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Tensor Triangulated Categories in Algebraic Geometry

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Eidesstattliche Erklärung

Gemäß §23 (6) der Diplomprüfungsordnung versichere ich hiermit, diese Arbeit selbstständig verfasst zu haben. Ich habe alle bei der Erstellung dieser Arbeit benutzten Hilfsmittel und Hilfen angegeben.

Abstract

We use a tensor structure on a triangulated category to turn the Grothendieck group into a ring and characterize all dense "‘subrings" of the triangulated category by subrings of the Grothendieck ring. Further, we apply the spectrum construction to study the connection between the number of Fourier-Mukai partners of a given scheme and the number of tensor structures on its derived category. In the last chapter we compute the spectra of some tensor triangulated categories which arise in algebraic geometry.

Contents

Chapter 1

Introduction

The focus of this study are tensor triangulated categories in algebraic geometry. The starting point was Balmer's paper [3] which defined a ringed space for any given tensor triangulated category. The most important example of such structures in algebraic geometry is the derived category of (quasi-)coherent sheaves on a scheme X. One of the fundamental results in [3] was that the spectrum of the bounded derived category of perfect complexes $\mathrm{D}^{\mathrm{perf}}(X)$ on a noetherian scheme X is actually isomorphic to X. Thus the scheme structure is completely encoded in the homological data described by the derived category. More generally this approach provides a possibility to extract geometry from homological data.
The study is divided into four chapters. The first one covers the basics about tensor triangulated categories, in particular about derived categories. Most facts presented are standard however some proofs are included since there are no suitable references. The main facts of the spectrum construction from [3] are given along with some new definitions and propositions which underline the similarity of tensor triangulated categories to (semi-)rings.
Chapter 2 deals with the Grothendieck group which is defined for any triangulated category. Since the studied categories have an additional structure this allows us to make the Grothendieck group into a commutative ring. We discuss the connections between the various construction of the Grothendieck group respectively ring for a scheme X and give an explicit description of the Grothendieck ring of the projective line.
In Chapter 3 we apply the spectrum construction to study the connection between the number m of Fourier-Mukai-partners of a given scheme X and the number n of tensor structures on the derived category of X. This question is interesting since a tensor structure which does not correspond to a Fourier-Mukai-partner of X gives a new ringed space associated to X via the spectrum construction and one may ask when this ringed space is a scheme or a variety. We prove that $m \leq n$ and give explicit examples where $m < n$.
In the fourth and final chapter we turn to examples. First we present examples of prime ideals in homotopy categories before turning to geometric examples: We compute spectra of "subrings" of derived categories of certain schemes. Since these "subrings" are not of the form $\mathrm{D}^{\mathrm{perf}}(X)$ for a scheme X these examples are not covered by the results in [3] and are presumably the first computations of this sort in the geometrical context.

Chapter 2

Preliminaries

2.1 Triangulated categories

Definition 2.1.1. Let $\mathcal{T}$ be an additive category with an additive automorphism $T : \mathcal{T} \to \mathcal{T}$. $T^n(A)$ will sometimes be abbreviated as $A[n]$ and $T^n(f)$ as $f[n]$. A sequence of objects and morphisms in $\mathcal{T}$:

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$$

will be called a triangle and sometimes written as (A, B, C, u, v, w). C is called *a cone* of u. A morphism of triangles is a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & TA \arrow[d, "Tf"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA' \end{tikzcd}$$

Two triangles are called isomorphic if all the vertical arrows are isomorphisms. $\mathcal{T}$ with the so called *translation* or *shift* functor T is called a *triangulated category* if there is a class of so called *exact* (or *distinguished*) *triangles* fulfilling the following axioms:

TR1:
- Every morphism $A \xrightarrow{u} B$ can be completed to an exact triangle.
- The sequence $A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow TA$ is an exact triangle.
- Any triangle isomorphic to an exact one is itself exact.

TR2: A triangle $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$ is exact if and only if the *rotated* triangle $B \xrightarrow{v} C \xrightarrow{w} TA \xrightarrow{-Tu} TB$ is exact.

TR3: Given two exact triangles and two morphisms $f : A \to A'$ and $g : B \to B'$ such that $gu = u'f$ there exists a morphism $h : C \to C'$ such that the following diagram is a morphism of triangles:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, dashed, "h"] & TA \arrow[d, "Tf"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA' \end{tikzcd}$$

TR4: For all commutative diagrams

$$\begin{tikzcd} & X_2 \arrow[dr, "u_1"] & \\ X_1 \arrow[ur, "u_3"] \arrow[rr, "u_2"] & & X_3 \end{tikzcd}$$

and distinguished triangles

$$X_1 \xrightarrow{u_3} X_2 \xrightarrow{v_3} Z_3 \xrightarrow{w_3} X_1[1]$$

$$X_2 \xrightarrow{u_1} X_3 \xrightarrow{v_1} Z_1 \xrightarrow{w_1} X_2[1]$$

$$X_1 \xrightarrow{u_2} X_3 \xrightarrow{v_2} Z_2 \xrightarrow{w_2} X_1[1]$$

there exist two morphisms:

$$m_1 : Z_1 \longrightarrow Z_2$$

$$m_3 : Z_2 \longrightarrow Z_1$$

such that $(\mathrm{id}_{X_1}, u_1, m_1)$ and $(u_3, \mathrm{id}_{X_3}, m_3)$ are morphisms of triangles and

$$Z_3 \xrightarrow{m_1} Z_2 \xrightarrow{m_3} Z_1 \xrightarrow{v_3[1]w_1} Z_3[1]$$

is a distinguished triangle.

Remark 2.1.2. Let (A, B, C, u, v, w) be an exact triangle in a triangulated category $\mathcal{T}$ and let E be an object in $\mathcal{T}$. Then the induced sequences of abelian groups are exact:

$$\mathrm{Hom}(E, A) \xrightarrow{u_*} \mathrm{Hom}(E, B) \xrightarrow{v_*} \mathrm{Hom}(E, C)$$

$$\mathrm{Hom}(C, E) \xrightarrow{u^*} \mathrm{Hom}(B, E) \xrightarrow{v^*} \mathrm{Hom}(A, E)$$

TR2 gives that the sequences $\mathrm{Hom}(E, B) \xrightarrow{v_*} \mathrm{Hom}(E, C) \xrightarrow{w_*} \mathrm{Hom}(E, A[1])$ and $\mathrm{Hom}(A[-1], E) \xrightarrow{v^*} \mathrm{Hom}(C, E) \xrightarrow{w^*} \mathrm{Hom}(B, E)$ are exact as well and therefore one obtains long exact sequences.

Remark 2.1.3. If $f : A \longrightarrow B$ is a morphism in a triangulated category $\mathcal{T}$ then TR1 gives the existence of a cone of f. It follows from 2.1.2 that if two of the vertical arrows in a morphism of triangles are isomorphisms then so is the third and therefore a cone is unique up to a non-canonical isomorphism.

Definition 2.1.4. Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories. An additive functor $F : \mathcal{T} \to \mathcal{T}'$ is called a *triangulated* or *exact functor* if it commutes with the translation functors, i.e. the functors $F \circ T$ and $T' \circ F$ are isomorphic, and if F transforms exact triangles into exact triangles.

Definition 2.1.5. Let $\mathcal{T}$ be a triangulated category and $\mathcal{C}$ be a full additive subcategory. $\mathcal{C}$ is called a *triangulated subcategory* if we have the following: Whenever two of the three objects of an exact triangle belong to $\mathcal{C}$ then so does the third.

Remark 2.1.6. It is possible to replace 2.1.5 with the following equivalent condition which is quite useful when dealing with triangulated subcategories of the derived category of a scheme: $\mathcal{C}$ is closed under shifts, isomorphisms and taking cones. We will give the proof since there does not seem to be a suitable reference.

Proof. ($\Rightarrow$) Let A be an object of $\mathcal{C}$ such that $f : A \xrightarrow{\sim} B$. Then we have the following commutative diagram of triangles:

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & A \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & TA \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "f"] & B \arrow[r] & 0 \arrow[r] & TA \end{tikzcd}$$

Since these triangles are isomorphic and the upper one is exact the lower one is exact as well and therefore $B \in \mathcal{C}$. Now let $A \xrightarrow{f} B$ be a morphism between objects in $\mathcal{C}$. It follows easily from TR1 that $\mathcal{C}$ is closed under taking cones. Finally TR2 gives that

$$A \longrightarrow 0 \longrightarrow TA \xrightarrow{-\mathrm{id}} TA$$

is an exact triangle and therefore TA is an object in $\mathcal{C}$.
($\Leftarrow$) Let

$$A \xrightarrow{f} B \longrightarrow \mathrm{cone}(f) \longrightarrow TA$$

be a triangle. Then we have to consider three cases:

- $A, B \in \mathcal{C}$. Then $\mathrm{cone}(f)$ is determined up to a non-canonical isomorphism and therefore is an object in $\mathcal{C}$.
- $B, \mathrm{cone}(f) \in \mathcal{C}$. Then TR2 gives that

  $$B \longrightarrow \mathrm{cone}(f) \longrightarrow TA \longrightarrow TB$$

  is an exact triangle so $TA \in \mathcal{C}$ and therefore $A \in \mathcal{C}$.
- $A, \mathrm{cone}(f) \in \mathcal{C}$. Then by TR2 again

  $$T^{-1}\mathrm{cone}(f) \longrightarrow A \longrightarrow B \longrightarrow \mathrm{cone}(f)$$

  is a triangle and therefore $B \in \mathcal{C}$.

□

Definition 2.1.7. Let $\mathcal{T}$ be an additive category. An additive subcategory $\mathcal{C}$ is called *thick* if it is closed under direct summands, i.e. $A \oplus B \in \mathcal{C}$ implies that $A,\ B \in \mathcal{C}$.
$\mathcal{C}$ is called *dense* or *cofinal* if for every object A in $\mathcal{T}$ there exists an object A' in $\mathcal{T}$ such that $A \oplus A' \in \mathcal{C}$.

Remark 2.1.8. Let $\mathcal{C}$ be a dense and thick subcategory. Let A be an object in $\mathcal{T}$. Then there exists an object A' in $\mathcal{T}$ such that $A \oplus A' \in \mathcal{C}$ since $\mathcal{C}$ is dense. But then $A \in \mathcal{C}$, since $\mathcal{C}$ is thick, so $\mathcal{C} = \mathcal{T}$.

Definition 2.1.9. A *tensor triangulated category* is a triple $(\mathcal{T}, \otimes, \mathbf{1})$ consisting of a triangulated category $\mathcal{T}$, a distinguished object $\mathbf{1}$ and a covariant bifunctor $\otimes$ which is exact in every variable, i.e. the functors $- \otimes A$ and $A \otimes -$ are exact for all objects A, and there are four natural isomorphisms:

$$A \otimes \mathbf{1} \cong A \qquad \mathbf{1} \otimes A \cong A \qquad A \otimes (B \otimes C) \cong (A \otimes B) \otimes C \qquad A \otimes B \cong B \otimes A.$$

A full additive subcategory $\mathcal{C}$ of a tensor triangulated category $\mathcal{T}$ is a *tensor triangulated subcategory* if it is a triangulated subcategory and if the tensor product in $\mathcal{T}$ restricts to a tensor product in $\mathcal{C}$.
A *tensor triangulated functor* F between tensor triangulated categories $\mathcal{T}$ and $\mathcal{T}'$ is a triangulated functor sending the unit to the unit, $F(\mathbf{1}_{\mathcal{T}}) = \mathbf{1}_{\mathcal{T}'}$, and respecting the given tensor structures, i.e. $F(A \otimes_{\mathcal{T}} B) = F(A) \otimes_{\mathcal{T}'} F(B)$ for all objects A and B in $\mathcal{T}$.

Definition 2.1.10. Let $\mathcal{T}$ and $\mathcal{T}'$ be tensor triangulated categories. Then $\mathcal{T} \oplus \mathcal{T}'$ is again a tensor triangulated category with triangulation and tensor structure defined componentwise.

Convention: Whenever it will be necessary to avoid set-theoretical difficulties we will assume our categories to be essentially small, i.e. having an isomorphy set of objects.

2.2 Complexes and homotopies

Recall that an *abelian category* $\mathcal{A}$ is an additive category in which:

1. Every morphism $f : B \to C$ has a kernel and cokernel.

2. Every monic arrow is a kernel and every epi arrow is a cokernel (Recall that $f : B \to C$ is called *monic* if $fg_1 \neq fg_2$ for every $g_1 \neq g_2 : A \to B$ and $f : B \to C$ is called *epi* if $h_1 f \neq h_2 f$ for every $h_1 \neq h_2 : C \to D$).

Example 2.2.1. Let R be a commutative ring with identity. Then the category $\mathcal{A} = R-\mathrm{Mod}$ of R-modules is an abelian category. The (sub)category $R-\mathrm{mod}$ of finitely generated R-modules is always addtive and it is abelian if and only if R is a noetherian ring.

Example 2.2.2. Let X be a scheme. The category $\mathbf{Sh}(X)$ of sheaves of $\mathcal{O}_X$-modules on X is an abelian category. The same is true for $\mathbf{Qcoh}(X)$, the category of quasi-coherent sheaves on X. If $X = \mathrm{Spec}(R)$ is affine then $\mathbf{Qcoh}(X)$ is equivalent to $R-\mathrm{Mod}$ (see [11], chapter II.5, corollary 5.5).

Example 2.2.3. Let X be a noetherian scheme. Then $\mathbf{Coh}(X)$, the category of coherent sheaves of $\mathcal{O}_X$-modules on X is abelian. Similarly to the previous example this category is equivalent to $R-\mathrm{mod}$ if $X = \mathrm{Spec}(R)$.

Let $\mathcal{A}$ be an abelian category. Recall that a *complex* $A^\bullet$ is a sequence of objects and morphisms in $\mathcal{A}$ of the form

$$\dots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \dots$$

such that $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$. A morphism $f : A^\bullet \to B^\bullet$ between complexes is a commutative diagram

$$\begin{tikzcd}
\dots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r, "d_A^{i+1}"] \arrow[d, "f^{i+1}"] & \dots \\
\dots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r, "d_B^{i+1}"] & \dots
\end{tikzcd}$$

These morphisms are also called *chain maps*. So we have a *category of complexes* $\mathrm{Kom}(\mathcal{A})$ of an abelian category $\mathcal{A}$ where the objects are complexes and the morphisms are chain maps. $\mathrm{Kom}(\mathcal{A})$ is an abelian category. If $\mathcal{A}$ is additive then $\mathrm{Kom}(\mathcal{A})$ is only additive as well.

Remark 2.2.4. Mapping an object $A \in \mathcal{A}$ to the complex $A^\bullet$ with $A^0 = A$ and $A^i = 0\ \forall\, i \neq 0$ identifies $\mathcal{A}$ with a full subcategory of $\mathrm{Kom}(\mathcal{A})$. Such a complex is sometimes called a *0-complex.*

Definition 2.2.5. Let $A^\bullet$ be a complex.

1. The *cohomology* $H^i(A^\bullet)$ of $A^\bullet$ is defined as

   $$H^i(A^\bullet) = \ker(d^i)/\mathrm{im}(d^{i-1})$$

   A complex morphism $f : A^\bullet \to B^\bullet$ induces a natural homomorphism $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)\ \forall\, i \in \mathbb{Z}$.

2. f is called a *quasi-isomorphism* if the induced map $H^i(f)$ is an isomorphism for all $i \in \mathbb{Z}$.

3. $A^\bullet$ is called *acyclic* or *exact* if $H^i(A^\bullet) = 0\ \forall\, i \in \mathbb{Z}$.

4. $A^\bullet$ is called *split* if there exists a collection of morphisms $s^i : C^i \to C^{i-1}$, $i \in \mathbb{Z}$, such that $d^i s^{i-1} d^i = d^i\ \forall\, i \in \mathbb{Z}$. A split complex which is acyclic is called *split exact.*

Remark 2.2.6. Note that two complexes can have isomorphic cohomology but no chain map between them which is a quasi-isomorphism. Take for example $A^\bullet = [\dots \to 0 \to \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/5\mathbb{Z} \to 0 \to \dots]$ where π denotes the canonical projection and $\mathbb{Z}$ sits in degree 0. Furthermore take $B^\bullet$ to be the 0-complex $\mathbb{Z}$. These complexes have isomorphic cohomology, $\mathbb{Z} \cong 5\mathbb{Z}$ in degree 0 and 0 everywhere else, but there is no map from $\mathbb{Z}$ to $\mathbb{Z}$ which induces the above isomorphism on cohomology.

Definition 2.2.7. Two chain maps $f, g : A^\bullet \to B^\bullet$ are called *homotopic*, $f \sim g$, if there are morphisms $h^i : A^i \to B^{i-1}$, $i \in \mathbb{Z}$, such that $f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$.
The *homotopy category of complexes* $\mathrm{K}(\mathcal{A})$ is defined to be the category with $\mathrm{Ob}(\mathrm{K}(\mathcal{A}))=\mathrm{Ob}(\mathrm{Kom}(\mathcal{A}))$ and $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A^\bullet, B^\bullet)/\sim$.

See ([9], III.1.2) for details.
Furthermore we define the categories $\mathrm{K}^{+}(\mathcal{A})$ and $\mathrm{K}^{-}(\mathcal{A})$ to be the full subcategories of $\mathrm{K}(\mathcal{A})$ consisting of those complexes $A^\bullet$ such that $A^i = 0$ for $i \ll 0$ and $i \gg 0$ respectively. $\mathrm{K}^b(\mathcal{A})$ is defined to be the sull subcategory consisting of bounded complexes.

Remark 2.2.8. Since there are no homotopies between maps of 0-complexes $\mathcal{A}$ can be considered as a full subcategory in $\mathrm{K}(\mathcal{A})$.

Definition 2.2.9. Let $A^\bullet$ be a complex and $f : A^\bullet \to B^\bullet$ be a chain map in $\mathrm{Kom}(\mathcal{A})$.

1. Let $n \in \mathbb{Z}$. Then $A^\bullet[n]$ is the complex with $(A^\bullet[n])^i := A^{i+n}$ and $d^i_A[n] := (-1)^n d^{i+n}_A$. The *shift* of f is the chain map $f[n] : A^\bullet[n] \to B^\bullet[n]$ given by $f[n]^i := f^{i+n}$. This defines a shift functor $T : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$, $A^\bullet \mapsto A^\bullet[1]$, which is clearly an equivalence. The same is true for $\mathrm{K}(\mathcal{A})$.

2. The *mapping cone* of f is the complex $C(f)$ defined by

$$C(f)^i := A[1]^i \oplus B^i = A^{i+1} \oplus B^i \qquad \text{and} \qquad d^i_{C(f)} := \begin{pmatrix} -d^{i+1}_A & 0 \\ f^{i+1} & d^i_B \end{pmatrix}$$

There are two natural complex morphisms $i : B^\bullet \to C(f)$ and $\pi : C(f) \to A^\bullet[1]$. Having established what a mapping cone is we can now introduce triangles in $\mathrm{K}(\mathcal{A})$ as being isomorphic to a sequence of the form:

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{i} C(f) \xrightarrow{\pi} A^\bullet[1]$$

We then get the

Theorem 2.2.10. $\mathrm{K}(\mathcal{A})$ *is a triangulated category* (for proof see e.g. [9], IV.1.9-14 or [24], ch.10). *The full subcategories* $\mathrm{K}^*(\mathcal{A})$*, where* $* = +, -, b$ *are triangulated.*

Remark 2.2.11. If $F : \mathcal{T} \to \mathcal{T}'$ is an exact functor between triangulated categories and $\mathcal{C}$ is a triangulated subcategory of $\mathcal{T}$ it is in general not true that the image of $\mathcal{C}$ under F is a triangulated subcategory in $\mathcal{T}'$. As an example consider $\mathcal{T}' =\mathrm{K}(\mathcal{A})$ where $\mathcal{A} = \mathbb{K} - \mathrm{Mod}$ is the category of $\mathbb{K}$-vector spaces. As $\mathcal{T} = \mathcal{C}$ we consider the category of complexes where the complexes only have $\mathbb{K}^n$ ($n \in \mathbb{N}$) as objects. Then the inclusion functor is exact, but $\mathcal{T}$ is not a triangulated subcategory of $\mathcal{T}'$ since it is not closed under isomorphisms. However the *essential image* of F, i.e. the subcategory of $\mathcal{T}'$ consisting of all objects isomorphic to some object of the image of F, is always a triangulated subcategory as can be easily shown.

Definition 2.2.12. Let $\mathcal{T}$ be a triangulated, $\mathcal{A}$ an abelian category and $H : \mathcal{T} \to \mathcal{A}$ an additive functor. H is called *(co)homological* if for every exact triangle $A \longrightarrow B \longrightarrow C \longrightarrow TA$ the induced sequence of objects in $\mathcal{A}$ $H(A) \longrightarrow H(B) \longrightarrow H(C)$ is exact.

Example 2.2.13. The functors Hom(- ,E) and Hom(E, -) are cohomological for all objects E as was remarked in 2.1.2.
If $\mathcal{T} = \mathrm{K}(\mathcal{A})$ then the homology functor H^i is a cohomological functor for all $i \in \mathbb{Z}$. Similarly to 2.1.2 one gets a long exact sequence.

Remark 2.2.14. Let $f : A^\bullet \to B^\bullet$ be a chain map. Then f is a quasi-isomorphism if and only if the mapping cone of f is an acyclic complex. This follows immediately from the long exact sequence in 2.2.13.

Definition 2.2.15. Let $\mathcal{A} = R - \mathrm{Mod}$ and consider the homotopy category $\mathrm{K}(\mathcal{A})$. The tensor product $\otimes_R$ in $\mathcal{A}$ induces a *tensor product of complexes* defined by:

$$(A^\bullet \otimes B^\bullet)^n := \bigoplus_{i+j=n} A^i \otimes_R B^j \qquad \text{and}$$

$$d^n_{A\otimes B}(a^i \otimes b^j) := d^i_A(a^i) \otimes b^j + (-1)^i a^i \otimes d^j_B(b^j).$$

The following proposition is well-known but since we could not find its proof in the literature we include (a part of) it here.

Proposition 2.2.16. *The above defined tensor product makes* $\mathrm{K}(\mathcal{A})$ *a tensor triangulated category.*

Proof. The unit is the 0-complex R and the defined tensor product is obviously symmetric. It remains to prove that the tensor product is associative and exact (in the sense of 2.1.4). First the associativity: Let $A^\bullet, B^\bullet$ and $C^\bullet$ be complexes. Then:

$$[(A^\bullet \otimes B^\bullet) \otimes C^\bullet]^n = \bigoplus_{i+j=n} (A^\bullet \otimes B^\bullet)^i \otimes C^j = \bigoplus_{i+j=n} \left(\bigoplus_{k+l=i} A^k \otimes B^l \right) \otimes C^j =$$

$$= \bigoplus_{k+l+j=n} A^k \otimes B^l \otimes C^j = \bigoplus_{k+m=n} A^k \otimes \left(\bigoplus_{l+j=m} B^l \otimes C^j \right) = \bigoplus_{k+m=n} A^k \otimes (B^\bullet \otimes C^\bullet)^m$$

$$= (A^\bullet \otimes (B^\bullet \otimes C^\bullet))^n$$

So the objects in the complexes are equal. Let us have a look at the differentials:

$$d^n_{(A^\bullet \otimes B^\bullet)\otimes C^\bullet}((a^k \otimes b^l) \otimes c^j) = d^{k+l}_{A^\bullet \otimes B^\bullet}(a^k \otimes b^l) \otimes c^j + (-1)^{k+l}(a^k \otimes b^l) \otimes d^j_{C^\bullet}(c^j) =$$

$$d^k_{A^\bullet}(a^k) \otimes b^l \otimes c^j + (-1)^k a^k \otimes d^l_{B^\bullet}(b^l) \otimes c^j + (-1)^{k+l} a^k \otimes b^l \otimes d^j_{C^\bullet}(c^j) = (*)$$

On the other hand:

$$(*) = d^k_{A^\bullet}(a^k) \otimes b^l \otimes c^j + (-1)^k a^k \otimes d^l_{B^\bullet}(b^l) \otimes c^j + (-1)^k (-1)^l a^k \otimes b^j \otimes d^j_{C^\bullet}(c^j) =$$

$$d^k_{A^\bullet}(a^k) \otimes (b^l \otimes c^j) + (-1)^k a^k \otimes d^{l+j}_{B^\bullet \otimes C^\bullet}(b^l \otimes c^j) = d^n_{A^\bullet \otimes (B^\bullet \otimes C^\bullet)}(a^k \otimes (b^l \otimes c^j))$$

The exactness is proved with similar (long but easy) calculations. □

Remark 2.2.17. 2.2.15 is not valid for arbitrary abelian categories. For example consider a noetherian ring R so that the category $\mathcal{A} = R - \mathrm{mod}$ is abelian. The tensor product of two complexes in $\mathrm{K}(R - \mathrm{mod})$ need not be a complex of finitely generated R-modules. Take e.g. $R = \mathbb{Z}$ and $A^\bullet := [\ldots \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \ldots] \in \mathrm{Ob}(\mathrm{K}(\mathcal{A}))$. The tensor complex $A^\bullet \otimes A^\bullet$ is not an object in $\mathrm{K}(\mathcal{A})$. However the above construction is possible if e.g. $\mathcal{A} = \mathbf{Sh}(X)$ is the category of sheaves of $\mathcal{O}_X$-modules on a scheme. $\mathcal{A} = \mathbf{Coh}(X)$ is possible if we take the bounded homotopy category.

2.3 Derived categories

Definition 2.3.1. Let $\mathcal{A}$ be an abelian category and consider the category of complexes $\mathrm{Kom}(\mathcal{A})$. The *derived category* $\mathrm{D}(\mathcal{A})$ is a category together with a functor $Q : \mathrm{Kom}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ such that:

1. $Q(f)$ is an isomorphism for every quasi-isomorphism f.

2. For every category $\mathcal{B}$ with a functor $F : \mathrm{Kom}(\mathcal{A}) \to \mathcal{B}$ which transforms quasi-isomorphisms into isomorphisms there exists a unique functor $G : \mathrm{D}(\mathcal{A}) \to \mathcal{B}$ such that $F = G \circ Q$.

In order to prove the existence of the derived category and to be able to describe its structure one first passes over to the homotopy category $\mathrm{K}(\mathcal{A})$. Then one *localizes* $\mathrm{K}(\mathcal{A})$ with respect to the quasi-isomorphisms. This process is described more generally in the following

Definition 2.3.2. Let $\mathcal{T}$ be a triangulated category and $\mathcal{C}$ a thick triangulated subcategory. The *Verdier localization* of $\mathcal{T}$ at $\mathcal{C}$, denoted by $\mathcal{T}/\mathcal{C}$, is the category obtained by formally inverting those morphisms whose cone is an object in $\mathcal{C}$. It is a triangulated category with triangles defined as being isomorphic to the image of a triangle in $\mathcal{T}$ via the natural functor $\mathcal{T} \to \mathcal{T}/\mathcal{C}$. It has a universal property, namely: Every functor to a triangulated category which makes morphisms whose cone is in $\mathcal{C}$ to isomorphisms factors through $\mathcal{T}/\mathcal{C}$.

It turns out that the derived category $\mathrm{D}(\mathcal{A})$ can be obtained by localizing $\mathrm{K}(\mathcal{A})$ with respect to quasi-isomorphisms. The category $\mathcal{C}$ is in this case the category of all acyclic complexes, cf. 2.2.14 and note that any acyclic complex $A^\bullet$ is a cone of the quasi-isomorphism $0 \to A^\bullet$. The localization then gives that:

$$\mathrm{Ob}(\mathrm{D}(\mathcal{A})) = \mathrm{Ob}(\mathrm{K}(\mathcal{A})) = \mathrm{Ob}(\mathrm{Kom}(\mathcal{A}))$$

and that a morphism between objects $A^\bullet$ and $B^\bullet$ in $\mathrm{D}(\mathcal{A})$ can be represented by a *roof* of the form:

$$\begin{tikzcd} & C^\bullet \arrow[dl, "s"'] \arrow[dr, "f"] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

with s being a morphism whose cone is in $\mathcal{C}$ (in our case s is a quasi-isomorphism) and f a chain map. For details see [9], III.2 or [19], chapter 2. Note the similarity to the construction of the localization of a ring.

Remark 2.3.3. One can also consider $\mathrm{D}^*(\mathcal{A})$ with $^* = +, -, b$. The natural functors $\mathrm{D}^*(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ define equivalences of $\mathrm{D}^*(\mathcal{A})$ with the full triangulated subcategories of $\mathrm{D}(\mathcal{A})$ formed by complexes $A^\bullet$ with $\mathrm{H}^i(A^\bullet) = 0$ for $i \ll 0, i \gg 0$ and $|i| \gg 0$. There is also an equivalence between $\mathcal{A}$ and the subcategory in $\mathrm{D}(\mathcal{A})$ consisting of complexes $A^\bullet$ with $\mathrm{H}^i(A^\bullet) = 0$ for $i \neq 0$. Such a complex is often called a *H_0-complex.*

Example 2.3.4. Let $\mathcal{A}$ be a *semi-simple* abelian category, i.e. a category where every exact triple is isomorphic to a triple of the form

$$0 \longrightarrow X \xrightarrow{i} X \oplus Y \xrightarrow{p} Y \longrightarrow 0$$

where i denotes the inclusion and p the projection. Then $\mathrm{D}(\mathcal{A})$ is equivalent to the category $\mathrm{Kom}_0(\mathcal{A})$ whose objects are *cyclic* complexes (all differentials are zero). An example for a semi-simple category is $\mathcal{A} = \mathbb{K} - \mathrm{Mod}$. A counter-example is the category of abelian groups.

Example 2.3.5. Denote by $\mathcal{I}$ and $\mathcal{P}$ the full additive subcategories of an abelian category $\mathcal{A}$ containing all injective respectively projective objects. Define $\mathrm{K}^*(\mathcal{I})$ and $\mathrm{K}^*(\mathcal{P})$ where $^* = +, -, b, \emptyset$ in the obvious way. If $\mathcal{A}$ contains enough injectives/projectives then the natural functors $i : \mathrm{K}^+(\mathcal{I}) \to \mathrm{D}^+(\mathcal{A})$ respectively $i : \mathrm{K}^-(\mathcal{P}) \to \mathrm{D}^-(\mathcal{A})$ are equivalences.

Remark 2.3.6. If $0 \longrightarrow A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \longrightarrow 0$ is an exact sequence in $\mathrm{Kom}(\mathcal{A})$ then there is a morphism $C^\bullet \xrightarrow{w} A^\bullet[1]$ making the sequence (A, B, C, u, v, w) a triangle in $\mathrm{D}(\mathcal{A})$. Furthermore every triangle in $\mathrm{D}(\mathcal{A})$ is isomorphic to one obtained in this way (see [9], IV.2.8). This is not true for $\mathrm{K}(\mathcal{A})$.

Definition 2.3.7. Let X be a noetherian scheme. A complex $A^\bullet$ on X of sheaves of $\mathcal{O}_X$-modules is called *perfect* if for every point $x \in X$ there is a neighbourhood $U \subset X$ such that $A^\bullet_{|U}$ is quasi-isomorphic to a bounded complex of locally free $\mathcal{O}_X$-modules. We denote by $\mathrm{D}^{\mathrm{perf}}(X)$ the full subcategory of perfect complexes in $\mathrm{D}(\mathbf{Sh}(X))$. It is in fact a thick triangulated subcategory.

Remark 2.3.8. X is smooth if and only if $\mathrm{D}^{\mathrm{perf}}(X)$ is equivalent to $\mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X))$.

Convention: If X is a scheme we denote by $\mathrm{D}^*(X)$ (where $* = +, -, \mathrm{b}, \emptyset$) the derived category $\mathrm{D}^*(\mathbf{Coh}(X))$.

2.4 Derived functors

If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor between abelian categories, then a natural question to ask is how to define an extension of F on the level of derived categories. This will not work in general but only for functors F which are at least half-exact (in the abelian sense). Their extensions will be exact functors in the triangulated sense. The extension of such a functor F to the level of the homotopy category is straightforward by applying F to the objects of a complex. The passing to the derived world is trickier: If we take an acyclic complex $A^\bullet$ in $\mathrm{K}(\mathcal{A})$ it becomes isomorphic to 0 in $\mathrm{D}(\mathcal{A})$ so the extension of F on the derived level must take $A^\bullet$ to 0 in $\mathrm{D}(\mathcal{B})$, i.e. an acyclic complex. Thus we see that we cannot define the *derived functor* of F as easily as $\mathrm{K}(F)$, its extension to the homotopy category, because in general such a functor would not transform acyclic complexes into acyclic ones. The basic idea in the construction of the derived functor is the following: Take an arbitrary complex $A^\bullet$ in $\mathrm{D}(\mathcal{A})$ replace it with an isomorphic complex $\widetilde{A}^\bullet$ which has *adapted* objects and then apply F to $\widetilde{A}^\bullet$ term by term. We will not delve into the general theory but just mention that a class of objects $\mathcal{R}$ in $\mathcal{A}$ is adapted to a left/right exact functor F if F maps any acyclic complex from $\mathrm{Kom}^+(\mathcal{R})/\mathrm{Kom}^-(\mathcal{R})$ to an acyclic complex and any object in $\mathcal{A}$ is a subobject/quotient of an object in $\mathcal{R}$. Let us consider the functors which will be of interest in the following. Our description follows [13].

Tensor product. Let X be a projective scheme over a field $\mathbb{K}$ and $\mathcal{F} \in \mathbf{Coh}(X)$. Then we have the right exact functor $\mathcal{F} \otimes (\ )$ and we are looking for its *left derived functor*. By assumption on X every sheaf admits a resolution by locally free sheaves. Using the fact that for an acyclic complex $E^\bullet$ the complex $\mathcal{F} \otimes E^\bullet$ is still acyclic we conclude that the class of locally free sheaves is adapted to $\mathcal{F} \otimes (\ )$, i.e. we get the left derived functor

$$\mathcal{F} \otimes^L (\ ) : \mathrm{D}^-(X) \longrightarrow \mathrm{D}^-(X)$$

If X is smooth of dimension n then any coherent sheaf admits a locally free resolution of length n so $\mathcal{F} \otimes^L (\ )$ is defined on the bounded derived category. In the more general situation, i.e. if we consider a complex $\mathcal{F}^\bullet$, one has to show that the subcategory of complexes of locally-free sheaves is adapted to $\mathcal{F}^\bullet \otimes (\ )$. The situation is summarized in the following diagram:

$$\begin{tikzcd}
\mathrm{D}^-(X) \times \mathrm{D}^-(X) \arrow[r, "\otimes^L"] & \mathrm{D}^-(X) \\
\mathrm{D}^{\mathrm{b}}(X) \times \mathrm{D}^{\mathrm{b}}(X) \arrow[u, hook] \arrow[r, "\text{X smooth}"] & \mathrm{D}^{\mathrm{b}}(X) \arrow[u, hook]
\end{tikzcd}$$

Let us mention that for a scheme X which is not necessarily projective one uses flat sheaves instead of locally free ones.

Local Homs. Let X be a noetherian scheme, then for any $\mathcal{F} \in \mathbf{Qcoh}$ we have the left exact functor

$$\mathcal{H}om(\mathcal{F}, \ ) : \mathbf{Qcoh}(X) \to \mathbf{Qcoh}(X)$$

The class of injective objects in $\mathbf{Qcoh}(X)$ is adapted to this functor so we get the *right derived* functor $R\mathcal{H}om$. The general situation is summarized in the following diagram

$$\begin{tikzcd}
\mathrm{D}^-(\mathbf{Qcoh}(X))^{\mathrm{opp}} \times \mathrm{D}^+(\mathbf{Qcoh}(X)) \arrow[r, "R\mathcal{H}om"] & \mathrm{D}^+(\mathbf{Qcoh}(X)) \\
\mathrm{D}^-(X)^{\mathrm{opp}} \times \mathrm{D}^+(X) \arrow[u, hook] \arrow[r] & \mathrm{D}^+(X) \arrow[u, hook] \\
\mathrm{D}^{\mathrm{b}}(X)^{\mathrm{opp}} \times \mathrm{D}^{\mathrm{b}}(X) \arrow[u, hook] \arrow[r, "\text{X regular}"] & \mathrm{D}^{\mathrm{b}}(X) \arrow[u, hook]
\end{tikzcd}$$

Inverse image. If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces one defines the *inverse image* functor $f^* : \mathbf{Sh}_{\mathcal{O}_Y} \to \mathbf{Sh}_{\mathcal{O}_X}$ to be the composition of the exact functor f^{-1} and the right exact functor $\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)}$, i.e. $f^*(\mathcal{F}) = \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{F})$. Thus, f^* is right exact and it is possible to construct its left derived functor Lf^* using flat sheaves. If f is a flat morphism then f^* is exact and therefore does not need to be derived.

Convention: From now on functors between derived categories will always be assumed to be derived. We will omit the extra symbols, e.g. $\otimes^L$ will be denoted by $\otimes$ when working in the derived world.

2.5 The spectrum of a tensor triangulated category

The main references for the following section are [3] and [4].

Definition 2.5.1. Let $\mathcal{T}$ be a tensor triangulated category. A full additive subcategory $\mathcal{C}$ is called a *thick tensor ideal* if:

(i) $\mathcal{C}$ is a thick triangulated subcategory;

(ii) For all $C \in \mathcal{C}$ and for all $D \in \mathcal{T}$ one has that $C \otimes D \in \mathcal{C}$.

$\mathcal{P}$ is called a *prime ideal* if $\mathcal{P}$ is a proper thick tensor ideal and if:

$$A \otimes B \in \mathcal{P} \Longrightarrow A \in \mathcal{P} \text{ or } B \in \mathcal{P}$$

Now the *spectrum* of $\mathcal{T}$ is defined to be the set of all prime ideals in $\mathcal{T}$:

$$\mathrm{Spc}(\mathcal{T}) = \{\mathcal{P}, \mathcal{P} \text{ prime in } \mathcal{T}\}$$

Remark 2.5.2. Note the similarity to the definition of primes in commutative algebra. However, some of the following definitions will not have this property.

Having associated a set to a tensor triangulated category we now endow this set with a topology:

Definition 2.5.3. Let $A \in \mathcal{T}$. The *support* of A is defined to be the subset

$$\mathrm{supp}(A) = \{\mathcal{P} \in \ \mathrm{Spc}(\mathcal{T}) \mid A \notin \mathcal{P}\} \subset \mathrm{Spc}(\mathcal{T})$$

Accordingly let S be an arbitrary family of objects in $\mathcal{T}$ and define:

$$Z(S) = \{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid S \cap \mathcal{P} = \emptyset\} = \bigcap_{A \in S} \mathrm{supp}(A).$$

The collection of these subsets defines the closed subsets of the so called *Zariski topology* on $\mathrm{Spc}(\mathcal{T})$. The sets $U(A) = \mathrm{Spc}(\mathcal{T}) \setminus \mathrm{supp}(A)$ define a basis of open subsets for this topology.

Remark 2.5.4. Let $\mathcal{P}$ be a prime in $\mathcal{T}$ and consider the *Verdier localization* $\mathcal{T}/\mathcal{P}$ of $\mathcal{T}$ in $\mathcal{P}$. Since we have the exact triangle $0 \longrightarrow P \longrightarrow P \longrightarrow 0$ ($P \in \mathcal{P}$) all objects of $\mathcal{P}$ become isomorphic to zero in $\mathcal{T}/\mathcal{P}$. Using this description one has

$$\mathrm{supp}(A) = \{\mathcal{P} \in \ \mathrm{Spc}(\mathcal{T}) \mid A \neq 0 \text{ in } \mathcal{T}/\mathcal{P}\}$$

Remark 2.5.5. Let $\mathcal{T}$ be a non-trivial tensor triangulated category and $A \in \mathcal{T}$. We list some properties of the spectrum and the support:

(i) $\mathrm{Spc}(\mathcal{T}) \neq \emptyset$;

(ii) For every proper thick $\otimes$-ideal $\mathcal{J}$ there exists a maximal proper thick $\otimes$-ideal $\mathcal{P}$ with $\mathcal{J} \subset \mathcal{P}$ and $\mathcal{P}$ is prime;

(iii) $\mathrm{supp}(A) = \emptyset \Longleftrightarrow \exists\ \geq 1$ such that $A^{\otimes n} = 0$;

(iv) $\mathrm{supp}(A) = \mathrm{Spc}(\mathcal{T}) \iff \langle A \rangle = \mathcal{T}$ where $\langle A \rangle$ denotes the smallest thick $\otimes$-ideal containing A, i.e. the intersection of all thick $\otimes$-ideals containing A;

(v) $\mathrm{supp}(A \oplus B) = \mathrm{supp}(A) \cup \mathrm{supp}(B)$;

(vi) $\mathrm{supp}(TA) = \mathrm{supp}(A)$;

(vii) If $A \longrightarrow B \longrightarrow C \longrightarrow TA$ is a triangle then $\mathrm{supp}(A) \subset \mathrm{supp}(B) \cup \mathrm{supp}(C)$;

(viii) $\mathrm{supp}(A \otimes B) = \mathrm{supp}(A) \cap \mathrm{supp}(B)$.

"Dual" properties hold for the open complements.

Proposition 2.5.6. *([3], Proposition 2.9.) Let $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$. Then $\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{Q} \subset \mathcal{P}\}$.*

Proof. Set $S := \mathcal{T} \setminus \mathcal{P}$. Then $\mathcal{P} \in Z(S)$ and if $\mathcal{P} \in Z(S')$ then $S' \subset S$ and therefore $Z(S) \subset Z(S')$. It follows that $Z(S)$ is the smallest closed subset containing $\mathcal{P}$, i.e. its closure in $\mathrm{Spc}(\mathcal{T})$. □

Remark 2.5.7. This result shows that not everything is the same as in commutative algebra where the closure of a prime ideal consists of all prime ideals containing it. However ([3], Proposition 2.11.) tells us that for every prime $\mathcal{P}$ there is a minimal prime $\mathcal{P}'$ such that $\mathcal{P}' \subset \mathcal{P}$.

Proposition 2.5.8. *([3], Proposition 2.18.) Let $Z \neq \emptyset$ be an irreducible closed subset in $\mathrm{Spc}(\mathcal{T})$. Then $Z = \overline{\{\mathcal{P}\}}$ where $\mathcal{P} = \{A \in \mathcal{T} \mid U(A) \cap Z \neq \emptyset\}$. Therefore if $\mathrm{Spc}(\mathcal{T})$ is noetherian then it is a Zariski space.*

Proposition 2.5.9. *([3], Proposition 3.6.) The spectrum is functorial: Given a $\otimes$-triangulated functor $F : \mathcal{T} \to \mathcal{T}'$ the map*

$$\mathrm{Spc}(F) : \mathrm{Spc}(\mathcal{T}') \to \mathrm{Spc}(\mathcal{T}), \qquad \mathcal{Q} \mapsto F^{-1}(\mathcal{Q})$$

is well-defined, continuous and for all $A \in \mathcal{T}$ we have $\mathrm{Spc}(F)^{-1}(\mathrm{supp}_{\mathcal{T}}(A)) = \mathrm{supp}_{\mathcal{T}'}(F(A))$. Given another $\otimes$-triangulated functor $G : \mathcal{T}' \to \mathcal{T}''$ we have $\mathrm{Spc}(G \circ F) = \mathrm{Spc}(F) \circ \mathrm{Spc}(G)$.

Remark 2.5.10. In general there are two possibilities to define $F^{-1}(\mathcal{Q})$:

(i) $F^{-1}(\mathcal{Q}) = \{A \in \mathcal{T} \mid F(A) \in \mathcal{Q}\}$,

(ii) $F^{-1}(\mathcal{Q}) = \{A \in \mathcal{T} \mid \exists\, B \in \mathcal{Q} :\ F(A) \cong B\}$.

Since $\mathcal{Q}$ is closed under isomorphisms these two definitions coincide.

Proof. Let $\mathcal{Q} \in \mathrm{Spc}(\mathcal{T}')$. Then $F^{-1}(\mathcal{Q})$ is clearly a thick additive subcategory in $\mathcal{T}$ since F is an additive functor. Next let $A \longrightarrow B \longrightarrow C \longrightarrow TA$ be a triangle in $\mathcal{T}$ such that $A, B \in F^{-1}(\mathcal{Q})$. Then

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow F(TA) = T(F(A))$$

is a triangle in $\mathcal{T}'$ with $F(A), F(B) \in \mathcal{Q}$ so that $F(C) \in \mathcal{Q}$ and therefore $C \in F^{-1}(\mathcal{Q})$. Now let $A \otimes B \in F^{-1}(\mathcal{Q})$. Then $F(A \otimes B) = F(A) \otimes F(B) \in \mathcal{Q}$, so, since $\mathcal{Q}$ is prime, w.l.o.g. $F(A) \in \mathcal{Q}$ and $A \in F^{-1}(\mathcal{Q})$, i.e. $F^{-1}(\mathcal{Q})$ is prime. The proof that $F^{-1}(\mathcal{Q})$ is a thick $\otimes$-ideal is similar. So $\mathrm{Spc}(F)$ is well-defined. Now:

$$\mathrm{Spc}(F)^{-1}(\mathrm{supp}_{\mathcal{T}}(A)) = \mathrm{Spc}(F)^{-1}\{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid A \notin \mathcal{P}\} =$$

$$= \left\{\mathcal{Q} \in \mathrm{Spc}(\mathcal{T}') \mid A \notin F^{-1}(\mathcal{Q})\right\} = \{\mathcal{Q} \in \mathrm{Spc}(\mathcal{T}') \mid F(A) \notin \mathcal{Q}\} = \mathrm{supp}_{\mathcal{T}'}(F(A))$$

Since the subsets $\mathrm{supp}(-)$ form a basis for the topology on $\mathrm{Spc}(\mathcal{T})$ it follows that $\mathrm{Spc}(F)$ is continuous. Finally if F, G are both $\otimes$-triangulated then so is $G \circ F$ and the rest is clear. □

Next we quote some results from [3] which will be used later on.

Proposition 2.5.11. *([3], Corollary 3.7.) If $F, F' : \mathcal{T} \to \mathcal{T}'$ are isomorphic $\otimes$-triangulated functors, then* $\mathrm{Spc}(F) = \mathrm{Spc}(F')$.

Corollary 2.5.12. *If $F : \mathcal{T} \to \mathcal{T}'$ is an equivalence, then* $\mathrm{Spc}(F)$ *is a homeomorphism.*

Proposition 2.5.13. *([3], Proposition 3.11.) Let $\mathcal{I}$ be a thick $\otimes$-ideal in $\mathcal{T}$ and consider the $\otimes$-triangulated* localization functor $q : \mathcal{T} \to \mathcal{T}/\mathcal{I}$. *Then the map* $\mathrm{Spc}(q)$ *induces a homeomorphism between* $\mathrm{Spc}(\mathcal{T}/\mathcal{I})$ *and the subset* $\{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \mid \mathcal{I} \subset \mathcal{P}\}$ *of* $\mathrm{Spc}(\mathcal{T})$.

Proposition 2.5.14. *([3], Proposition 3.13.) Let $\mathcal{C}$ be a full dense triangulated subcategory in $\mathcal{T}$ having the same unit. Then the map* $\mathrm{Spc}(i) : \mathrm{Spc}(\mathcal{T}) \to \mathrm{Spc}(\mathcal{C})$*, where i is the inclusion functor, is a homeomorphism.*

Definition 2.5.15. Let $\mathcal{T}$ be a $\otimes$-triangulated category and $A \in \mathcal{T}$.

i. A is called a *unit* if there exists a $B \in \mathcal{T}$ such that $A \otimes B \cong \mathbf{1}$;

ii. A is called a *t-generator* if $\langle A \rangle = \mathcal{T}$, i.e. the smallest thick $\otimes$-ideal containing A equals to $\mathcal{T}$;

iii. A is called a *zero-divisor* if there exists $B \neq 0 \in \mathcal{T}$ such that $A \otimes B \cong 0$.

Remark 2.5.16. Let us list some properties of the above defined objects.

- The product of two units is again a unit.
- Let A be a unit. Then so is $T(A)$ since if $A \otimes B \cong \mathbf{1}$ then $T(A) \otimes T^{-1}(B) \cong \mathbf{1}$.
- Let A be a t-generator. Then for all objects $B \in \mathcal{T}$ we have that $A \oplus B$ is a t-generator.

Remark 2.5.17. The above definitions are of course inspired by commutative algebra. Certain properties turn out to be the same as in ring theory (and so are the proofs), e.g.:

- A zero divisor is not a unit.
- If A is a unit, then A is a t-generator.

But the converse of the second statement is not true. Consider e.g. a principal ideal domain R and the homotopy category $\mathrm{K}(R\mathrm{-Mod})$. Then R^2, considered as a 0-complex is a t-generator, but it cannot be a unit: Assume for a contradiction that there exists a complex $A^\bullet$ such that $A^\bullet \otimes R^2 \cong \mathbf{1}$ in $\mathrm{K}(R - \mathrm{Mod})$. In particular the isomorphy is a quasi-isomorphy. Therefore $R \cong H_0(A^\bullet \otimes R^2) = H_0(A^\bullet) \otimes R^2 = H_0(A^\bullet)^2$, contradiction. See chapter 3 for more examples.
Therefore some statements of commutative algebra have to be adjusted, e.g.: It is again true that every element which is not a unit is contained in a maximal thick prime ideal. In our setting this statement is also true for every non-t-generator.

Proposition 2.5.18. *Let $\mathcal{T}$ be a tensor triangulated category such that $\mathrm{Spc}(\mathcal{T})$ is irreducible. Let $A \longrightarrow B \longrightarrow C \longrightarrow T(A)$ be a triangle. Then if one of the objects is a t-generator at least one of the other two also has this property.*

Proof. W.l.o.g. we assume that A is the given t-generator. Then $\mathrm{Spc}(\mathcal{T}) = \mathrm{supp}(A) = \mathrm{supp}(B) \cup \mathrm{supp}(C)$. Since $\mathrm{Spc}(\mathcal{T})$ is irreducible one of the two sets has to be $\mathrm{Spc}(\mathcal{T})$, i.e. either B or C is a t-generator. Rotation of the triangle gives the result. $\square$

Example 2.5.19. Consider $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1_{\mathbb{C}})$. The exact sequence in $\mathbf{Coh}(X)$

$$0 \longrightarrow \mathcal{O}(k) \longrightarrow \mathcal{O}(k+1) \longrightarrow k(x) \longrightarrow 0$$

(where $k(x)$ is the skyscraper sheaf in a closed point $x \in \mathbb{P}^1_{\mathbb{C}}$) gives rise to an exact triangle in $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1_{\mathbb{C}})$. It will follow from 2.5.23 that skyscraper sheaves are not t-generators in $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1_{\mathbb{C}})$ so the statement of the previous proposition cannot be strengthened.

Remark 2.5.20. The previous proposition is not true if we replace "t-generator" by "unit". To see this consider the homotopy category of a principal ideal domain R. Then the exact sequence $0 \longrightarrow R \longrightarrow R^3 \longrightarrow R^2 \longrightarrow 0$ (with the obvious maps) gives rise to the triangle $R \longrightarrow R^3 \longrightarrow R^2 \longrightarrow R[1]$ in $\mathrm{D}(R - \mathrm{mod})$ (see [9], Lemma IV.1.13) and R^2, R^3 are not invertible in $\mathrm{D}(R - \mathrm{mod})$, see [8].

Proposition 2.5.21. *Let $F : \mathcal{T} \to \mathcal{T}'$ be a $\otimes$-triangulated functor. If $A \in \mathcal{T}$ is a unit/t-generator, then so is $F(A)$.*

Proof. The first statement follows immediately from $F(A \otimes B) \cong F(A) \otimes F(B)$ and $F(\mathbf{1}_{\mathcal{T}}) = \mathbf{1}'_{\mathcal{T}}$, the second from $\mathrm{Spc}(F)^{-1}(\mathrm{supp}_{\mathcal{T}}(A)) = \mathrm{supp}_{\mathcal{T}'}(F(A))$. $\square$

Definition 2.5.22. Let $Y \subset \mathrm{Spc}(\mathcal{T})$. Then the following subcategory is a thick $\otimes$-ideal:

$$\mathcal{T}_Y := \{A \in \mathcal{T} \mid \mathrm{supp}(A) \subset Y\} = \bigcap_{\mathcal{P} \in \mathrm{Spc}(\mathcal{T}) \setminus Y} \mathcal{P}.$$

Theorem 2.5.23. *([3], Corollary 5.6.) Let X be a noetherian scheme and $\mathcal{T} := \mathrm{D}^{\mathrm{perf}}(X)$ the derived category of perfect complexes over X. Then there is a homeomorphism $f : X \xrightarrow{\sim} \mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X))$ defined by:*

$$f(x) = \left\{A \in \mathrm{D}^{\mathrm{perf}}(X) \mid A_x \cong 0 \text{ in } \mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x})\right\} \qquad \text{for all } x \in X.$$

Moreover for any perfect complex $A \in \mathrm{D}^{\mathrm{perf}}(X)$ *the closed subset* $\mathrm{supph}(A) := \{y \in X \mid A_y \text{ is not acyclic}\} \subset X$ *corresponds via* f *to the closed subset* $\mathrm{supp}(A) \subset \mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X))$.

Remark 2.5.24. It is possible to equip $\mathrm{Spc}(\mathcal{T})$ with a structure sheaf as follows: Let $U \subset \mathrm{Spc}(\mathcal{T})$ be an open subset, $Z = \mathrm{Spc}(\mathcal{T}) \setminus U$ and consider $\mathcal{T}_Z = \{A \in \mathcal{T} \mid \mathrm{supp}(A) \subset Z\} = \{A \in \mathcal{T} \mid \mathrm{supp}(A) \cap U = \emptyset\} = \mathcal{T}^U$. Then we can consider the localization $\mathcal{T}/\mathcal{T}_Z$ and the image of the unit via the localization, denoted by $\mathbf{1}_U$. We know from [2] that $\mathrm{End}_{\mathcal{T}/\mathcal{T}_Z}(\mathbf{1}_U)$ is a commutative ring. If $U' \subset U$ then $Z \subset Z'$ and therefore $\mathcal{T}_Z \subset \mathcal{T}_{Z'}$. Hence from the universal property of localization we get a canonical functor $\mathcal{T}/\mathcal{T}_Z \to \mathcal{T}/\mathcal{T}_{Z'}$. By this we have defined the following presheaf $\widetilde{\mathcal{O}}_{\mathrm{Spc}(\mathcal{T})}$ of rings:

$$U \mapsto \mathrm{End}_{\mathcal{T}/\mathcal{T}_Z}(\mathbf{1}_U)$$

The sheafification of this presheaf will be denoted by $\mathcal{O}_{\mathrm{Spc}(\mathcal{T})}$. So we have made $\mathrm{Spc}(\mathcal{T})$ into a ringed space which will be denoted by $\mathrm{Spec}(\mathcal{T})$. If $\mathcal{T} = \mathrm{D}^{\mathrm{perf}}(X)$ as above then $\mathcal{O}_{\mathrm{Spc}(\mathcal{T})}$ identifies with $\mathcal{O}_X$, i.e. $\mathrm{Spec}(\mathcal{T})$ is isomorphic to $(X, \mathcal{O}_X)$ as a ringed space. This is due to the fact that the canonical functor $\mathrm{D}^{\mathrm{perf}}(X)/\mathrm{D}^{\mathrm{perf}}(X)^U \to \mathrm{D}^{\mathrm{perf}}(U)$ induced by the restriction functor $\mathrm{D}^{\mathrm{perf}}(X) \to \mathrm{D}^{\mathrm{perf}}(U)$ is fully faithful (see [23]).

The next two propositions are from [7].

Proposition 2.5.25. *Let* $\mathcal{P} \in \mathrm{Spc}(\mathcal{T})$*. Then the stalk* $\mathcal{O}_{\mathrm{Spc}(\mathcal{T}),\mathcal{P}}$ *equals to* $\mathrm{End}_{\mathcal{T}/\mathcal{P}}(\mathbf{1})$.

Proof. First we will show that $\mathcal{P} = \bigcup_{\mathcal{P} \in U} \mathcal{T}^U$.
"$\subset$" Let $A \in \mathcal{P}$. Then $\mathcal{P} \in U_0$, where U_0 denotes the complement of $\mathrm{supp}(A)$, and therefore $A \in \mathcal{T}^{U_0}$.
"$\supset$" Assume $A \notin \mathcal{P}$. Then by definition $\mathcal{P} \in \mathrm{supp}(A)$ and therefore there is no open subset U containing $\mathcal{P}$ such that $\mathrm{supp}(A) \cap U = \emptyset$, i.e. $A \notin \bigcup_{\mathcal{P} \in U} \mathcal{T}^U$.
Now

$$\mathcal{O}_{\mathrm{Spc}(\mathcal{T}),\mathcal{P}} = \varinjlim_{\mathcal{P} \in U} \mathcal{O}_{\mathrm{Spc}(\mathcal{T})}(U) = \varinjlim_{\mathcal{P} \in U} \mathrm{End}_{\mathcal{T}/\mathcal{T}^U}(\mathbf{1}) = \\ = \mathrm{End}_{\mathcal{T}/\bigcup_{\mathcal{P} \in U}(\mathcal{T}^U)}(\mathbf{1}) = \mathrm{End}_{\mathcal{T}/\mathcal{P}}(\mathbf{1})$$

□

Proposition 2.5.26. *If* $F : \mathcal{T} \to \mathcal{T}'$ *is a tensor triangulated functor, then* $f := \mathrm{Spc}(F) : \mathrm{Spc}(\mathcal{T}') \to \mathrm{Spc}(\mathcal{T})$ *is a morphism of ringed spaces.*

Proof. We have to define a natural morphism of sheaves $\mathcal{O}_{\mathrm{Spc}(\mathcal{T})} \to f_*\mathcal{O}_{\mathrm{Spc}(\mathcal{T}')}$. Consider an open subset $U \subset \mathrm{Spc}(\mathcal{T})$ and $f^{-1}(U) \subset \mathrm{Spc}(\mathcal{T}')$.
Claim: We have that $F(\mathcal{T}_Z) \subset \mathcal{T}'_{Z'}$ where $Z = \mathrm{Spc}(\mathcal{T}) \setminus U$ and $Z' = \mathrm{Spc}(\mathcal{T}') \setminus f^{-1}(U)$.
Assume the converse: Then there exists $A \in \mathcal{T}$ such that $\mathrm{supp}(F(A)) \cap f^{-1}(U) \neq \emptyset$. Let $\mathcal{Q} \in \mathrm{Spc}(\mathcal{T}')$ be an element of the intersection, i.e. $\mathcal{Q} \in f^{-1}(U)$ and $F(A) \notin \mathcal{Q}$. Then $A \notin F^{-1}(\mathcal{Q})$ and $f(\mathcal{Q}) = F^{-1}(\mathcal{Q}) \in U$ so that $F^{-1}(\mathcal{Q}) \in U \cap \mathrm{supp}(A)$ which is a contradiction.
So we have that $F(\mathcal{T}_Z) \subset \mathcal{T}_{Z'}$ and therefore F induces a functor $\mathcal{T}/\mathcal{T}_Z \to \mathcal{T}'/\mathcal{T}'_{Z'}$ and for every open $U \subset \mathrm{Spc}(\mathcal{T})$ we have a morphism $\widetilde{\mathcal{O}}_{\mathrm{Spc}(\mathcal{T})}(U) \to f_*\widetilde{\mathcal{O}}_{\mathrm{Spc}(\mathcal{T}')}(U)$ which is clearly compatible with restrictions. This morphism of presheaves then induces the desired morphism of sheaves. □

Chapter 3

The Grothendieck ring

3.1 Definition and properties

Let $\mathcal{T}$ be a triangulated category. Recall that the *Grothendieck group* $K_0(\mathcal{T})$ is defined to be the quotient of the free abelian group on the set of isomorphism classes of $\mathcal{T}$ by the *Euler relations*: [B]=[A]+[C] whenever there exists a triangle in $\mathcal{T}$:

$$A \longrightarrow B \longrightarrow C \longrightarrow TA$$

It has the following universal property: Whenever there is a function f from the set of isomorphism classes of objects in $\mathcal{T}$ to an abelian group G such that the Euler relations hold, then f factors through $K_0(\mathcal{T})$, i.e. there is a unique group homomorphism $\overline{f}$ so that we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{T} \arrow[rr, "f"] \arrow[dr, "{[\,]}"'] & & G \\ & K_0(\mathcal{T}) \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

From the exact triangle $A \longrightarrow A \oplus B \longrightarrow B \longrightarrow T(A)$ one gets $[A] + [B] = [A \oplus B]$ and from the triangle $A \longrightarrow 0 \longrightarrow T(A) \longrightarrow T(A)$ we have $[T(A)] = -[A]$. Furthermore $[0] = 0$. If $F : \mathcal{T} \to \mathcal{T}'$ is a triangulated functor, then F induces a group homomorphism between $K_0(\mathcal{T})$ and $K_0(\mathcal{T}')$ by sending $[A]$ to $[F(A)]$.

Proposition 3.1.1. *Let $\mathcal{T}$ be a tensor triangulated category. Then one can introduce a commutative multiplication on the Grothendieck group $K_0(\mathcal{T})$ by setting:*

$$[A] * [B] := [A \otimes B]$$

This definition makes the Grothendieck group a commutative ring with identity.

Proof. One first needs to check that the operation is well-defined. Consider for example an exact triangle

$$A' \longrightarrow B \longrightarrow C' \longrightarrow TA'$$

so that $[B] = [A'] + [C']$ in $K_0(\mathcal{T})$. Now tensoring the triangle with an object A gives, since the tensor product is an exact functor, a triangle

$$A' \otimes A \longrightarrow B \otimes A \longrightarrow C' \otimes A \longrightarrow TA' \otimes A$$

So in $K_0(\mathcal{T})$ one has:

$$[A] * ([A'] + [C']) = [A] * [A'] + [A] * [C'] = [A \otimes A'] + [A \otimes C'] = [B \otimes A]$$

as was to be shown.
The unit element in $\mathcal{T}$ serves as the unit in the ring $K_0(\mathcal{T})$. Associativity, commutativity and distributivity are clear. □

It was shown in [22] that there is a one-to-one correspondence between dense triangulated subcategories of a triangulated category $\mathcal{T}$ and subgroups of $K_0(\mathcal{T})$. Since now the triangulated category has an additional structure we get the following

Proposition 3.1.2. *There is a bijective correspondence between dense tensor-triangulated subcategories $\mathcal{C}$ in $\mathcal{T}$ containing the unit element and subrings R of $K_0(\mathcal{T})$.*

Proof. As in [22] one considers $\mathrm{im}(K_0(\mathcal{C}))$ in $K_0(\mathcal{T})$ for a given subcategory $\mathcal{C}$. To a subring R one considers the full subcategory $\mathcal{C}_R$ consisting of objects A such that $[A] \in R$. We know from [22] that these assignments provide the correspondence on the level of groups. Let us see that the additional structures are preserved:
The subgroup $\mathrm{im}(K_0(\mathcal{C}))$ contains the unit since $\mathcal{C}$ does. Let $[A], [B] \in \mathrm{im}(K_0(\mathcal{C}))$. Then $[A] * [B] = [A \otimes B] \in \mathrm{im}(K_0(\mathcal{C}))$ since $\mathcal{C}$ is closed under $\otimes$. So $\mathrm{im}(K_0(\mathcal{C}))$ is indeed a subring of $K_0(\mathcal{T})$.
On the other hand we already know that $\mathcal{C}_R$ is a dense triangulated subcategory in $\mathcal{T}$. As before it is clear that it contains the unit and it is closed under $\otimes$ since for $A, B \in \mathcal{C}$ one has $[A \otimes B] = [A] * [B] \in R$, so $A \otimes B \in \mathcal{C}$. □

Remark 3.1.3. If we consider an abelian category $\mathcal{A}$, e.g. the category $\mathbf{Coh}(X)$ of coherent sheaves on a (noetherian) scheme X, then there are several ways to define the Grothendieck group of $\mathcal{A}$. Following the above approach we could consider $\mathcal{T} = \mathrm{D^b}(\mathcal{A})$. On the other hand one can consider $\mathcal{A}$ and define the Grothendieck group to be the free abelian group $K_0(\mathcal{A})$ generated by all objects where we factor out the subgroup generated by relations:

$$\mathcal{F} - \mathcal{F}' - \mathcal{F}''$$

whenever there is an exact sequence $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$. Denote the image of an object $\mathcal{F}$ in $K_0(\mathcal{A})$ by $\psi(\mathcal{F})$. As in the above case there is a universal property, namely:
Every additive function λ, i.e. $\lambda(\mathcal{F}) = \lambda(\mathcal{F}') + \lambda(\mathcal{F}'')$ whenever there is an exact sequence as above, from $\mathcal{A}$ to an abelian group G factors through $K_0(\mathcal{A})$. Now considering an object of $\mathcal{A}$ as a 0-complex in $\mathcal{T}$ and setting $\lambda = [\,] : \mathcal{A} \to K_0(\mathcal{T})$ we see that λ is an additive function because of the Euler relations and therefore

we get a homomorphism $\Phi : K_0(\mathcal{A}) \to K_0(\mathcal{T}), \Phi(\psi(\mathcal{F})) = [\mathcal{F}]$. Now considering a complex $A^\bullet \in \mathcal{T}$ set, in the same notation as above,

$$f : \mathcal{T} \to K_0(\mathcal{A}), \qquad A^\bullet \mapsto \sum_{i=-\infty}^{\infty} (-1)^i \psi(A^i)$$

If $\Delta = \; A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow A^\bullet[1]$ is a triangle in $\mathcal{T}$ then it is isomorphic to a triangle of the form $\Delta' = \; X^\bullet \longrightarrow Y^\bullet \longrightarrow \quad \longrightarrow Z^\bullet \longrightarrow X^\bullet[1]$ such that the sequence of complexes $0 \longrightarrow X^\bullet \longrightarrow Y^\bullet \longrightarrow Z^\bullet \longrightarrow 0$ is exact in $\mathrm{Kom}(\mathcal{A})$ (see 2.3.6). The Euler relation holds for Δ' and therefore for Δ so we get a homomorphism $\Theta : K_0(\mathcal{T}) \to K_0(\mathcal{A}), \Theta([A^\bullet]) = \sum_{i=-\infty}^{\infty} (-1)^i \psi(A^i)$. Obviously $\Theta \circ \Phi = \mathrm{id}$. The other identity will follow from 3.2.1.

Remark 3.1.4. Note that the introduction of multiplication for the Grothendieck group is in general not straightforward on the level of the abelian category $\mathbf{Coh}(X)$, since the usual tensor product of sheaves is not exact. However for any noetherian scheme X one can consider the group $K^0(X)$ defined in essentially the same way as $K_0(X)$ but using locally free coherent sheaves. If X is a noetherian, integral, separated, regular scheme then the natural map $\Xi : K^0(X) \to K_0(X)$ is an isomorphism (see [11], Chapter III, Ex.6.9). Apparently $K^0(X)$ is a ring with the multiplication being induced by the tensor product (which is now possible since the considered sheaves are locally free and therefore tensorizing with them is exact). Then $\Phi \circ \Xi : K^0(X) \to K_0(\mathcal{T})$ is a ring isomorphism. It is an isomorphism of abelian groups anyway, and furthermore $\Phi \circ \Xi$ sends the unit to the unit and the derived product of 0-complexes of locally free sheaves is the usual tensor product so $\Phi \circ \Xi$ is compatible with the multiplication.

Example 3.1.5. Let $X = \mathbb{A}^1_{\mathbb{K}}$ for an arbitrary field $\mathbb{K}$. Then we know that $K_0(X) = \mathbb{Z}$ using the rank of a (locally free) sheaf. Now 3.1.2 shows that there are no proper dense tensor triangulated subcategories in $\mathrm{D}(X)$ since there are no proper subrings in $\mathbb{Z}$.
More generally we have that $K_0(\mathbb{A}^n_{\mathbb{K}}) = \mathbb{Z}$ for all $n \in \mathbb{N}$ since we can compute the Grothendieck group using locally free sheaves which correspond to projective modules over $\mathbb{K}[x_1, ..., x_n]$ and these are free by a theorem which was proven independently by Quillen and Suslin.

Remark 3.1.6. If X is a smooth projective variety over $\mathbb{C}$ then the morphism groups in $\mathcal{T} = \mathrm{D^b}(X)$ are in fact $\mathbb{C}$-vector spaces. Furthermore we have that for every pair of objects $E, F \in \mathcal{T}$ $\dim_{\mathbb{C}}(\bigoplus_i \mathrm{Hom}_{\mathcal{T}}(E, F[i])) < \infty$. It is possible to define a bilinear form, called the *Euler form*, on $K_0(\mathcal{T})$ via

$$\chi(E, F) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{T}}(E, F[i]).$$

One now defines two objects E_1, E_2 to be *numerically equivalent*, $E_1 \sim_{\mathrm{num}} E_2$, if $\chi(E_1, F) = \chi(E_2, F) \; \forall \; F \in \mathcal{T}$. The group $N(\mathcal{T}) := K_0(\mathcal{T}) / \sim_{\mathrm{num}}$ is usually called the *numerical Grothendieck group* and has finite rank in our case. Now using

$$\chi(E \otimes G, F) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{T}}(E \otimes G, F[i]) =$$

$$= \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{T}}(E, \mathcal{H}om(G, F[i])) = 0$$

we see that $N(\mathcal{T})$ inherits the ring structure introduced on $K_0(\mathcal{T})$.

3.2 The Grothendieck ring of the projective line

Lemma 3.2.1. *For all complexes $A^\bullet \in \mathcal{T}$ we have $[A^\bullet] = \sum_{i=-\infty}^{\infty}(-1)^i[A^i]$ in $K_0(\mathcal{T})$ where $[A^i]$ denotes the class of the 0-complex A^i.*

Proof. By induction on the number of non-zero terms of $A^\bullet$. The claim is trivial for a 0-complex. For a complex $A^\bullet = [\ A^k \xrightarrow{d} A^{k+1}\ ](k \in \mathbb{Z}$ and A^k sits in degree k) note that $A^\bullet = \mathrm{cone}(f)$ where $f : A^k[-(k+1)] \xrightarrow{d} A^{k+1}[-(k+1)]$. Therefore $[A^\bullet] = (-1)^k A^k + (-1)^{k+1} A^{k+1}$. To complete the induction consider $A^\bullet = [\ 0 \longrightarrow A^k \xrightarrow{d} A^{k+1} \longrightarrow \dots\ ]$ where k denotes the index such that $A^l = 0\ \forall\, l < k$. Now denote by $\tilde{A}^\bullet$ the complex with $\tilde{A}^n = A^n\ \forall\, n > k$ and $A^n = 0$ otherwise. Then $A^\bullet = \mathrm{cone}(f)$ where $f : A^k[-(k+1)] \xrightarrow{d} \tilde{A}^\bullet$. By induction the claim is true for $\tilde{A}^\bullet$ and therefore for $A^\bullet$. ☐

Example 3.2.2. Consider $X = \mathbb{P}^1_{\mathbb{C}}$ and $\mathcal{T} = \mathrm{D}(\mathbf{Coh}(\mathbb{P}^1_{\mathbb{C}}))$. Recall that every coherent sheaf $\mathcal{F}$ on $\mathbb{P}^1_{\mathbb{C}}$ is of the form $\mathcal{F} = \bigoplus_{i=1}^k \mathcal{O}(n_i) \oplus \bigoplus_{j=1}^l \Theta(P_j)$ where $\Theta(P_j)$ denotes a torsion sheaf in a point P_j (note that the P_j are not necessarrily distinct). Furthermore recall that the *rank* of a coherent sheaf $\mathcal{F}$ on an integral noetherian scheme is defined to be

$$\mathrm{rk}(\mathcal{F}) := \dim_{\mathcal{O}_\xi} \mathcal{F}_\xi$$

where ξ denotes the generic point. Obviously the rank of a locally free sheaf of rank r is r and the rank of a skyscraper sheaf (or more generally of a torsion sheaf, i.e. a sheaf whose stalk at the generic point is zero) is zero.
Recall from ([11], II, Ex.6.11) that $K_0(X) = \mathbb{Z} \oplus \mathbb{Z}$ using degree (of the determinant line bundle of a coherent sheaf) and rank. However, the above defined multiplication is not the usual componentwise multiplication in $\mathbb{Z} \oplus \mathbb{Z}$:

Proposition 3.2.3. *Let $\mathcal{T} = \mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(\mathbb{P}^1_{\mathbb{C}}))$. The multiplication on the ring $K_0(\mathcal{T}) = \mathbb{Z} \oplus \mathbb{Z}$ is given by*

$$(a, b) * (c, d) = (ad + bc, bd)$$

Proof. By 3.1.4 we know that $K_0(\mathcal{T}) \cong K^0(\mathbb{P}^1_{\mathbb{C}})$ so it will be sufficient to prove the proposition for direct sums of locally free sheaves. Let $\mathcal{F} = \mathcal{O}(n_1) \oplus \dots \oplus \mathcal{O}(n_k)$ and $\mathcal{G} = \mathcal{O}(m_1) \oplus \dots \oplus \mathcal{O}(m_l)$. Then their images in $K^0(X)$ are $(n_1 + \dots + n_k, k) = (\deg(\mathcal{F}), k)$ and $(m_1 + \dots m_l, l) = (\deg(\mathcal{G}), l)$. The tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ equals to $\bigoplus_{j=1}^l \mathcal{O}(n_1 + m_j) \dots \bigoplus_{j=1}^l \mathcal{O}(n_k + m_j)$ and its image in $K^0(X)$ is $(l \cdot n_1 + \deg(\mathcal{G}) + \dots l \cdot n_k + \deg(\mathcal{G}), k \cdot l) = (l \cdot \deg(\mathcal{F}) + k \cdot \deg(\mathcal{G}), k \cdot l)$ as was to be shown. ☐

Remark 3.2.4. A more general description of $K^0(\mathbb{P}^n)$ for an arbitrary $n \in \mathbb{N}$ can be found in [15].

Corollary 3.2.5. *The only subrings R of $K_0(\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1_{\mathbb{C}}))$ are $R = n\mathbb{Z} \oplus \mathbb{Z}$, $n \in \mathbb{N}$.*

Proof. Since R is an abelian subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ it is of the form $R = n\mathbb{Z} \oplus m\mathbb{Z}$, $m, n \in \mathbb{N}$. Now R is a subring and contains the unit element which in this case is $(0, 1)$. Therefore $m = 1$. $\square$

Chapter 4

D-Equivalences and tensor structures

Definition 4.0.6. Let X be a noetherian scheme. A scheme Y is called D-*equivalent* to X if there is an equivalence of triangulated categories $\mathrm{D^b}(X) \cong \mathrm{D^b}(Y)$. We denote by $\mathrm{FM}(X)$ the set of isomorphy classes of D-equivalent schemes of X. These will be also called Fourier-Mukai partners of X.

Remark 4.0.7. The first example of non-isomorphic schemes which are FM-partners, namely an abelian variety and its dual, was given by Mukai in [17].

We will now consider the map $\phi : \mathrm{FM}(X) \to \mathrm{TS}(X)$ from the set of FM-partners to the set of tensor structures (in the sense of definition 2.1.9) on $\mathrm{D^b}(X)$ defined in the following way: Let Y be a Fourier-Mukai partner of X. Define a new product on $\mathrm{D^b}(X)$ by setting:

$$A *_Y B := F^{-1}(F(A) \otimes_Y F(B))$$

This clearly defines a tensor structure on $\mathrm{D^b}(X)$ with the unit being $F^{-1}(\mathbf{1}_{\mathrm{D^b}(Y)})$.

Proposition 4.0.8. *ϕ is injective.*

Proof. Let Y and Y' be two non-isomorphic Fourier-Mukai partners of X and assume that the induced tensor structures $\star_Y$ and $\star_{Y'}$ are equal. By definition of these structures the functors

$$F : (\mathrm{D^b}(X), \star_Y) \to (\mathrm{D^b}(Y), \otimes_Y)$$

and

$$F' : (\mathrm{D^b}(X), \star_{Y'}) \to (\mathrm{D^b}(Y'), \otimes_{Y'})$$

are *tensor*-triangulated equivalences. Therefore they induce maps on the spectra and these are isomorphisms. Since the identity functor $\mathrm{id} : (\mathrm{D^b}(X), \star_Y) \to (\mathrm{D^b}(X), \star_{Y'})$ is also a tensor-triangulated equivalence by assumption we conclude that $Y \cong Y'$, contradiction. $\square$

Proposition 4.0.9. *ϕ is not surjective.*

Proof. Let X be connected. We will prove that X is not a Fourier-Mukai partner of $X \amalg X$, yet there exists a tensor structure on $\mathrm{D^b}(X \amalg X)$ such that the spectrum of $\mathrm{D^b}(X \amalg X)$ with this structure is X. The proof consists of three steps.
Step 1. Since $\mathrm{D^b}(X \amalg X) \cong \mathrm{D^b}(X) \oplus \mathrm{D^b}(X)$ we set:

$$(A, B) * (C, D) := (A \otimes_X C, A \otimes_X D \oplus B \otimes_X C)$$

(we will omit X from now on). The associativity and comutativity of this product is proved in the same way as in commutative algebra where one defines a new multiplication on the product of two rings in a similar way (with $\otimes$ being the multiplication and $\oplus$ being the addition). The unit element is $\mathbf{1} = (\mathcal{O}_X, 0)$. It remains to show that the defined product is exact in every variable. Because of the commutativity it suffices to show the exactness of the functor $- * (A', B')$ where (A', B') is an arbitrary object in $\mathrm{D^b}(X \amalg X)$. The additivity is clear. Next we show that $- * (A', B')$ is compatible with the shift functor:

$$((A, B)[1]) * (A', B') = (A[1], B[1]) * (A', B') = (A[1] \otimes A', A[1] \otimes B' \oplus B[1] \otimes A') =$$

$$((A \otimes A')[1], (A \otimes B')[1] \oplus (B \otimes A')[1]) = ((A, B) * (A', B'))[1]$$

Now consider a triangle in $\mathrm{D^b}(X \amalg X)$:

$$(A, B) \longrightarrow (C, D) \longrightarrow (E, F) \longrightarrow (A, B)[1] = (A[1], B[1])$$

The product with (A', B') gives:

$$(A \otimes A', A \otimes B' \oplus B \otimes A') \longrightarrow (C \otimes A', C \otimes B' \oplus D \otimes A') \longrightarrow$$

$$(E \otimes A', E \otimes B' \oplus F \otimes A') \longrightarrow (A[1] \otimes A', A[1] \otimes B' \oplus B[1] \otimes A') \; .$$

One now sees immediately that in the first argument we just have the triangle $A \longrightarrow C \longrightarrow E \longrightarrow A[1]$ tensorized with A' in $\mathrm{D^b}(X)$ which is a triangle since the tensor product in $\mathrm{D^b}(X)$ is exact. Similarly, in the second argument we have the direct sum of the triangles

$$A \otimes B' \longrightarrow C \otimes B' \longrightarrow E \otimes B' \longrightarrow A[1] \otimes B'$$

and

$$B \otimes A' \longrightarrow D \otimes A' \longrightarrow F \otimes A' \longrightarrow B[1] \otimes A'$$

in $\mathrm{D^b}(X)$ which is again a triangle. Therefore our functor is exact and we have indeed defined a tensor structure.
Step 2. Let us have a look at the spectrum of $\mathrm{D^b}(X \amalg X)$ with $*$. Let $P \in \mathrm{Spc}(\mathrm{D^b}(X \amalg X))$: Since P is a triangulated thick subcategory in $\mathrm{D^b}(X \amalg X)$ it is immediate that P decomposes into a direct product $P = P_1 \oplus P_2$ where P_1, P_2 are thick triangulated subcategories in $\mathrm{D^b}(X)$. Because P is a prime $\otimes$-ideal one sees immediately from the definition of the tensor structure that P_1 is a prime in $\mathrm{D^b}(X)$. Since P is a $\otimes$-ideal we have

$$(A, B) * (C, D) \in P \qquad \forall (A, B) \in \mathrm{D^b}(X \amalg X), \; \forall \; (C, D) \in P$$

Therefore $(A \otimes D \oplus B \otimes C) \in P_2 \;\forall (A, B) \in \mathrm{D^b}(X \amalg X)$ and since P_2 is thick one has $A \otimes D \;\forall\, D \in P_2 \;\forall\, A \in \mathrm{D^b}(X)$ i.e. P_2 is a thick $\otimes$-ideal in $\mathrm{D^b}(X)$. Assume that $P_2 \neq \mathrm{D^b}(X)$. Then P_2 does not contain the unit element $\mathbf{1}$ in $\mathrm{D^b}(X)$ and therefore $(0, \mathbf{1}) \notin P$. But $(0, \mathbf{1}) * (0, \mathbf{1}) = (0, 0) \in P$, contradiction. Therefore $P_2 = \mathrm{D^b}(X)$ and every prime P in $\mathrm{Spc}(\mathrm{D^b}(X \amalg X))$ is of the form $Q \oplus \mathrm{D^b}(X)$ where $Q \in \mathrm{Spc}(\mathrm{D^b}(X))$. Now one defines a functor

$$F : \left(\mathrm{D^b}(X \amalg X) = \mathrm{D^b}(X) \oplus \mathrm{D^b}(X), *\right) \longrightarrow \left(\mathrm{D^b}(X), \otimes_X\right),\ (A, B) \mapsto A$$

This functor is obviously $\otimes$-triangulated and therefore induces

$$\mathrm{Spc}(F) : \mathrm{Spc}(\mathrm{D^b}(X)) \to \mathrm{Spc}(\mathrm{D^b}(X \amalg X)),\ Q \mapsto F^{-1}(Q) = Q \oplus \mathrm{D^b}(X)$$

which is the wanted isomorphism.
Step 3. X is not a Fourier-Mukai partner of $X \amalg X$. Assume the converse, then there exists an exact equivalence $F : \mathrm{D^b}(X) \oplus \mathrm{D^b}(X) \to \mathrm{D^b}(X)$. The category $\mathrm{D^b}(X) \oplus \mathrm{D^b}(X)$ is decomposable into the two triangulated subcategories $\mathrm{D^b}(X) \oplus 0$ and $0 \oplus \mathrm{D^b}(X)$ (for the definition of decomposable see [13], chapter 1). Since F is exact the essential "images" of these subcategories are triangulated subcategories in $\mathrm{D^b}(X)$ which define a decomposition of this category. But X is connected and therefore $\mathrm{D^b}(X)$ is indecomposable ([13], proposition 3.10). It follows w.l.o.g. that $F(\mathrm{D^b}(X) \oplus 0)$ contains no object non-isomorphic to zero. But then $F^{-1}F((\mathbf{1}, 0)) \cong F^{-1}((0, 0)) = (0, 0)$, contradiction, since there is no isomorphism between $\mathbf{1}$ and 0 in $\mathrm{D^b}(X)$. $\square$

Now we will apply this approach to a slightly different situation. Let X and Y be two irreducible projective schemes such that $\#\mathrm{FM}(X) = \#\mathrm{FM}(Y)$=1 and there exists a non-trivial smooth morphism $f : X \to Y$. We will use this data to show that the map $\phi : \mathrm{FM}(Y \amalg X) \to \mathrm{TS}(Y \amalg X)$ is not surjective. In order to do this we will first the following

Lemma 4.0.10. *Let V and W be FM-partners, m_V, m_W be the number of connected components of V and W. Then $m_V = m_W$ and the equivalence between $\mathrm{D^b}(V)$ and $\mathrm{D^b}(W)$ restricts to equivalences between the components.*

Proof. First assume that $m_V = 1$. Then V is connected so $\mathrm{D^b}(V)$ is indecomposable and an argument similar to that used in Step 3 of 4.0.9 shows that W must be connected as well.
Next assume that $m_V = 2$ and $m_W \geq 3$. W.l.o.g. we consider the case $m_W = 3$. So we have the connected components V_1, V_2 of V, W_1, W_2, W_3 of W and an equivalence $F : \mathrm{D^b}(V_1) \oplus \mathrm{D^b}(V_2) \to \mathrm{D^b}(W_1) \oplus \mathrm{D^b}(W_2) \oplus \mathrm{D^b}(W_3)$. Consider the exact functor

$$F \circ I : \mathrm{D^b}(V_1) \to \mathrm{D^b}(W_1) \oplus \mathrm{D^b}(W_2) \oplus \mathrm{D^b}(W_3)$$

where $I : \mathrm{D^b}(V_1) \to \mathrm{D^b}(V_1) \oplus \mathrm{D^b}(V_2)$ is the natural (exact) embedding. This functor defines an equivalence of $\mathrm{D^b}(V_1)$ and its essential image $F \circ I(\mathrm{D^b}(V_1))$. Since V_1 is connected we conclude w.l.o.g. that $F \circ I(\mathrm{D^b}(V_1)) \subset \mathrm{D^b}(W_1)$ and therefore its essential image is a triangulated subcategory in $\mathrm{D^b}(W_1)$. The same argument applied to F^{-1} and $\mathrm{D^b}(W_1)$ shows that $F^{-1}(\mathrm{D^b}(W_1))$ must be contained in $\mathrm{D^b}(V_1)$ or $\mathrm{D^b}(V_2)$. But since F and F^{-1} are quasi-inverse to each other so for every object A in $\mathrm{D^b}(V_1)$ we have $F \circ F^{-1}(A) \cong A$ and there are no morphisms between objects in $\mathrm{D^b}(V_1)$ and $\mathrm{D^b}(V_2)$ we conclude that $F^{-1}(\mathrm{D^b}(W_1)) \subset \mathrm{D^b}(V_1)$. So our equivalences restrict to $F : \mathrm{D^b}(V_1) \to \mathrm{D^b}(W_1)$ and $F^{-1} : \mathrm{D^b}(W_1) \to \mathrm{D^b}(V_1)$ which are equivalences as well. That the functors are fully faithful is clear and as to the essential surjectivity: Let B be an arbitrary object in $\mathrm{D^b}(W_1)$ and assume that B is not isomorphic to $F(A)$ for all A in $\mathrm{D^b}(V_1)$. But then $F \circ F^{-1}(B) \cong B$ which lies in the isomorphy hull of $F(\mathrm{D^b}(V_1))$, contradiction. Similarly for F^{-1}. Thus the restricted F gives an equivalence between $\mathrm{D^b}(V_1)$ and $\mathrm{D^b}(W_1)$. Because of that one immediately gets that w.l.o.g. $F(\mathrm{D^b}(V_2)) \subset \mathrm{D^b}(W_2)$ and then an equivalence of $\mathrm{D^b}(V_2)$ and $\mathrm{D^b}(W_2)$ via F. Therefore $\mathrm{D^b}(W_3) = 0$, $W_3 = \emptyset$ and therefore $m_V = m_W = 2$. Induction proves the claim. $\square$

It is immediate from the lemma that if $\#\mathrm{FM}(X) = \#\mathrm{FM}(Y)$=1 then we also have $\#\mathrm{FM}(Y \amalg X) = 1$. Yet there is more than one tensor structure on $\mathrm{D}(Y \amalg X)$. Define:

$$(A, B) * (C, D) := (A \otimes_Y C, f^*(C) \otimes_X B \oplus f^*(A) \otimes_X D)$$

and note that this is indeed a tensor structure since f^* is compatible with $\otimes$ and is an exact functor. Therefore literally the same proof as in Step 2 of 4.0.9 applies.

Unfortunately the spectrum of $\mathrm{D^b}(Y \amalg X)$ is not as easily described as in 4.0.9: Let P be a thick prime ideal in $\mathrm{D^b}(Y \amalg X)$. Then, as before, it is immediate that P is a product of thick triangulated subcategories P_1 and P_2 in $\mathrm{D^b}(Y)$ and $\mathrm{D^b}(X)$ respectively. It is also clear that $P_1 \in \mathrm{Spc}(\mathrm{D^b}(Y))$. Since P is prime we again have that $\mathbf{1}_X \in P_2$. Therefore $f^*(\mathrm{D^b}(Y)) \subset P_2$. Since P_2 is thick we also have that $f^*(A) \otimes B \; \forall \, B \in P_2$ and $\forall \, A \in \mathrm{D^b}(Y)$.

One possible approach to describe the spectrum of $\mathrm{D^b}(Y \amalg X)$ in this case would be to find conditions on f and/or X, Y so that f^* is an equivalence (then the spectrum would be $\mathrm{Spc}(\mathrm{D^b}(Y))$). It is unclear how often this will be possible: For example assume X and Y to be smooth projective varieties over $\mathbb{C}$ and f^* to be an equivalence. Then f^* is a *Fourier-Mukai-transform* i.e. there exists an object $P \in \mathrm{D^b}(X \times Y)$ such that $f^*(E) = p_x(P \otimes p_y^*(E)) \; \forall \, E \in \mathrm{D^b}(Y)$ where p_x, p_y denote the projections. But $f^*(\mathcal{O}_Y) = \mathcal{O}_X$ and the right hand side rarely has this property.

Of course we can weaken our assumptions on f^*. If $f^*(\mathrm{D^b}(Y))$ were to contain elements which generate $\mathrm{D^b}(X)$ as a triangulated category the situation would become as clear as in the previous case. For example consider a map $f : \mathbb{P}^n_{\mathbb{C}} \to Y$ with Y smooth and projective. If the induced map on the Picard groups is surjective then $\mathcal{O}(-n), \ldots, \mathcal{O}$ are elements in $f^*(\mathrm{D^b}(Y))$ and since these elements generate $\mathrm{D^b}(\mathbb{P}^n_{\mathbb{C}})$ as a triangulated category we are done.

Chapter 5

Examples and applications

5.1 Examples of prime ideals

Example 5.1.1. Let us consider the general construction in a specific example, namely take $X = \mathrm{Spec}(\mathbb{Z})$. Then $\mathrm{D}^{\mathrm{perf}}(X) = \mathrm{K}^{\mathrm{b}}(\mathbb{Z} - \mathrm{proj})$, where $\mathrm{K}^{\mathrm{b}}(\mathbb{Z} - \mathrm{proj})$ denotes the homotopy category of bounded complexes of finitely generated projective $\mathbb{Z}$-modules. Since a projective $\mathbb{Z}$-module is free, we get:

$$\mathrm{D}^{\mathrm{perf}}(X) = \mathrm{K}^{\mathrm{b}}(\mathbb{Z} - \mathrm{mod}_f^f)$$

where $\mathbb{Z}{-}\mathrm{mod}_f^f$ denotes free $\mathbb{Z}$-modules of finite rank. The homeomorphism of X and $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X))$ sends a prime ideal $\mathcal{P}$ in X to the class of all those complexes $A^\bullet$ such that $A^\bullet_{\mathcal{P}} \cong 0$ in $\mathrm{D}^{\mathrm{b}}(\mathbb{Z}_{\mathcal{P}})$. Thus we see that the prime ideal corresponding to $\mathcal{P}$ contains all those complexes which become acyclic when localized in $\mathcal{P}$. For example take $\mathcal{P} = (2)$ then the complex $[\ldots 0 \longrightarrow \mathbb{Z} \xrightarrow{5} \mathbb{Z} \longrightarrow 0 \ldots]$ becomes exact when localized in (2). In general a necessary condition for this is that

$$\forall n \in \mathbb{Z}: \ \mathrm{rk}(\mathrm{im}(d^{n-1})) = \mathrm{rk}(\ker(d^n))$$

Another way of stating this is:
For an arbitrary $n \in \mathbb{Z}$ let

$$\mathrm{im}(d^{n-1}) = \mathrm{span}_{\mathbb{Z}}(v_1, ..., v_k)$$

and

$$\ker(d^n) = \mathrm{span}_{\mathbb{Z}}(w_1, ..., w_k)$$

Then $A^\bullet$ becomes acyclic at A^n when localized in $\mathcal{P}$ iff

$$\mathrm{span}_{\mathbb{Z}_P}(v_1, ..., v_k) = \ker(d^n) = \mathrm{span}_{\mathbb{Z}_P}(w_1, ..., w_k)$$

Now let us have a look at the generic point of $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X))$. From the above discussion we see that it contains all those complexes which become acyclic when tensorized with $\mathbb{Q}$. On the other hand ([3], Proposition 2.18) shows that the generic point x of an irreducible closed subset in $\mathrm{Spc}(\mathcal{T})$ is given by:

$$x = \{A \in \mathcal{T} \text{ such that } U(A) \cap Z \neq \emptyset\}$$

Since in our case $Z = \mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X))$ we get:

$$x = \{A \in \mathcal{T} \text{ such that } U(A) \neq \emptyset\}$$

Since $U(A) \neq \emptyset$ iff A does not generate $\mathcal{T}$ as a thick $\otimes$-ideal we have that only complexes which do not become acyclic when tensorized with $\mathbb{Q}$ generate $\mathrm{K}^{\mathrm{b}}(\mathbb{Z}_f^f)$ as a thick tensor ideal.

In the following we give an example of a prime ideal in an "ungeometric" situation.

Proposition 5.1.2. $\mathcal{P} = \{A^\bullet \in \mathcal{T}, A^\bullet \otimes \mathbb{Q} = 0\}$ *is a prime ideal in* $\mathcal{T} = \mathrm{K}(\mathbb{Z}-$ mod).

Proof. It is clear that $\mathcal{P}$ is thick and closed under shifts and isomorphisms. It is a $\otimes$-ideal because $\otimes$ is commutative and associative: If $A^\bullet \in \mathcal{P}$ and $B^\bullet \in \mathcal{T}$ then

$$(A^\bullet \otimes B^\bullet) \otimes \mathbb{Q}^\bullet = (A^\bullet \otimes \mathbb{Q}^\bullet) \otimes B^\bullet = 0$$

To see that $\mathcal{P}$ is prime consider two complexes $A^\bullet$ and $B^\bullet$ which are not in $\mathcal{P}$. This means that the complexes of $\mathbb{Q}$-vector spaces $A^\bullet \otimes \mathbb{Q}$ and $B^\bullet \otimes \mathbb{Q}$ are not homotopy equivalent to the zero complex. This in turn means that these complexes are not acyclic. It then follows from the Künneth formula that $A^\bullet \otimes B^\bullet$ is again not acyclic (see the proof of 5.2.1). ☐

Remark 5.1.3. It is not clear whether the spectrum of $\mathcal{T}$ is just a point. For example the full subcategory of complexes of finite abelian groups $\mathcal{G}$ is a $\otimes$-ideal, prime, is closed under shifts and thick. But it is not closed under isomorphisms: Consider the complexes

$$A^\bullet = [\,\dots \longrightarrow 0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \longrightarrow \dots]$$

and

$$B^\bullet = [\,\dots \longrightarrow 0 \longrightarrow \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} \longrightarrow 0 \longrightarrow \dots]\,.$$

Define $f^\bullet : A^\bullet \to B^\bullet$ to be the natural injection in each term and $g^\bullet : B^\bullet \to A^\bullet$ to be the natural projection in each term. These are clearly chain maps. Then $g^\bullet \circ f^\bullet = \mathrm{id}$ and $f^\bullet \circ g^\bullet$ sends (x, y) to $(0, y)$ so $f^\bullet \circ g^\bullet - \mathrm{id}$ sends (x, y) to $(-x, 0)$. By setting $s : \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z},\ (x, y) \mapsto (-x, 0)$ we see that $f^\bullet \circ g^\bullet - \mathrm{id}$ is homotopic to zero so that $A^\bullet$ and $B^\bullet$ are isomorphic in $\mathcal{T}$ and therefore $\mathcal{G}$ is not closed under isomorphisms.

5.2 Spectrum of the homotopy category of vector spaces

Theorem 5.2.1. *Fix an arbitrary field* $\mathbb{K}$. *We consider* $\mathcal{T} = K(\mathcal{A})$ *where* $\mathcal{A} = \mathbb{K} - \mathrm{Mod}$. *We know from 2.2.15 that this is a tensor triangulated category. Then:*

1. *A proper thick* $\otimes$*-ideal* $\mathcal{I}$ *in* $\mathcal{T}$ *cannot contain a complex which is not acyclic.*

2. *The identity map of an acyclic complex of $\mathbb{K}$-vector spaces is homotopic to the zero map.*

3. *The isomorphy hull of the zero complex is a prime ideal.*

From all this it follows that the spectrum of $\mathcal{T}$ is just a point.

Proof. **1.** Suppose $\mathcal{I}$ contains a complex $C^\bullet$ which is not acyclic. Then there is an index n such that $\operatorname{im}(d^{n-1})$ is a proper sub vector space in $\ker(d^n)$. Take a vector $x \in C^n$ which is in the kernel but not in the image. Take a basis of $\operatorname{im}(d^{n-1})$, complete it via x to a basis B' of $\ker(d^n)$ and then complete B' to a basis B'' of C^n. Denote by B the set $B'' \setminus \{x\}$. Now we have:

$$C^n \cong \mathbb{K} \cdot x \oplus \operatorname{span}_{\mathbb{K}} B \cong \mathbb{K} \oplus V$$

Modulo this isomorphim $d^n = 0 \oplus \tilde{d}^n$ where $\tilde{d}^n$ denotes the restriction of d^n to $\operatorname{span}(B)$. Since $\operatorname{im}(d^{n-1}) \subset \operatorname{span}(B)$ we see that we can decompose write $C^\bullet = \hat{C}^\bullet \oplus \mathbb{K}^\bullet$ where $\hat{C}^\bullet = C^\bullet$ except in degree n, $\hat{C}^n = V$ and $\hat{d}^n = \tilde{d}^n$ and $\mathbb{K}^\bullet = \mathbf{1}[-n]$. Now since $\mathcal{I}$ is thick it contains $\mathbb{K}^\bullet$; since it is triangulated it contains $\mathbf{1}$ and then $\mathcal{I} = \mathcal{T}$ because $\mathcal{I}$ is a $\otimes$-ideal. So we have a contradiction.
2. Consider an acyclic complex $C^\bullet$ and an arbitrary $n \in \mathbb{Z}$:

$$[\ldots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \ldots]$$

Choose a basis B' of $\operatorname{im}(d^n) \subset C^{n+1}$ and complete it to a basis B'' of C^{n+1}. Define $s^n : C^{n+1} \to C^n$ to be zero for all $b'' \in B'' \setminus B'$ and for each $b' \in B'$ define $s^n(b')$ to be an arbitrary element $\widetilde{b} \in C^n$ such that $d(\widetilde{b}) = b$. This gives a linear map such that $dsd = d$ and identifies $\operatorname{im}(d^n)$ with a subspace $U = \operatorname{span}(\widetilde{B})$ of C^n (as B' is identified with $\widetilde{B}$). Now from the homomorphy theorem and the exactness we get: $C^n / \operatorname{im}(d^{n-1}) \cong C^n / \ker(d^n) \cong \operatorname{im}(d^n) \cong U$ so that $C^n \cong \operatorname{im}(d^{n-1}) \oplus U$. We have to show that $\operatorname{id} = ds + sd$ for all $n \in \mathbb{Z}$. Denote by B a basis of $\operatorname{im}(d^{n-1})$. Then every element in C^n can be written as:

$$x = \sum_i \lambda_i b_i + \sum_j \mu_j \widetilde{b}_j$$

where $b_i \in B \; \forall\, i$ and $\widetilde{b}_j \in \widetilde{B} \; \forall\, j$. Now

$$ds(x) = d(\sum_i \lambda_i s(b_i)) = \sum_i \lambda_i b_i$$

and similarly

$$sd(x) = s(\sum_j \mu_j d(\widetilde{b}_j)) = \sum_j \mu_j \widetilde{b}_j$$

so that $ds + sd = \operatorname{id}$.
3. Denote the isomorphy hull of the zero complex by $\widetilde{0}$. Considered as a (full) subcategory $\widetilde{0}$ is clearly closed under shifts, isomorphisms and taking cones so it is a triangulated subcategory. It is clear that it is thick as well. To see that $\widetilde{0}$ is a prime $\otimes$-ideal we use the Künneth formula which, since we are dealing with vector spaces, yields: If $A^\bullet$ and $B^\bullet$ are two complexes then one has

$$H_n(A^\bullet \otimes B^\bullet) = \bigoplus_{i+j=n} H_i(A^\bullet) \otimes H_j(B^\bullet).$$

This formula immediately gives that $\widetilde{0}$ is a tensor ideal. Further, assume that $A^\bullet$ and $B^\bullet$ are not acyclic. Then there exist i, j such that $H_i(A^\bullet) \neq 0$ and $H_j(B^\bullet) \neq 0$ so the Künneth formula states that the complex $A^\bullet \otimes B^\bullet$ is also not acyclic and therefore not zero in $\mathcal{T}$ so $\widetilde{0}$ is indeed prime. $\square$

Remark 5.2.2. If we equip $\mathrm{Spc}(\mathrm{K}(\mathbb{K}-\mathrm{Mod}))$ with a structure sheaf as in 2.5.24 then the global sections are the endomorphism ring of the 0-complex $\mathbb{K}$, i.e. $\mathbb{K}$. Therefore $\mathrm{Spec}(\mathrm{K}(\mathbb{K}-\mathrm{Mod}))$ is isomorphic to $\mathrm{Spec}(\mathbb{K})$ as a scheme.

Remark 5.2.3.

1. The above proof shows that $\mathrm{Spc}(\mathrm{K}(\mathbb{K}-\mathrm{Mod}))$ is a point. From [3] we already knew that for $X = \mathrm{Spec}(\mathbb{K})$ we have

$$X = \mathrm{Spec}(\mathrm{D}^{\mathrm{perf}}(X)) = \mathrm{Spec}(\mathrm{D}^{\mathrm{b}}(\mathbf{Coh}(X))) = \mathrm{Spec}(\mathrm{K}^{\mathrm{b}}(\mathbb{K}-\mathrm{mod}))$$

The above result is therefore a slight generalisation.

2. For more general rings the zero complex need not be a prime ideal. Consider e.g. $\mathrm{K}(\mathbb{Z}-\mathrm{Mod})$ and the 0-complexes $A^\bullet = (\mathbb{Z}/2\mathbb{Z})$ and $B^\bullet = (\mathbb{Z}/3\mathbb{Z})$. Then $A^\bullet \otimes B^\bullet = 0$ although $A^\bullet \neq 0$ and $B^\bullet \neq 0$ in $\mathrm{K}(\mathbb{Z}-\mathrm{Mod})$.

3. In general the class of all acyclic complexes need not be a $\otimes$-ideal. Again consider $\mathrm{K}(\mathbb{Z}-\mathrm{Mod})$ and the acyclic complex:

$$C^\bullet = [\ldots \to 0 \to \mathbb{Z} \stackrel{\cdot 2}{\to} \mathbb{Z} \stackrel{\pi}{\to} \mathbb{Z}/2\mathbb{Z} \to 0 \to \ldots]$$

where π denotes the canonical projection. Tensorizing $C^\bullet$ with $A^\bullet$ as in remark 2. yields the complex:

$$C^\bullet \otimes A^\bullet = [\ldots \to 0 \to \mathbb{Z}/2\mathbb{Z} \stackrel{0}{\to} \mathbb{Z}/2\mathbb{Z} \stackrel{\mathrm{id}}{\to} \mathbb{Z}/2\mathbb{Z} \to 0 \to \ldots]$$

which is not acyclic. This shows in particular that the triangulated functor $\mathrm{K}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ is in general not *tensor*-triangulated.

5.3 Subrings in $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^1_{\mathbb{C}})$

Consider $X = \mathbb{P}^1_{\mathbb{C}}$ and $\mathcal{T} = \mathrm{D}^{\mathrm{b}}(\mathbb{P}^1_{\mathbb{C}})$. We know that as a triangulated category $\mathcal{T}$ is generated by $\mathcal{O}(-1)$ and $\mathcal{O}$, see [5]. This means that the smallest triangulated category containing $\mathcal{O}(-1)$ and $\mathcal{O}$ and which is closed under shifts and taking cones is equivalent to $\mathcal{T}$. So basically, up to isomorphism, every object in $\mathcal{T}$ can be reached by shifting $\mathcal{O}(-1)$ and $\mathcal{O}$, taking a cone of an arbitrary morphism and repeating this procedure a finite number of times. Now quite a natural question is to ask what one gets if $\mathcal{O}(-1)$ is replaced by $\mathcal{O}(k)$, $k \neq -1$. Since we are interested in "subrings" of $\mathcal{T}$ we will also ask this category to be closed under the tensor product.

Definition 5.3.1. $\mathcal{T}_n := \langle \mathcal{O}, \mathcal{O}(n) \rangle_\otimes :=$ the smallest tensor triangulated subcategory of $\mathcal{T}$ containing $\mathcal{O}$ and $\mathcal{O}(n)$.

Recall that since X is a regular variety of dimension 1 for every coherent sheaf $\mathcal{F}$ on X there exists a resolution

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{E}_i$ are locally free sheaves (see [11], III, Ex. 6.9). Then one can define the *determinant* of $\mathcal{F}$ to be the line bundle

$$\det(\mathcal{F}) = (\Lambda^{r_0}\mathcal{E}_0) \otimes (\Lambda^{r_1}\mathcal{E}_1)^{-1} \in \mathrm{Pic}(X)$$

where r_i is the rank of $\mathcal{E}_i$ and Λ denotes the exterior power. Obsiously the determinant of a locally free sheaf is its highest exterior power, e.g. $\det(\mathcal{O}(2) \oplus \mathcal{O}(7)) = \mathcal{O}(9)$. The determinant of a skyscraper sheaf $k(P)$ of a closed point P is $\mathcal{O}(1)$ since for $P = (a : b)$ we have the exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \xrightarrow{a \cdot x_1 - b \cdot x_0} \mathcal{O} \longrightarrow k(P) \longrightarrow 0$$

where the last map is the evaluation.
Now since $\det(\mathcal{F}) \in \mathrm{Pic}(X)$ it is of the form $\mathcal{O}(k)$ for a $k \in \mathbb{Z}$. We define the *degree* of $\mathcal{F}$ to be k. This degree function determines an isomorphism between $\mathrm{Pic}(X)$ and $\mathbb{Z}$.
If $A^\bullet$ is a complex in $\mathcal{T}$ we define the degree of $A^\bullet$ to be the (finite) sum:

$$\deg(A^\bullet) = \sum_{i=-\infty}^{\infty} (-1)^i \deg(A^i)$$

This defines a group homomorphism $K_0(\mathcal{T}) \to \mathbb{Z}$ since this map is the composition of the group homomorphisms $K_0(\mathcal{T}) \xrightarrow{\Theta} K_0(X) \xrightarrow{\det} \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z}$ (for the definition of Θ see remark 3.1.3). Defining the *rank* of $A^\bullet$ to be

$$\mathrm{rk}(A^\bullet) = \sum_{i=-\infty}^{\infty} (-1)^i \mathrm{rk}(A^i)$$

we get a ring homomorphism $K_0(\mathcal{T}) \overset{(\deg,\mathrm{rk})}{\longrightarrow} \mathbb{Z} \oplus \mathbb{Z}$ where the multiplication in $\mathbb{Z} \oplus \mathbb{Z}$ is described by proposition 3.2.3.

Definition 5.3.2. $\mathcal{C}_n = \{A^\bullet \in \mathcal{T} \mid \deg(A^\bullet) \equiv 0 \pmod{n}\}$.

Remark 5.3.3. $\mathcal{C}_n$ is a dense "subring" in $\mathcal{T}$, i.e. a tensor triangulated dense subcategory containing the unit. To prove this it will be sufficient to see that $\mathcal{C}_n = \mathcal{C}_R$ where $R = n\mathbb{Z} \oplus \mathbb{Z}$ using notation of 3.1.2. Now what we actually have to prove is that the condition on the degree allows arbitrary ranks. To see this consider, for an arbitrary $i \in \mathbb{N}$, the sheaf $\mathcal{F} = \mathcal{O}(-1)^i \oplus k(P)^i$ which has degree 0 and rank i. Considering $\mathcal{F}$ as a complex and shifting it to the left we get a complex with degree 0 and rank $-i$.

Lemma 5.3.4. *Consider $\mathcal{T} = \mathrm{K^b}(\mathcal{A})$ or $\mathcal{T} = \mathrm{D^b}(\mathcal{A})$ (where $\mathcal{A}$ is an abelian category), $\mathcal{C}$ a full triangulated subcategory of $\mathcal{T}$ and $A^\bullet \in \mathcal{T}$ a complex such that all terms A^i of $A^\bullet$ are in $\mathcal{C}$. Then $A^\bullet \in \mathcal{C}$.*

Proof. We use the same technique as in the proof of proposition 3.2.1 and the fact that $\mathcal{C}$ is closed under shifts and taking cones. □

Theorem 5.3.5. *The categories $\mathcal{T}_n$ and $\mathcal{C}_n$ are equal.*

Proof. Due to proposition 3.1.2 is will be sufficient to see that $\mathcal{T}_n$ is dense in $\mathcal{T}$ because its image in $K_0(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ is obviously $n\mathbb{Z} \oplus \mathbb{Z}$. Since $\mathcal{T}_n$ is closed under the tensor product the collection $\{\mathcal{O}(a \cdot n) \mid a \in \mathbb{N}\}$ is in $\mathcal{T}_n$. Now for $m, k \in \mathbb{Z}$ we consider the following exact sequences in $\mathbf{Coh}(\mathbb{P}^1_{\mathbb{C}})$ (see [10]):

$$(1)\ \ 0 \longrightarrow \mathcal{O}(k)^{m-k-1} \longrightarrow \mathcal{O}(k+1)^{m-k} \longrightarrow \mathcal{O}(m) \longrightarrow 0 \quad \text{if } m > k+1$$

$$(2)\ \ 0 \longrightarrow \mathcal{O}(m) \longrightarrow \mathcal{O}(k)^{k-m+1} \longrightarrow \mathcal{O}(k+1)^{k-m} \longrightarrow 0 \quad \text{if } m < k$$

Consider the case $n > 1$ first. Set $m = n$ and $k = 2n$. Then the second sequence reads:

$$0 \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}(2n)^{n+1} \longrightarrow \mathcal{O}(2n+1)^{n} \longrightarrow 0$$

This exact sequence gives a triangle in $\mathcal{T}$ and as the first two terms are in $\mathcal{T}_n$ by assumption then so is the third, namely $\mathcal{O}(2n+1)^n$. Now set $m = -n$ and $k = 2n$, then (2) reads

$$0 \longrightarrow \mathcal{O}(-n) \longrightarrow \mathcal{O}(2n)^{3n+1} \longrightarrow \mathcal{O}(2n+1)^{2n} \longrightarrow 0$$

and by the same argument as before $\mathcal{O}(-n) \in \mathcal{T}_n$. Therefore $\{\mathcal{O}(b \cdot n) \mid b \in \mathbb{Z}\} \subset \mathcal{T}_n$. Now $\mathcal{O}(1)^n = \mathcal{O}(-2n) \otimes \mathcal{O}(2n+1)^n \in \mathcal{T}_n$. In the next step we set $m = n$ and $k = 2n-1$ and get

$$0 \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}(2n-1)^{n} \longrightarrow \mathcal{O}(2n)^{n-1} \longrightarrow 0$$

so $\mathcal{O}(2n-1)^n \in \mathcal{T}_n$ and therefore $\mathcal{O}(-1)^n \in \mathcal{T}_n$. It follows that for all $l \in \mathbb{Z}$ there exists an $x_l \in \mathbb{N}$ such that $\mathcal{O}(l)^{(x^l)} \in \mathcal{T}_n$, namely $x_l = n^p$ where $p \equiv |l| (\mathrm{mod}(n))$. This can be seen by considering $\mathcal{O}(1)^n \otimes \mathcal{O}(1)^n = \mathcal{O}(2)^{(n^2)}$, $\mathcal{O}(-1)^n \otimes \mathcal{O}(-1)^n = \mathcal{O}(-2)^{(n^2)}$, $\mathcal{O}(1)^n \otimes \mathcal{O}(n) = \mathcal{O}(n+1)^n$ etc.
Now take an arbitrary coherent sheaf $\mathcal{F}$. Considering the locally free resolution

$$0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

and using that $\mathcal{E}_0$ and $\mathcal{E}_1$ are direct sums of twisted sheaves $\mathcal{O}(k)$ we look at the exact sequence

$$0 \longrightarrow \mathcal{E}_1^s \longrightarrow \mathcal{E}_0^s \longrightarrow \mathcal{F}^s \longrightarrow 0$$

where s is big enough so all direct summands of $\mathcal{E}_0$ and $\mathcal{E}_1$ are elements in $\mathcal{T}_n$ and therefore $\mathcal{E}_1^s$ and $\mathcal{E}_0^s$ are elements in $\mathcal{T}_n$. Then $\mathcal{F}$ is also an element in $\mathcal{T}_n$. So we proved the following:
If $n > 1$ then for all coherent sheaves $\mathcal{F}$ there exists a coherent sheaf $\mathcal{G}$ such that $\mathcal{F} \oplus \mathcal{G} \in \mathcal{T}_n$.
Now we will show that $\mathcal{T}_n = \mathcal{T}_{-n}\ \forall\, n \in \mathbb{N}$. We have already seen that $\mathcal{T}_{-n} \subset \mathcal{T}_n\ \forall\, n > 1$. Now let $n = 1$. Consider sequence (1) with $m = 1$ and $k = -1$. Then we get:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}^2 \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

and therefore $\mathcal{O}(-1) \in \mathcal{T}_1$ so $\mathcal{T}_1 = \mathcal{T}_{-1} = \mathcal{T} = \mathcal{C}_1$.
Finally let $-n < -1$. Consider sequence (1) again, now with $m = -n$, $k = -2n$:

$$0 \longrightarrow \mathcal{O}(-2n)^{n-1} \longrightarrow \mathcal{O}(-2n+1)^{n} \longrightarrow \mathcal{O}(-n) \longrightarrow 0$$

Then $\mathcal{O}(-2n+1)^n \in \mathcal{T}_{-n}$. Now set $m = n$ and $k = -2n$:

$$0 \longrightarrow \mathcal{O}(-2n)^{3n-1} \longrightarrow \mathcal{O}(-2n+1)^{3n} \longrightarrow \mathcal{O}(n) \longrightarrow 0$$

so $\mathcal{O}(n) \in \mathcal{T}_{-n}$ and therefore $\mathcal{T}_n \subset \mathcal{T}_{-n}$ as desired.
So we have shown that for every complex $A^\bullet \in \mathcal{T}$ there exists a complex $B^\bullet \in \mathcal{T}$ such that $A^\bullet \oplus B^\bullet \in \mathcal{T}_n$. The previous lemma therefore gives that $\mathcal{T}_n$ is dense in $\mathcal{T}$ and because $\mathcal{T}_n$ and $\mathcal{C}_n$ define the same subring of $K_0(X)$ they are equal by proposition 3.1.2. ☐

Corollary 5.3.6. $\mathrm{Spc}(\mathcal{T}_n) = \mathrm{Spc}(\mathcal{T}) = \mathbb{P}^1_{\mathbb{C}}$.

Remark 5.3.7. If $k(P)$ is the skyscraper sheaf of a closed point then the exact sequence

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow k(P) \longrightarrow 0$$

gives that the smallest subring of $\mathcal{T}$ containing $\mathcal{O}$ and $k(P)$ is $\mathcal{T}$.

Question: Do we have that $\mathrm{Spec}(\mathcal{T}_n) = \mathrm{Spec}(\mathcal{T})$ as ringed spaces? Probably not since this would mean that there is no "smallest" subcategory of $\mathcal{T}$ possessing all information to recover the scheme structure of $\mathbb{P}^1_{\mathbb{C}}$.

5.4 Semistable sheaves on an elliptic curve

In the following C will denote an elliptic curve, i.e. a smooth projective curve of genus 1 over an algebraically closed field $\mathbb{K}$. Recall the following

Definition 5.4.1. Let $\mathcal{F}$ be a torsion free coherent sheaf on C. $\mathcal{F}$ is called *semistable* if for every non-trivial subsheaf $\mathcal{F}'$ of $\mathcal{F}$ we have that

$$\frac{\deg(\mathcal{F}')}{\mathrm{rk}(\mathcal{F}')} =: \mu(\mathcal{F}') \leq \mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}$$

Equivalently, $\mathcal{F}$ is semistable if for every quotient sheaf $\mathcal{F}''$ we have that $\mu(\mathcal{F}'') \geq \mu(\mathcal{F})$. $\mathcal{F}$ is called *stable* if the inequality is strict. We call μ the *slope* of $\mathcal{F}$.

Remark 5.4.2. Since we consider sheaves on a curve, a semistable sheaf is automatically locally free. An example of a stable sheaf is a line bundle. For more general stability concepts of sheaves see [14].

Now let $\lambda \in \mathbb{Q}$. We will consider the full subcategory $\mathcal{A}_\lambda$ of $\mathbf{Coh}(C)$ consisting of semistable sheaves with slope λ. We will need the following

Lemma 5.4.3. *Let* $0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$ *be an exact sequence of coherent sheaves on* C*. Then the slopes are either strictly increasing* or *strictly decreasing* or *all equal.*

Proof. Let $\mu(\mathcal{F}_1) = \frac{d_1}{r_1}$, $\mu(\mathcal{F}_2) = \frac{d_2}{r_2}$ and $\mu(\mathcal{F}_3) = \frac{d_3}{r_3}$. Note that $d_3 = d_2 - d_1$ and $r_3 = r_2 - r_1$. Assume that $\mu(\mathcal{F}_1) < \mu(\mathcal{F}_2)$. Then we have:

$$\mu(\mathcal{F}_1) < \mu(\mathcal{F}_2) \Longleftrightarrow d_1 r_2 < d_2 r_1 \Longleftrightarrow d_2 r_2 - d_2 r_1 < d_2 r_2 - d_1 r_2 \Longleftrightarrow$$

$$\frac{d_2}{r_2} = \mu(\mathcal{F}_2) < \frac{d_2 - d_1}{r_2 - r_1} = \mu(\mathcal{F}_3)$$

The other cases are dealt with similarly. ☐

Corollary 5.4.4. *$\mathcal{A}_\lambda$ is closed under kernels and cokernels.*

Proof. A morphism $f : \mathcal{F} \to \mathcal{G}$ between semistable sheaves of slope λ gives us two exact sequences, namely

$$0 \longrightarrow \ker(f) \longrightarrow \mathcal{F} \longrightarrow \mathrm{im}(f) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{im}(f) \longrightarrow \mathcal{G} \longrightarrow \mathrm{coker}(f) \longrightarrow 0$$

Using the semistability of $\mathcal{F}$ and $\mathcal{G}$ and the previous lemma we deduce that the slopes of $\ker(f)$, $\mathrm{coker}(f)$ and $\mathrm{im}(f)$ are again equal to λ. Then the semistability of $\ker(f)$ and $\mathrm{coker}(f)$ follows immediately from the definitions. □

Proposition 5.4.5. *The category $\mathcal{A}_\lambda$ is closed under extensions in* $\mathbf{Coh}(C)$. *In particular, it is abelian.*

Proof. See [21], chapter 14. □

Remark 5.4.6. Consider $\lambda = 0$. Then the tensor product of two sheaves with slope 0 has again slope 0 (see 3.2.3) and it is semistable (see [14], theorem 3.1.4).

We will now consider the bounded derived category $\mathcal{T} := \mathrm{D}^b(\mathcal{A}_0)$. This category is tensor triangulated by the previous remark. Since the tensor product of locally free sheaves is exact in the abelian sense we do not need to derive it and the tensor product in $\mathcal{T}$ is the usual tensor product of complexes. Our aim is to describe the spectrum of $\mathcal{T}$. First we recall the

Definition 5.4.7. A sheaf $\mathcal{F}$ is called *indecomposable* if it cannot be written as a direct sum of two proper subsheaves. By definition every sheaf can be written as a direct sum of indecomposable sheaves.

Remark 5.4.8. In the following we will use some results on existence and properties of (semi)stable sheaves over an elliptic curve, see [1] for the original proofs or [12] for proofs of these results using Fourier-Mukai-transforms:

- For every $r \in \mathbb{N}$ there is a unique (up to isomorphy) semistable vector bundle E_r of rank r and degree 0.
- For all $r \in \mathbb{N}$ there is a short exact sequence

$$(*) \qquad 0 \longrightarrow \mathcal{O}_C \longrightarrow E_r \longrightarrow E_{r-1} \longrightarrow 0$$

In order to describe the spectrum we will need two lemmas.

Lemma 5.4.9. *If $\mathcal{F}$ is a decomposable semi-stable bundle with slope 0 and $\mathcal{F} = \bigoplus_j \mathcal{F}_j$ where $\mathcal{F}_j$ are all indecomposable then $\mu(\mathcal{F}_j) = 0 \,\forall\, j$.*

Proof. If one of the $\mathcal{F}_j$ has a negative slope, i.e. a negative degree, then another one has to have a positive degree and therefore a positive slope. This contradicts the semi-stability of $\mathcal{F}$. □

Lemma 5.4.10. *A complex $A^\bullet$ in $\mathcal{T}$ with indecomposable homology objects is a t-generator in $\mathcal{T}$, i.e. $\mathcal{J} = \langle A^\bullet \rangle = \mathcal{T}$ where $\langle A^\bullet \rangle$ denotes the smallest thick $\otimes$-ideal containing $A^\bullet$.*

Proof. First note that the homological dimension of $\mathcal{T}$ is ≤ 1 and therefore every complex is isomorphic to the cyclic (all differentials are zero) complex of its homology objects. We will show that each complex with the above property which is not isomorphic to the zero complex generates $\mathcal{T}$ as a thick $\otimes$-ideal. Since we are only interested in the ideal generated by our complex $A^\bullet$ we replace it by the isomorphic object $\bigoplus_{i\in\mathbb{Z}} H^i(A^\bullet)[-i]$. Because $\mathcal{J}$ is thick, it contains $H^i(A^\bullet)\ \forall\, i \in \mathbb{Z}$. Now these objects are by assumption indecomposable semi-stable bundles of slope 0 and at least one object is not equal to zero.

- If one of the homology objects is $\mathcal{O}$, i.e. E_1, $\mathcal{J}$ contains the unit and therefore $\mathcal{T} = \mathcal{J}$.
- Next we consider the case $r = 2$. Then $E_2 \otimes E_4 = E_3 \oplus E_5$ (see [12]) and since $E_3 \oplus E_5 \in \mathcal{J}$ and $\mathcal{J}$ is thick it contains E_3. Using the exact sequence $(*)$ for $r = 3$ we get a triangle in $\mathcal{T}$ such that two of the three objects are elements of $\mathcal{J}$. Therefore the third, i.e. $\mathcal{O}$, is contained in $\mathcal{J}$.
- $r \geq 3$. We consider $E_2 \otimes E_r = E_{r-1} \oplus E_{r+1}$ and with a similar sequence as in the case above we have that $\mathcal{O} \in \mathcal{J}$.

□

Theorem 5.4.11. *The spectrum of $\mathcal{T}$ is a point.*

Proof. Immediate from the previous two lemmas. It follows that the unique prime ideal of $\mathcal{T}$ is the isomorphy hull of the zero complex. That this is indeed a prime ideal can also be seen directly, e.g. as follows:
Take two complexes which are not isomorphic to zero in $\mathcal{T}$. They are isomorphic to the direct sum of their homology objects, each one has a homology object $\neq 0$ and this object is a semi-stable vector bundle of slope 0. Tensorizing these and localizing in a point $x \in C$ we see that the tensor product is again not acyclic since we get the tensor product of two non-zero free modules somewhere in the homology and therefore cannot be isomorphic to zero in $\mathrm{D}(C_x)$. □

Remark 5.4.12. In $\mathrm{D^b}(C)$ the isomorphy hull of the zero complex is not a prime ideal. To see this consider the homeomorphism in 2.5.23. If we show that for every $x \in C$ there is a non-acyclic complex (in particular not isomorphic to zero in $\mathrm{D^b}(C)$) which becomes exact localized in x we are done. But such a complex exists, e.g. the 0-complex $k(y)$ with $x \neq y$ (and y not the generic point of C). Therefore we see that 0 is not of the form $f(x)$ for some $x \in C$.

Proposition 5.4.13. *As a scheme* $\mathrm{Spec}(\mathcal{T})$ *is isomorphic to* $\mathrm{Spec}(\mathbb{K})$.

Proof. We will consider the global sections of the structure sheaf of $\mathrm{Spc}(\mathcal{T})$. In the notation of 2.5.24 we take $U = \mathrm{Spc}(\mathcal{T})$, then $\mathcal{T}^U = 0$. Since a morphism whose cone is 0 is an isomorphism we see that localizing in 0 is trivial. Therefore $\Gamma(\mathrm{Spc}(\mathcal{T}), \mathcal{O}_{\mathrm{Spc}(\mathcal{T})}) = \mathrm{End}_{\mathcal{T}}(\mathcal{O}_C) = \mathrm{End}_{\mathbf{Coh}(C)}(\mathcal{O}_C) = \mathbb{K}$. □

Remark 5.4.14. If we consider the natural inclusion functor $I : \mathcal{T} \to \mathrm{D^b}(C)$ then $\mathcal{T}$ is not a triangulated subcategory of $\mathrm{D^b}(C)$ because it is not closed under isomorphisms, take for example the complex $[k(P) \xrightarrow{i} \mathcal{O}_C \oplus k(P)]$ which in $\mathrm{D^b}(C)$ is isomorphic to its homology, i.e. $\mathcal{O}_C \in \mathcal{T}$. Taking the isomorphy hull of $\mathcal{T}$ in $\mathrm{D^b}(C)$ we get a triangulated category which will be denoted by $\mathcal{T}'$ and is generated by $\mathcal{O}_C$ as a triangulated category. Indeed, we use sequence $(*)$ in 5.4.10 for $r = 2$. This sequence gives a triangle in $\mathrm{D^b}(C)$ and after appropriate rotation of the triangle we see that E_2 is a cone of a certain morphism $f : \mathcal{O}_C[-1] \to \mathcal{O}_C$, i.e. E_2 is in the triangulated subcategory generated by $\mathcal{O}_C$. Inductively we get $E_r \,\forall\, r \in \mathbb{N}$. All other semistable bundles of slope 0 are direct sums of the E_r. Using lemma 5.3.4 we see that the triangulated subcategory which is generated by $\mathcal{O}_C$ contains $\mathcal{T}$ and must therefore be $\mathcal{T}'$.

5.5 Homogeneous bundles on abelian varieties

Definition 5.5.1. Let A be an abelian variety and for a fixed $x \in A$ denote by $T_x : A \to A$ the *translation* map sending an element $a \in A$ to $a + x$. A vector bundle E on A is called *homogeneous* if $T_x^*(E) \cong E \,\forall\, x \in A$.

In his book [21] Polishchuk showed that there is an equivalence between sheaves with finite support on A and homogeneous bundles on $\widehat{A}$, the dual abelian variety. Therefore the category of homogeneous bundles is an abelian category which will be denoted by $\mathcal{A}$. Furthermore $\mathcal{A}$ has a tensor product since for two homogeneous vector bundles E, E' and an arbitrary point $x \in A$ we have $E \otimes E' \cong T_x^*(E) \otimes T_x^*(E') \cong T_x^*(E \otimes E')$. Having established this we can consider $\mathrm{D^b}(\mathcal{A}) := \mathcal{T}$ which is of course tensor triangulated. Denote its isomorphy hull in $\mathrm{D^b}(A)$ again by $\mathcal{T}$.

Proposition 5.5.2. *Let $I : \mathcal{T} \to \mathrm{D^b}(A)$ be the inclusion functor. Then the induced map on spectra $\mathrm{Spc}(I) : \mathrm{Spc}(\mathrm{D^b}(A)) \to \mathrm{Spc}(\mathcal{T})$ sends everything a point.*

Proof. Indeed, take an element $A^\bullet$ in $\mathcal{T}$. By definition, its homology objects are homogeneous bundles. Assume that there exists a point $x \in A$ such that $A^\bullet_x$ is acyclic, i.e. $A^\bullet \in \mathrm{Spc}(I)(\mathcal{P}) = I^{-1}(\mathcal{P}) = \mathcal{P} \cap \mathcal{T}$ for a prime ideal $\mathcal{P}$ in $\mathrm{Spc}(\mathrm{D^b}(A))$. Then we have that the homology objects are zero when localized in x. But these objects are locally free sheaves over an integral scheme and therefore they must have been zero. This proves at once that the class of all acyclic complexes (i.e. the isomorphy hull of the zero complex) in $\mathcal{T}$ is a prime ideal since for every prime $\mathcal{P}$ in $\mathrm{Spc}(\mathrm{D^b}(A))$ we have that the prime $I^{-1}(\mathcal{P}) = \mathcal{P} \cap \mathcal{T} \subset \{\text{acyclic complexes}\}$ and since all these complexes are isomorphic to each other we have that $I^{-1}(\mathcal{P}) = \{\text{acyclic complexes}\}$. In particular, the class of all acyclic complexes is the only closed point in $\mathrm{Spc}(\mathcal{T})$. $\square$

Remark 5.5.3. The proposition provides us with a large class of objects which are t-generators but not units. By [8] we know that the set of all units in $\mathrm{D^b}(A)$ is isomorphic to $\mathrm{Pic}(A) \oplus \mathbb{Z}$. Now the proposition shows that every complex of homogeneous bundles is a t-generator but is not a unit if at least one of the objects has rank ≥ 2.

Remark 5.5.4. Consider an elliptic curve, i.e. $\dim(A) = 1$. Then we have that

$$\{\text{homogeneous bundles}\} = \{\text{semi-stable bundles with slope } 0\}.$$

Therefore in this case the spectrum is only a point.

Proof. "$\subseteq$" Let F be a unipotent bundle, i.e. there exists a filtration $0 \subset F_1 \subset \ldots \subset F_n = F$ such that $F_{i+1}/F_i \cong \mathcal{O}_A$. Obviously F_2 is an extension of $\mathcal{O}_A$ by $\mathcal{O}_A$ and is therefore a semi-stable bundle with slope 0. Using induction we have that F also has this property. Now an arbitrary homogeneous bundle F is of the form

$$(*) \qquad F = \bigoplus_i L_i \otimes F_i$$

where the F_i are unipotent bundles and the L_i are line bundles of degree 0 (see [16]).
"$\supseteq$" Now let E_r be the unique indecomposable semi-stable bundle with rank r and degree 0. Then E_r fits into an exact sequence

$$0 \longrightarrow E_{r-1} \longrightarrow E_r \longrightarrow \mathcal{O}_A \longrightarrow 0$$

(this is the dual sequence of the one mentioned in the previous section using that the E_r are self-dual)
Using induction we have that E_{r-1} is unipotent with filtration E^k_{r-1}. Then the filtration (E^k_{r-1}, E_{r-1}) shows that E_r is unipotent and an arbitrary semi-stable bundle is a direct sum of the E_r and hence homogeneous (put $L_i = \mathcal{O}_A$ in $(*)$). $\square$

Conjecture: The spectrum of the bounded derived category of homogeneous bundles on an abelian variety of dimension $n \geq 2$ is a point.

5.6 Linearized sheaves on a projective variety

Definition 5.6.1. Let X be a smooth projective variety over $\mathbb{C}$ with an action by a finite group G. A linearization of a coherent sheaf $\mathcal{F}$ on X is given by isomorphisms $\lambda_g : \mathcal{F} \xrightarrow{\sim} g^*\mathcal{F}$ for every $g \in G$ satisfying $\lambda_1 = \mathrm{id}$ and $\lambda_{gh} = h^*\lambda_g \circ \lambda_h$.

It follows that the category $\mathcal{A}$ of G-linearized sheaves is an abelian category with enough injectives. We refer to [20] for further properties. We will consider $\mathrm{D^G}(X) := \mathrm{D^b}(\mathcal{A})$. Note that this category is tensor-triangulated. Our goal is to describe its spectrum. In order to do this we recall the following definition taken from [3]

Definition 5.6.2. A *classifying support datum* on a tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$ is a pair (Y, σ) where Y is a topological space and σ an assignment associating to any object $A \in \mathcal{T}$ a closed subset $\sigma(A) \subset Y$ subject to the following rules:

(i) $\sigma(0) = \emptyset$ and $\sigma(\mathbf{1}) = Y$;

(ii) $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B)$;

(iii) $\sigma(TA) = \sigma(A)$;

(iv) $\sigma(A) \subset \sigma(B) \cup \sigma(C)$ for any triangle $A \longrightarrow B \longrightarrow C \longrightarrow TA$;

(v) $\sigma(A \otimes B) = \sigma(A) \cap \sigma(B)$;

(vi) Y is noetherian and every non-empty irreducible slosed subset $Z \subset Y$ has a unique generic point;

(vii) There is a bijection $\psi : \{W \subset Y \mid W = \bigcup_\alpha W_\alpha \text{ with } Y_\alpha \text{ closed}\} \xrightarrow{\sim} \{\mathcal{J} \subset \mathcal{T} \mid \mathcal{J} \text{ radical thick } \otimes\text{-ideal}\}$ defined by $W \mapsto \{A \in \mathcal{T} \mid \sigma(A) \subset W\}$ with inverse φ given by $\mathcal{J} \mapsto \sigma(\mathcal{J}) := \bigcup_{A \in \mathcal{J}} \sigma(A)$.

If (Y, σ) is a classifying support datum on $\mathcal{T}$ then ([3], Theorem 5.2.) gives that there is a homeomorphism $f : Y \to \mathrm{Spc}(\mathcal{T})$ where $f(x) := \{A \in \mathcal{T} \mid x \notin \sigma(A)\}$. Our aim is to use this theorem with appropriately chosen (Y, σ). First let us introduce some notation: The canonical continuous projection $X \to X/G$ will be denoted by π and for a complex $A^\bullet \in \mathrm{D^G}(X)$ $\mathrm{supph}(A^\bullet)$ will denote the homological support of $A^\bullet$, i.e. the union of the supports of the homology sheaves of $A^\bullet$. The proof of the following theorem will tacitly use the fact that the forgetful functor for : $\mathrm{D^G}(X) \to \mathrm{D^b}(X)$ is tensor-triangulated.

Proposition 5.6.3. *The pair* $(Y := X/G, \sigma := \pi \circ \mathrm{supph})$ *is a classifying support datum on* $\mathrm{D^G}(X)$.

Proof. First note that σ is well-defined: Since $\mathcal{A}$ is an abelian category the homology objects of a complex in $\mathrm{D^G}(X)$ are G-linearised and therefore their supports are G-invariant closed subsets in X. It follows that $\mathrm{supph}(A^\bullet)$ is G-invariant and closed because $\mathrm{D^G}(X)$ is homologically bounded and $A^\bullet$ has only finitely many homology objects. Therefore $\pi^{-1}(\pi(\mathrm{supph}(A^\bullet)) = \mathrm{supph}(A^\bullet)$ which by definition of the quotient topology means that $\pi(\mathrm{supph}(A^\bullet))$ is a closed subset in X/G. It is trivial that conditions (i)-(iii) are fulfilled for supph and therefore for σ. (iv) is true for supph because of the long exact homology sequence and therefore is true for σ as well. As to (v) note that $(A^\bullet \otimes B^\bullet)_x = A^\bullet_x \otimes B^\bullet_x$ for all $x \in X$ and now the desired property for supph follows from the Künneth formula since $A^\bullet_x$ and $B^\bullet_x$ are complexes of free $\mathcal{O}_x$-modules. Since $\sigma(U \cap V) = \sigma(U) \cap \sigma(V)$ for G-invariant (closed) subsets $U, V \subset X$ (v) is true for σ. Obviously (vi) is fulfilled in our setting so it remains to prove (vii). This property is proved in [22] for the pair (X, supph) on $\mathrm{D^{perf}}(X)$ and our proof will closely follow the proof presented there, with certain necessary modifications. The general idea is of course to use the fact that everything is true in $\mathrm{D^{perf}}(X)$. First note that since the sheaf of homomorphisms of two G-linearized sheaves is again G-linearized the category $\mathrm{D^G}(X)$ has an internal Hom-functor and therefore proposition 2.4. in [4] gives that all thick $\otimes$-ideals are automatically radical. It is clear that φ is well defined and the same property for ψ is easily checked using the fact that $\{A^\bullet \in \mathrm{D^{perf}}(X) \mid \mathrm{supph}(A^\bullet) \subset \pi^{-1}(W)\}$ is a thick $\otimes$-ideal. Obviously if $\mathcal{J} \subset \mathcal{J}'$ then $\varphi(\mathcal{J}) \subset \varphi(\mathcal{J}')$ and if $W \subset W'$ then $\psi(W) \subset \psi(W')$. Next let $\mathcal{J}$ be a thick $\otimes$-ideal in $\mathrm{D^G}(X)$, then $\psi\varphi(\mathcal{J}) = \psi(\bigcup_{A^\bullet \in \mathcal{J}} \pi(\mathrm{supph}(A^\bullet))) = \{B^\bullet \in \mathrm{D^G}(X) \mid \pi(\mathrm{supph}(B^\bullet)) \subset \bigcup_{A^\bullet \in \mathcal{J}} \pi(\mathrm{supph}(A^\bullet)))\}$ so $\mathcal{J} \subset \psi\varphi(\mathcal{J})$. For a closed subset W in X/G we obviously have $\varphi\psi(W) \subset W$. It remains to prove the reverse inclusions. Let us first prove that $W \subset \varphi\psi(W)$:
Since W is a union of closed subsets W_α it suffices to show that for every α there exists an $A^\bullet \in \mathrm{D^G}(X)$ such that $\pi(\mathrm{supph}(A^\bullet)) \subset W_\alpha$ so that $A^\bullet \in \psi(W)$. We know that there exists a perfect complex $E^\bullet$ such that $\mathrm{supph}(E^\bullet) \subset \pi^{-1}(W_\alpha)$. Now set $A^\bullet = \bigoplus_{g \in G} g^* E^\bullet$ and note that this is a G-linearized complex with $\mathrm{supph}(A^\bullet) \subset \pi^{-1}(W_\alpha)$ since $\pi^{-1}(W_\alpha)$ is a G-invariant subset in X. Then $A^\bullet$

is the desired complex.
To show that $\psi\varphi(\mathcal{J}) \subset \mathcal{J}$ let $A^\bullet \in \psi\varphi(\mathcal{J})$ be given. By definition there exist complexes $B^\bullet_\alpha \in \mathrm{D^G}(X)$ such that $\pi(\mathrm{supph}(A^\bullet)) \subset \bigcup_\alpha \pi(\mathrm{supph}(B^\bullet_\alpha))$. Considering these complexes as elements in $\mathrm{D^{perf}}(X)$ we have $\mathrm{supph}(A^\bullet) \subset \bigcup_\alpha \mathrm{supph}(B^\bullet_\alpha)$. Then there exist finitely many indices such that $\mathrm{supph}(A^\bullet) \subset \bigcup_{i=1}^n \mathrm{supph}(B^\bullet_i) = \mathrm{supph}(\bigoplus_{i=1}^n B^\bullet_i)$ (see the original proof in [22]). Now lemma 3.14. in [22] gives that the smallest thick $\otimes$-triangulated subcategory $\mathcal{C}$ in $\mathrm{D^{perf}}(X)$ containing $\bigoplus_{i=1}^n B^\bullet_i$ contains A as well. Consider $\mathcal{C} \cap \mathrm{D^G}(X) := (\mathrm{for})^{-1}(\mathcal{C})$, note that this is a (non-empty) thick $\otimes$-ideal in $\mathrm{D^G}(X)$, $\mathcal{C} \cap \mathrm{D^G}(X) \subset \mathcal{J}$ and $A^\bullet \in \mathcal{C} \cap \mathrm{D^G}(X)$. This completes the proof. □

Remark 5.6.4. It was shown in [6] that under certain assumptions the category $\mathrm{D^G}(X)$ is equivalent to the category $\mathrm{D^b}(\widetilde{X})$ where $\widetilde{X}$ is a crepant resolution of X/G. However, this equivalence is not *tensor* triangulated so our result is not a contradiction.

Proposition 5.6.5. *The map on spectra* $\mathrm{Spc(for)} : X \cong \mathrm{Spc}(\mathrm{D}(X)) \longrightarrow X/G = \mathrm{Spc}(\mathrm{D^G}(X))$ *induced by the* $\otimes$*-triangulated functor* for *corresponds to the projection* $\pi : X \to X/G$.

Proof. Let $x \in X$ be an arbitrary point and consider the corresponding prime ideal $\mathcal{P}_x \in \mathrm{Spc}(\mathrm{D^b}(X))$. Then

$$\mathrm{Spc(for)}(\mathcal{P}_x) = \mathcal{P}_x \ \cap \ \mathrm{D^G}(X) = \left\{A^\bullet \in \mathrm{D^G}(X) \mid A^\bullet_x \cong 0\right\} =: \widetilde{\mathcal{P}}_x$$

and $\widetilde{\mathcal{P}}_x$ corresponds precisely to the class $\overline{x}$ of x in X/G. □

Our next goal is to describe the structure sheaf of $\mathrm{Spc}(\mathrm{D^G}(X))$. Recall from [18] that under our assumptions on X and the G-action the pair $\left(X/G, \mathcal{O}_{X/G}\right)$ is actually a variety and $\mathcal{O}_{X/G} \cong (\pi_*\mathcal{O}_X))^G$, where $(\pi_*(\mathcal{O}_X))^G$ denotes the subsheaf of G-invariant sections in $\pi_*(\mathcal{O}_X)$. The question is: What is the connection between the structure sheaf defined via categories and $(\pi_*\mathcal{O}_X))^G$?

Abbreviate $\mathrm{D^G}(X)$ by $\mathcal{T}$ and X/G by Y. Recall from proposition 2.5.26 that the tensor triangulated functor for : $\mathcal{T} \to \mathrm{D^b}(X)$ induces a morphism of ringed spaces $\pi = f : \mathrm{Spec}(\mathrm{D^b}(X)) \to \mathrm{Spec}(\mathcal{T})$. For an arbitrary open subset $U \subset Y$ f induces a homomorphism $f^U : \Gamma(U, \mathcal{O}_{\mathrm{Spc}(\mathcal{T})}) \to \Gamma(U, f_*(\mathcal{O}_{\mathrm{Spc}(\mathrm{D^b}(X))}))$. Now $\Gamma(U, \mathcal{O}_{\mathrm{Spc}(\mathcal{T})})$ equals to the ring $\mathrm{End}_{\mathcal{T}/\mathcal{T}^U}(\mathbf{1})$ where $\mathcal{T}/\mathcal{T}^U$ denotes the localization of $\mathcal{T}$ in the thick $\otimes$-ideal

$$\mathcal{T}^U = \{A^\bullet \in \mathcal{T} \mid \pi(\mathrm{supph}(A^\bullet)) \subset Y \setminus U\} = \left\{A^\bullet \in \mathcal{T} \mid \mathrm{supph}(A^\bullet) \subset X \setminus \pi^{-1}(U)\right\}$$

and $\Gamma(U, f_*(\mathcal{O}_{\mathrm{Spc}(\mathrm{D}(X))}))$ equals to $\mathrm{End}_{\mathrm{D}(X)/\mathcal{C}}(\mathbf{1})$ where $\mathcal{C}$ is the thick $\otimes$-ideal

$$\mathcal{C} = \left\{B^\bullet \in \mathrm{D}(X) \mid \mathrm{supph}(B^\bullet) \subset \pi^{-1}(U)\right\}$$

i.e. $\mathcal{T}^U = \mathcal{C} \cap \mathcal{T}$. Now $\mathrm{End}_{\mathrm{D^b}(X)/\mathcal{C}}(\mathbf{1}) = \mathrm{End}_{\mathrm{D^b}(X)/\mathcal{C}}(\mathcal{O}_X) = \mathrm{End}_{\mathrm{D^b}(\pi^{-1}(U))}(\mathcal{O}_X) = \Gamma(\pi^{-1}(U), \mathcal{O}_X) = \Gamma(U, \pi_*(\mathcal{O}_X))$. The morphism f^U sends a roof $\mathcal{O}_X \xleftarrow{s} E^\bullet \xrightarrow{\alpha} \mathcal{O}_X$ in $\mathrm{D^G}(X)/\mathcal{T}^U$ to the same roof considered as an element in $\mathrm{D^b}(X)/\mathcal{C}$ which is then identified via the restriction map as an endomorphism of $\mathcal{O}_X$ in $\mathrm{D^b}(\pi^{-1}(U))$. Therefore we see that the image of f^U is contained in $\Gamma(U, \pi_*(\mathcal{O}_X))^G$ and furthermore f^U is onto this subset since any G-invariant section t can be considered as a G-invariant morphism t of $\mathcal{O}_X$ in $\mathrm{D^b}(\pi^{-1}(U))$ and then we just consider the roof $\mathcal{O}_X \overset{\mathrm{id}}{\leftarrow} \mathcal{O}_X \overset{t}{\to} \mathcal{O}_X$ in $\mathrm{D^G}(X)/\mathcal{T}^U$. After sheafification we see that we get a surjective morphism of sheaves $\mathcal{O}_{\mathrm{Spc}(\mathrm{D^G}(X))} \twoheadrightarrow (\pi_*(\mathcal{O}_X))^G$.

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