Dynkin or Euclidean diagram. We say that a non-zero vector is a *root* if it is an element in

$$\Delta = \{x \in \mathbb{Z}^n \mid q(x) \leq 1\}\,.$$

The following result summarizes some properties of roots.

Proposition 5.10. *If Γ is a Dynkin or Euclidean diagram, then the following holds.*

(1) *Each basis vector e_i is a root.*
(2) *If x is a root and $y \in \mathrm{rad} q$, then $-x, x+y$ are roots.*
(3) *Every root is either positive or negative.*
(4) *If Γ is Euclidean, then $\Delta / \mathrm{rad}\, q$ is a finite set.*
(5) *If Γ is Dynkin, then Δ is a finite set.*

Proof. (1) follows by inspection. Since $q(y \pm x) = q(y) + q(x) \pm (y, x) = q(x)$, (2) is clear. To see (3), let x be a root and write it as $x = x^+ - x^-$, where x^+, x^- both have non-negative entries and have disjoint support. The condition on the support implies that $(x^+, x^-) \leq 0$, and in turn

$$1 \geq q(x) = q(x^+) + q(x^-) - (x^+, x^-) \geq q(x^+) + q(x^-) \geq 0.$$

Therefore, either $q(x^+) = 0$ or $q(x^-) = 0$, since both numbers are integers. If both vectors are non-zero, then one of them has to be sincere by Lemma 5.8, which gives a contradiction.

Let us prove (4). Fix a vertex e. If x is a root whose component x_e at e is 0, then $\delta - x$ and $\delta + x$ are positive at e. Since $\delta \in \mathrm{rad}q$, (2) implies that $\delta \pm x$ are roots. By (3), they are positive. Hence,

$$\{x \in \Delta \mid x_e = 0\} \subset \{x \in \mathbb{Z}^n \mid -\delta \leq x \leq \delta\},$$

where for vectors v, w we write $v \leq w$ if $v_i \leq w_i$ for all i. The latter is a finite set. If $x \in \Delta$, then $x - x_e\delta$ belongs to the finite set $\{x \in \Delta \mid x_e = 0\}$.

Finally, a Dynkin diagram Γ can be obtained from a Euclidean diagram $\widetilde{\Gamma}$ by deleting some vertex e. Any root x of Γ can be viewed as a root for $\widetilde{\Gamma}$ with $x_e = 0$. Therefore, (5) follows from (4). $\square$

Lemma 5.11. *Let Q be a quiver whose underlying graph is Dynkin or Euclidean. If x is a positive root and $\sigma_i(x)$ is not positive, then $x = e_i$.*

Proof. Since σ_i preserves $(-,-)$, $\sigma_i(x)$ is a root. It is not positive by assumption, hence negative by (3) of the previous proposition. For each vertex $j \neq i$, we have $\sigma_i(x)_j = x_j$ (σ_i only changes the vector in the i-th coordinate), which has to be both positive and negative. Hence $x_j = 0$ and $x = e_i$. $\square$

Definition. Let Q be a quiver without oriented cycles and assume for simplicity that $1, \ldots, n$ is an admissible ordering of the vertices. The *Coxeter transformation* is the automorphism c of $\mathbb{Z}^n$ defined by $c(x) = \sigma_n \ldots \sigma_1(x)$.

Remark 5.12. This definition coincides with our previous one, that is, the Coxeter matrix of the previous section is the matrix of the Coxeter transformation in the canonical basis of $\mathbb{Z}^n$. For the proof see [1, Prop. VII.4.7].

Lemma 5.13. *Using the previous remark, we can translate statements from Section 4. For instance, $c(\dim(i)) = -\dim I(i)$ for all i and $\langle x, y\rangle = -\langle y, c(x)\rangle = \langle c(x), c(y)\rangle$, see Proposition 4.12.* $\square$

Lemma 5.14. *Let $x \in \mathbb{Z}^n$. Then $c(x) = x$ if and only if $x \in \mathrm{rad}q$.*

Proof.

$$c(x) = x \iff x_i = c(x)_i = \sigma_i(x)_i \ \forall i \iff (x, e_i) = 0 \ \forall i.$$

$\square$

If the underlying graph of the quiver Q is Dynkin or Euclidean, then c induces a permutation of the finite set $\Delta/\mathrm{rad}q$. In particular, for some $h > 0$, c^h is the identity on $\Delta/\mathrm{rad}q$. This already implies that c^h is the identity on $\mathbb{Z}^n/\mathrm{rad}q$, since $e_i \in \Delta$ for all i.

Lemma 5.15. *Let Q be of Dynkin type and $x \in \mathbb{Z}^n$. Then there exists an integer $r \geq 0$ such that $c^r(x)$ is not positive.*

Proof. Consider $y = \sum_{r=0}^{h-1} c^r(x)$ and note that it is fixed by c. By the previous lemma, $y \in \mathrm{rad}q$, hence $y = 0$ because Q is Dynkin. Therefore, $c^r(x)$ is not positive for some $r \geq 0$. □

Lemma 5.16. *Let Q be of Euclidean type and $x \in \mathbb{Z}^n$. Then (1) $c^r(x) > 0$ for all $r \in \mathbb{Z}$ implies that $c^h(x) = x$ and (2) if $c^h(x) = x$, then $\langle \delta, x \rangle = 0$.*

Proof. Let us prove (1). Suppose that $c^h(x) = x + v$ for some $0 \neq v \in \mathrm{rad}q$. By induction, $c^{lh}(x) = x + lv$ for all $l \in \mathbb{Z}$. Since v is sincere and either positive or negative, there exists an r such that $c^r(x)$ is not positive.

To see (2), let $y = \sum_{r=0}^{h-1} c^r(x)$. As before, $y \in \mathrm{rad}q$. But then

$$0 = \langle \delta, y \rangle = \sum_{r=0}^{h-1} \langle \delta, c^r(x) \rangle = h \langle \delta, x \rangle,$$

using that δ is radical, hence $\langle \delta, x \rangle = -\langle x, \delta \rangle$, and Lemma 5.13. □

We can now state the main result of this lecture.

Theorem 5.17. *Let Q be a connected quiver. There are only finitely many isomorphism classes of indecomposable representations of Q if and only if the underlying graph is of Dynkin type. More precisely, the assignment $M \longmapsto \mathrm{dim}M$ establishes a bijection between the isomorphism classes of indecomposable representations and the positive roots corresponding to Q.*

Proof. Let Q be a Dynkin diagram and choose an admissible ordering $i_1, \ldots, i_n$ of vertices of Q. Let M be an indecomposable representation of Q with dimension vector $x = \mathrm{dim}M$. Lemma 5.15 gives the existence of an integer r such that $\tau'(x) = (\sigma_{i_n} \ldots \sigma_{i_1})^r(x)$ is not positive. Consider $\tau = \sigma_{i_s} \ldots \sigma_{i_1} \tau'$, the shortest expression such that $\tau(x)$ is still not positive. Applying our reflection functors and using Lemma 5.6 we get that

$$S^+_{i_{s-1}} \ldots S^+_{i_1} (S^+_{i_n} \ldots S^+_{i_1})^r M \simeq S(i_s).$$

This implies

$$M \simeq (S^-_{i_1} \ldots S^-_{i_n})^r S^-_{i_1} \ldots S^-_{i_{s-1}} S(i_s),$$

since the functors S^+_i and S^-_i are inverse bijections of the sets of indecomposable representations apart from $S(i)$. Therefore, $\mathrm{dim}M$ is a positive root. Similarly one shows that if M' is indecomposable with $\mathrm{dim}M = \mathrm{dim}M'$, then $M \simeq M'$.

Conversely, let x be a positive root. Let

$$\tau = \sigma_{i_s} \ldots \sigma_{i_1} (\sigma_{i_n} \ldots \sigma_{i_1})^r$$

be the shortest expression such that $\tau(x)$ is not positive. Lemma 5.11 implies that

$$\sigma_{i_{s-1}} \dots \sigma_{i_1}(\sigma_{i_n} \dots \sigma_{i_1})^r = e_{i_s}.$$

Setting $M = (S^-_{i_1} \dots S^-_{i_n})^r S^-_{i_1} \dots S^-_{i_s-1} S(i_s)$, we can apply reflection functors to conclude that M is indecomposable and

$$\dim M = (\sigma_{i_1} \dots \sigma_{i_n})^r \sigma_{i_1} \dots \sigma_{i_{s-1}}(e_{i_s}) = x.$$

Since a Dynkin diagram has only finitely many roots, we established the "if" direction.

To see the other direction, we infer from Propositions 5.18 and 5.25 below that a Euclidean diagram has infinitely many indecomposable representations. So, let Q be a diagram not of Dynkin type. We already know that it contains a Euclidean subquiver Q'. Since any representation (N, φ_α) of Q' can be extended to a representation of Q by setting $N_i = 0$ for all $i \in Q_0 \setminus Q'_0$ and $\varphi_\alpha = 0$ for all $\alpha \in Q_1 \setminus Q'_1$, we conclude that Q has infinitely many indecomposable representations. □

To complete the proof of the theorem, it remains to show that any Euclidean diagram has infinitely many isomorphism classes of indecomposable representations. We will split the proof of the statement into two parts, dealing with the case of a diagram with oriented cycles first.

Proposition 5.18. *Let Q be a quiver of Euclidean type $\widetilde{A}_n$ with $n \geq 0$. There exist infinitely many isomorphism classes of indecomposable representations.*

Proof. The orientation ot the graph is not important in the following. Fix an arrow α_0. Define, for any $p \geq 1$, a representation $(M(p), \varphi_\alpha)$ as follows. For every vertex i we set $M_i = K^p$ and for every arrow $\alpha \neq \alpha_0$ let $\varphi_\alpha = \mathrm{id}$, while for α_0 the map φ_{α_0} is given as the Jordan block of size p with eigenvalue 0. A straightforward computation shows that $\mathrm{End}(M(p)) \simeq K[t]/t^p$, which is a local ring. Therefore, by Corollary 1.22, $M(p)$ is an indecomposable representation. Clearly, $M(p) \not\simeq M(p')$ for $p \neq p'$. □

From now on we will consider quivers Q without oriented cycles.

Let $i_1, \dots, i_n$ be an admissible ordering of the vertices of our finite quiver Q. The *Coxeter functor* with respect to this ordering is the functor

$$C^+ = S^+_{i_n} \dots S^+_{i_1} \simeq \mathrm{Rep}(Q, k) \longrightarrow \mathrm{Rep}(Q, k).$$

Similarly,

$$C^- = S^-_{i_1} \dots S^-_{i_n} \simeq \mathrm{Rep}(Q, k) \longrightarrow \mathrm{Rep}(Q, k).$$

If $r \in \mathbb{Z}_{>0}$, write $C^r = (C^+)^r$. If $r = 0$, set $C^r = \mathrm{id}$, and if $r \in \mathbb{Z}_{<0}$ write $C^r = (C^-)^{-r}$.

Lemma 5.19. *The functors C^+ and C^- do not depend on the choice of the admissible ordering of the vertices of Q.*

Proof. If i and j are sinks with respect to some orientation and they are not joined by an arrow, then $S^+_i S^+_j = S^+_j S^+_i$ as follows immediately from the definitions. Let $i_1, \dots, i_n$ and $i'_1, \dots, i'_n$ be two admissible orderings of Q and let $i_1 = i'_m$. In this case $i'_1, \dots, i'_m$ cannot be joined to i_1 by an arrow (This can be seen by induction: for instance, let

$m = 2$. So i'_1 is a sink and $i_1 = i'_2$ is a sink for $i'_1 Q$, hence there cannot be an arrow from i'_1 to i'_2 because that would contradict i'_2 being a sink for $i'_1 Q$ and there is no arrow from i'_2 to i'_1 because $i_1 = i'_2$ is a sink.) Therefore,

$$S^+_{i'_m} \dots S^+_{i'_1} = S^+_{i'_{m-1}} \dots S^+_{i'_1} S^+_{i_1}.$$

Applying the same argument to $i_2, i_3, \dots$ gives the claim. $\square$

Convention For simplicity, we will from now on assume that $1, \dots, n$ is an admissible ordering of the vertices of Q.

Lemma 5.20. *Let i be a vertex. Then*

$$\dim P(i) = \sigma_1 \dots \sigma_{i-1}(e_i) \quad \textit{and} \quad \dim I(i) = \sigma_n \dots \sigma_{i+1}(e_i).$$

Furthermore,

$$P(i) \simeq S^-_1 \dots S^-_{i-1} S(i) \quad \textit{and} \quad I(i) \simeq S^+_n \dots S^+_{i+1} S(i).$$

Proof. Since the proofs for $P(i)$ and $I(i)$ are dual, we only consider the case of projectives. Fix a vertex i. Firstly, one shows by induction that for any $0 \leq l < i$ the following holds (by definition $\sigma_0 = \mathrm{id}$):

$$\sigma_{i-l} \dots \sigma_{i-1}(e_i) = \sum_{j=0}^{l} \lambda_{i,i-j} e_{i-j},$$

where $\lambda_{i,i-j}$ is the number of paths starting in i and ending in $i - j$. For $l = i - 1$, we therefore get $\sigma_1 \dots \sigma_{i-1}(e_i) = \dim P(i)$, because, by admissibility, there are no paths from i to j if $j > i$.

To see the second statement, one checks by induction that for any $0 \leq l < i$ we have

$$\dim S^+_l \dots S^+_1 P(i) = \sigma_{l+1} \dots \sigma_{i-1}(e_i).$$

This implies that $S^+_{i-1} \dots S^+_1 P(i) \simeq S(i)$, hence $P(i) \simeq S^-_1 \dots S^-_{i-1} S(i)$. $\square$

Lemma 5.21. *Let M be an indecomposable representation of Q. If $M \simeq P(i)$ for some i, then $C^+ M \simeq 0$. Otherwise, $C^- C^+ M \simeq M$. Similarly, if $M \simeq I(i)$ for some i, then $C^- M \simeq 0$ and otherwise $C^+ C^- M \simeq M$.*

Proof. By the previous lemma, $P(i) \simeq S^-_1 \dots S^-_{i-1} S(i)$, hence

$$C^+ P(i) \simeq S^+_n \dots S^+_i S(i) \simeq 0.$$

If $M \not\simeq S^-_1 \dots S^-_{i-1} S(i)$ for all i, then

$$C^- C^+ M \simeq S^-_1 \dots S^-_n S^+_n \dots S^+_1 M \simeq M,$$

using that, for all i, $S^-_i S^+_i M \simeq M$ under our assumption. The proof of the second statement is the same. $\square$

Corollary 5.22. *Any indecomposable representation M satisfies one of the following conditions: (1) $M \simeq C^r P(i)$ for some i and $r \leq 0$ and this holds iff $C^r M \simeq 0$ for some $r > 0$, (2) $M \simeq C^r I(i)$ for some i and $r \geq 0$ and this holds iff $C^r M \simeq 0$ for some $r < 0$, or (3) $C^r M \not\simeq 0$ for all $r \in \mathbb{Z}$. An M satisfying (1) is called* preprojective*, if it satisfies (2) it is called* preinjective *and* regular *if (3) holds.* □

Lemma 5.23. *If $C^r P(i) \simeq C^s P(j) \neq 0$, then $i = j$ and $r = s$, and similarly for $I(k)$.*

Proof. If $C^r P(i) \simeq C^s P(j) \neq 0$, then $P(i) \simeq C^{s-r} P(j)$, so $s - r \leq 0$ by Lemma 5.21. The same argument gives that $r - s \leq 0$, hence $r = s$. Thus $P(i) \simeq P(j)$, so $i = j$. □

Definition. Let Q be a quiver of Euclidean type. The *defect* of a vector $x \in \mathbb{Z}^n$ is the number $\partial x = \langle \delta, x \rangle = -\langle x, \delta \rangle$. The defect ∂M of a representation M is defined to be $\partial \mathrm{dim} M$.

Lemma 5.24. *Let M be an indecomposable representation. Then M is preprojective iff $\partial M < 0$, preinjective iff $\partial M > 0$ and regular if and only if $\partial M = 0$.*

Proof. We know that $\mathrm{dim} C^r N = c^r \mathrm{dim} N$ for any representation N with $C^r N \neq 0$. If $M \simeq C^r P(i)$, then

$$\partial M = -\langle c^r(\mathrm{dim} P(i)), \delta \rangle = -\langle \mathrm{dim} P(i), \delta \rangle = -\delta_i < 0.$$

The preinjective case is proved analogously. For the regular case use Lemma 5.16. □

Proposition 5.25. *Let Q be a Euclidean diagram without oriented cycles and n vertices. Then $C^{-r} P(i)$ and $C^r I(i)$, $r \in \mathbb{N}$, $i \in Q_0$, give $2n$ infinite families of pairwise non-isomorphic representations.*

Proof. We only need to show that $C^{-r} P(i) \neq 0$ and $C^r I(i) \neq 0$ for all $r \geq 0$. If $C^{-r} P(i) = 0$, then $P(i)$ is preinjective. Since preprojectives and preinjectives have different defects by the previous lemma, the claim follows. □

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