EXERCISES, COMPLEX GEOMETRY, UNIVERSITY OF HAMBURG, WINTER SEMESTER 2015/2016

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Sheet 8

Exercise 1. (compare [2, Ex. 2.6.2]**)** Show that any oriented Riemann surface admits a natural almost complex structure.

Hint: You could first show that a two-dimensional euclidian vector space V with a fixed orientation admits a natural almost complex structure.

Exercise 2. (compare [2, Ex. 2.6.6]**)** Let X be a complex manifold. Show that the exterior product induces a multiplication on the full Dolbeault cohomology $\oplus_{p,q} H^{p,q}(X)$ and that this yields a $\mathbb{Z}^2$-graded algebra $\oplus_{p,q} H^{p,q}(X)$ for any complex manifold X.

Exercise 3. (compare [2, Ex. 2.6.7]**)** Let X be a complex manifold. Verify that the following definition of the *Bott-Chern cohomology*

$$H^{p,q}_{\mathrm{BC}}(X) = \frac{\{\alpha \in \mathcal{A}^{p,q}(X) \mid d\alpha = 0\}}{\partial\bar{\partial}\mathcal{A}^{p-1,q-1}}$$

makes sense. Show that there are natural maps

$$H^{p,q}_{\mathrm{BC}}(X) \longrightarrow H^{p,q}(X).$$

Exercise 4. (compare [2, Ex. 1.2.3]**)** Prove the following identities: $*\Pi^{p,q} = \Pi^{n-q,n-p}*$ and $[L, \mathbf{I}] = [\Lambda, \mathbf{I}] = 0$.

Exercise 5. (compare [2, Ex. 1.2.8]**)** Let $\alpha \in P^k$ and $r \geq s$. Prove the following formula:

$$\Lambda^s L^r \alpha = (-1)^s r(r-1)\dots(r-s+1)(n-k-r+1)\dots(n-k-r+s)L^{r-s}\alpha.$$

References

[1] R. Hartshorne, *Algebraic geometry*, Springer, New York, 1977.
[2] D. Huybrechts, *Complex geometry: An introduction*, Springer, Berlin (2005).
[3] M. Kashiwara and P. Shapira, *Sheaves on manifolds*, Springer, Berlin (1994).
[4] C. Schnell, *Complex manifolds*, available at `http://www.math.stonybrook.edu/~cschnell/`.
[5] C. Voisin, *Hodge theory and complex algebraic geometry*, Cambridge University Press, Cambridge (2002).